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Research article

Estimation-type results with respect to the parameterized (p, q)-integral inequalities

Chunyan Luo[1], **Tingsong Du**[2,*], **Muhammad Uzair Awan**[3] **and Yao Zhang**[1]

[1] Department of Mathematics, College of Science, China Three Gorges University, 443002 Yichang, China
[2] Three Gorges Mathematical Research Center, China Three Gorges University, 443002 Yichang, China
[3] Department of Mathematics, Government College University, Faisalabad, Pakistan

* **Correspondence:** Email: tingsongdu@ctgu.edu.cn.

Abstract: We establish a (p, q)-integral identity with parameters and certain new (p, q)-integral inequalities of different types through (p, q)-differentiable mappings. Many results obtained in this article provide significant extensions of other related results given in the literature. Furthermore, we construct three examples to illustrate the investigated results.

Keywords: (p, q)-integral inequalities; (p, q)-derivative; convexity
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1. Introduction

Throughout this paper, let $a < b$ and p, q be two constants which satisfy $0 < q < p \leq 1$.

In 2016, Tunç and Göv defined the (p, q)-derivative and (p, q)-integral as follows.

Definition 1. ([14]) Let $f : [a, b] \to \mathbb{R}$ be continuous, the (p, q)-derivative of f at $x \in [a, b]$ is defined by the expression

$$ {}_aD_{p,q}f(x) = \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)}, \quad x \neq a. $$

Since f is a continuous mapping, one has ${}_aD_{p,q}f(a) = \lim_{x\to a} {}_aD_{p,q}f(x)$.

Definition 2. ([14]) Let $f : [a, b] \to \mathbb{R}$ be continuous, the (p, q)-integral on $[a, x]$ is defined as

$$ \int_a^x f(t)\,{}_a\mathrm{d}_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\Big(\frac{q^n}{p^{n+1}}x + \Big(1 - \frac{q^n}{p^{n+1}}\Big)a\Big) $$

for $x \in [a, b]$. Moreover, if $c \in (a, x)$, then the (p, q)-integral on $[c, x]$ is delineated as

$$\int_c^x f(t)_a \mathrm{d}_{p,q}t = \int_a^x f(t)_a \mathrm{d}_{p,q}t - \int_a^c f(t)_a \mathrm{d}_{p,q}t.$$

In Definition 1 and Definition 2, if we take $p = 1$, then we get the definitions of q-derivative and q-integral, respectively.

In 2018, Kunt et al. presented the (p, q)-integrals version of the Hermite–Hadamard's inequality as follows.

Theorem 1 ([4]). *Let $f : [a, b] \to \mathbb{R}$ be convex, continuous and (p, q)-differentiable on $[a, b]$. Then we have*

$$f\Big(\frac{qa + pb}{p + q}\Big) \leq \frac{1}{p(b - a)} \int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x \leq \frac{qf(a) + pf(b)}{p + q}.$$

Clearly, if we put $p = 1$ in Theorem 1, then we obtain the q-integrals version of the Hermite–Hadamard's inequality. For recent results on the q-Hermite–Hadamard's inequality, see [5, 7–9, 17]. Besides this, we also refer to some recent related work with respect to other type quantum integral inequalities, for example, see [3, 6, 10, 12, 13, 16] and the references therein.

Here, our main purpose is to investigate the parameterized inequalities for (p, q)-integral operators. For this purpose, we will establish a (p, q)-integral identity with parameters. Using this (p, q)-integral identity, we present several (p, q)-integral inequalities for a class of (p, q)-differentiable mappings, which are related to convex mappings. In addition, we obtain some estimation-type results for (p, q)-integral inequalities by considering the boundedness and Lipschitz condition. Some relevant connections of the derived results in this paper with previous ones are also pointed out.

2. Auxiliary results

We need the following lemma.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous and (p, q)-differentiable function on (a, b). If ${}_aD_{p,q}f$ is integrable on $[a, b]$ and $\lambda, \mu \in [0, 1]$, then the following identity holds:*

$$\begin{aligned}\Lambda(\lambda, \mu; a, b) = (b - a)\Big\{ &\int_0^\mu (qt + \lambda\mu - \lambda)_a D_{p,q} f\big(tb + (1 - t)a\big)_0 \mathrm{d}_{p,q}t \\ &+ \int_\mu^1 (qt + \lambda\mu - 1)_a D_{p,q} f\big(tb + (1 - t)a\big)_0 \mathrm{d}_{p,q}t\Big\},\end{aligned}$$

where

$$\begin{aligned}\Lambda(\lambda, \mu; a, b) :=& \lambda[(1 - \mu)f(a) + \mu f(b)] + (1 - \lambda)f(\mu b + (1 - \mu)a) \\ &- \frac{1}{p(b - a)} \int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x.\end{aligned}$$

Proof. By identical transformation, we get

$$\begin{aligned}
&(b-a)\Big\{\int_0^{\mu}(qt+\lambda\mu-\lambda)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t \\
&\quad+\int_{\mu}^{1}(qt+\lambda\mu-1)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\Big\} \\
&\quad=(b-a)\Big\{\int_0^{1}(qt+\lambda\mu-1)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t \\
&\qquad+\int_0^{\mu}(1-\lambda)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\Big\}.
\end{aligned}\tag{2.1}$$

From Definition 1, we get

$$\begin{aligned}
&{}_aD_{p,q}f(tb+(1-t)a) \\
&=\frac{f(p[tb+(1-t)a]+(1-p)a)-f(q[tb+(1-t)a]+(1-q)a)}{t(p-q)(b-a)} \\
&=\frac{f(ptb+(1-pt)a)-f(qtb+(1-qt)a)}{t(p-q)(b-a)}.
\end{aligned}$$

Utilizing the above calculation and Definition 2, we have

$$\begin{aligned}
&\int_0^1 t\,{}_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t \\
&\quad=\int_0^1\frac{f(ptb+(1-pt)a)-f(qtb+(1-qt)a)}{(p-q)(b-a)}{}_0\mathrm{d}_{p,q}t \\
&\quad=\frac{1}{b-a}\Big\{\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big) \\
&\qquad-\frac{p}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f\Big(\frac{q^{n+1}}{p^{n+1}}b+\Big(1-\frac{q^{n+1}}{p^{n+1}}\Big)a\Big)\Big\} \\
&\quad=\frac{1}{b-a}\Big\{\frac{1}{p}f(b)+\Big(1-\frac{p}{q}\Big)\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}f\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big)\Big\} \\
&\quad=\frac{1}{b-a}\Big\{\frac{1}{q}f(b)-\frac{p-q}{q}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big)\Big\} \\
&\quad=\frac{f(b)}{q(b-a)}-\frac{1}{pq(b-a)^2}\int_a^{pb+(1-p)a}f(x)_a\mathrm{d}_{p,q}x,
\end{aligned}\tag{2.2}$$

$$\begin{aligned}
&\int_0^1 {}_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t \\
&=\frac{1}{b-a}\Big\{\sum_{n=0}^{\infty}f\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big)-\sum_{n=0}^{\infty}f\Big(\frac{q^{n+1}}{p^{n+1}}b+\Big(1-\frac{q^{n+1}}{p^{n+1}}\Big)a\Big)\Big\} \\
&=\frac{f(b)-f(a)}{b-a}
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&\int_0^{\mu} {}_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t \\
&= \frac{1}{b-a}\left\{\sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n}\mu b+\left(1-\frac{q^n}{p^n}\mu\right)a\right)-\sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{p^{n+1}}\mu b+\left(1-\frac{q^{n+1}}{p^{n+1}}\mu\right)a\right)\right\} \\
&= \frac{f(\mu b+(1-\mu)a)-f(a)}{b-a}.
\end{aligned} \tag{2.4}$$

Substituting (2.2), (2.3) and (2.4) into (2.1), we obtain the desired result. This ends the proof. □

Remark 1. Consider Lemma 1.
(i) Putting $\mu = 0$, we have

$$f(a)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x = (b-a)\int_0^1 (qt-1)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t. \tag{2.5}$$

(ii) Putting $\mu = 1$, we have

$$f(b)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x = (b-a)\int_0^1 qt\ {}_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t. \tag{2.6}$$

(iii) Putting $\mu = \frac{p}{p+q}$, we have

$$\begin{aligned}
&\lambda\frac{qf(a)+pf(b)}{p+q}+(1-\lambda)f\left(\frac{qa+pb}{p+q}\right)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x \\
&= (b-a)\left\{\int_0^{\frac{p}{p+q}}\left(qt-\frac{\lambda q}{p+q}\right)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\right. \\
&\quad \left.+\int_{\frac{p}{p+q}}^1\left(qt+\frac{p\lambda}{p+q}-1\right)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\right\}.
\end{aligned} \tag{2.7}$$

Remark 2. Consider Lemma 1.
(i) Putting $\lambda = 0$, we get

$$\begin{aligned}
&f(\mu b+(1-\mu)a)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x \\
&= (b-a)\left\{\int_0^{\mu} qt\ {}_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t+\int_{\mu}^1 (qt-1)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\right\}.
\end{aligned} \tag{2.8}$$

Specially, taking $\mu = \frac{p}{p+q}$, we obtain Lemma 3 presented by Kunt et al. in [4].
(ii) Putting $\lambda = \frac{1}{3}$, we get

$$\begin{aligned}
&\frac{1}{3}\Big[(1-\mu)f(a)+\mu f(b)+2f(\mu b+(1-\mu)a)\Big]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x \\
&= (b-a)\left\{\int_0^{\mu}\left(qt+\frac{1}{3}\mu-\frac{1}{3}\right)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\right. \\
&\quad \left.+\int_{\mu}^1\left(qt+\frac{1}{3}\mu-1\right)_aD_{p,q}f(tb+(1-t)a)_0\mathrm{d}_{p,q}t\right\}.
\end{aligned} \tag{2.9}$$

Specially, taking $\mu = \frac{p}{p+q}$, we obtain the Simpson-like integral identity

$$\begin{aligned}
&\frac{1}{3}\left[\frac{qf(a)+pf(b)}{p+q}+2f\left(\frac{qa+pb}{p+q}\right)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\
&= (b-a)\left\{\int_0^{\frac{p}{p+q}}\left(qt-\frac{q}{3p+3q}\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t\right. \qquad (2.10)\\
&\left.+\int_{\frac{p}{p+q}}^1\left(qt+\frac{p}{3p+3q}-1\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t\right\}.
\end{aligned}$$

(*iii*) Putting $\lambda = \frac{1}{2}$, we get

$$\begin{aligned}
&\frac{1}{2}\Big[(1-\mu)f(a)+\mu f(b)+f(\mu b+(1-\mu)a)\Big]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\
&= (b-a)\left\{\int_0^{\mu}\left(qt+\frac{1}{2}\mu-\frac{1}{2}\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t\right. \qquad (2.11)\\
&\left.+\int_{\mu}^1\left(qt+\frac{1}{2}\mu-1\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t\right\}.
\end{aligned}$$

Specially, taking $\mu = \frac{p}{p+q}$, we obtain the averaged midpoint-trapezoid-like integral identity

$$\begin{aligned}
&\frac{1}{2}\left[\frac{qf(a)+pf(b)}{p+q}+f\left(\frac{qa+pb}{p+q}\right)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\
&= (b-a)\left\{\int_0^{\frac{p}{p+q}}\left(qt-\frac{q}{2p+2q}\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t\right. \qquad (2.12)\\
&\left.+\int_{\frac{p}{p+q}}^1\left(qt+\frac{p}{2p+2q}-1\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t\right\}.
\end{aligned}$$

(*iv*) Putting $\lambda = 1$, we get

$$\begin{aligned}
&(1-\mu)f(a)+\mu f(b)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\
&= (b-a)\int_0^1 (qt+\mu-1)\,{}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t.
\end{aligned} \qquad (2.13)$$

Specially, taking $\mu = \frac{p}{p+q}$, we obtain the trapezoid-like integral identity

$$\begin{aligned}
&\frac{qf(a)+pf(b)}{p+q}-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x \\
&= (b-a)\int_0^1\left(qt+\frac{p}{p+q}-1\right){}_aD_{p,q}f(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t.
\end{aligned} \qquad (2.14)$$

Worth mentioning, to the best of our knowledge the above-obtained (p, q)-integral identities (2.5)-(2.14) are new in the literature.

3. Main results

In 2017, Kunt et al. established the (p, q)-Hermite–Hadamard inequality in the paper [4]. Here we give a new proof, which is more concise.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be convex, continuous and (p, q)-differentiable on (a, b). Then we have*

$$f\Big(\frac{qa+pb}{p+q}\Big) \le \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x \le \frac{qf(a)+pf(b)}{p+q}.$$

Proof. It is obvious that $\sum_{n=0}^{\infty}(1-\frac{q}{p})\frac{q^n}{p^n} = 1$, $0 < q < p \le 1$. Since Jensen's inequality defined on convex sets for infinite sums still remains true, utilizing this fact and Definition 2, we have

$$\begin{aligned} f\Big(\frac{qa+pb}{p+q}\Big) &= f\Big(\sum_{n=0}^{\infty}\Big(1-\frac{q}{p}\Big)\frac{q^n}{p^n}\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big)\Big) \\ &\le \sum_{n=0}^{\infty}\Big(1-\frac{q}{p}\Big)\frac{q^n}{p^n} f\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big) \\ &= \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x. \end{aligned}$$

Using Definition 2 and the convexity of f, we get

$$\begin{aligned} &\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x \\ &= \sum_{n=0}^{\infty}\Big(1-\frac{q}{p}\Big)\frac{q^n}{p^n} f\Big(\frac{q^n}{p^n}b+\Big(1-\frac{q^n}{p^n}\Big)a\Big) \\ &\le \sum_{n=0}^{\infty}\Big(1-\frac{q}{p}\Big)\frac{q^n}{p^n}\Big(\frac{q^n}{p^n}f(b)+\Big(1-\frac{q^n}{p^n}\Big)f(a)\Big) \\ &= \frac{qf(a)+pf(b)}{p+q}. \end{aligned}$$

The proof is completed. □

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be continuous and (p, q)-differentiable on (a, b), and let ${}_aD_{p,q}f$ be integrable on $[a, b]$. Then the following inequality*

$$\begin{aligned} &\Big|\Lambda(\lambda,\mu;a,b)\Big| \\ &\le (b-a)\Big\{\Big[\Phi_1(\lambda,\mu;p,q)+\Phi_2(\lambda,\mu;p,q)-\Phi_3(\lambda,\mu;p,q)\Big]\Big|{}_aD_{p,q}f(b)\Big| \\ &\quad+\Big[\Phi_4(\lambda,\mu;p,q)+\Phi_5(\lambda,\mu;p,q)-\Phi_6(\lambda,\mu;p,q)-\Phi_1(\lambda,\mu;p,q) \\ &\quad-\Phi_2(\lambda,\mu;p,q)+\Phi_3(\lambda,\mu;p,q)\Big]\Big|{}_aD_{p,q}f(a)\Big|\Big\} \end{aligned} \tag{3.1}$$

holds for all $\lambda, \mu \in [0,1]$ *if* $|{}_aD_{p,q}f|$ *is convex on* $[a,b]$*, where*

$$\Phi_1(\lambda,\mu;p,q) = \int_0^{\mu} t\Big|qt+\lambda\mu-\lambda\Big|{}_0\mathrm{d}_{p,q}t = \begin{cases} \frac{\mu^2(\lambda-\lambda\mu)}{p+q} - \frac{q\mu^3}{p^2+pq+q^2}, & (\lambda+q)\mu \le \lambda, \\ \frac{2(\lambda-\lambda\mu)^3}{q^2}\left(\frac{1}{p+q} - \frac{1}{p^2+pq+q^2}\right) \\ \quad + \frac{q\mu^3}{p^2+pq+q^2} - \frac{\mu^2(\lambda-\lambda\mu)}{p+q}, & (\lambda+q)\mu > \lambda, \end{cases} \tag{3.2}$$

$$\Phi_2(\lambda,\mu;p,q) = \int_0^{1} t\Big|qt+\lambda\mu-1\Big|{}_0\mathrm{d}_{p,q}t = \begin{cases} \frac{1-\lambda\mu}{p+q} - \frac{q}{p^2+pq+q^2}, & \lambda\mu+q \le 1, \\ \frac{2(1-\lambda\mu)^3}{q^2}\left(\frac{1}{p+q} - \frac{1}{p^2+pq+q^2}\right) \\ \quad + \frac{q}{p^2+pq+q^2} - \frac{1-\lambda\mu}{p+q}, & \lambda\mu+q > 1, \end{cases} \tag{3.3}$$

$$\Phi_3(\lambda,\mu;p,q) = \int_0^{\mu} t\Big|qt+\lambda\mu-1\Big|{}_0\mathrm{d}_{p,q}t = \begin{cases} \frac{\mu^2(1-\lambda\mu)}{p+q} - \frac{q\mu^3}{p^2+pq+q^2}, & (\lambda+q)\mu \le 1, \\ \frac{2(1-\lambda\mu)^3}{q^2}\left(\frac{1}{p+q} - \frac{1}{p^2+pq+q^2}\right) \\ \quad + \frac{q\mu^3}{p^2+pq+q^2} - \frac{\mu^2(1-\lambda\mu)}{p+q}, & (\lambda+q)\mu > 1, \end{cases} \tag{3.4}$$

$$\Phi_4(\lambda,\mu;p,q) = \int_0^{\mu} \Big|qt+\lambda\mu-\lambda\Big|{}_0\mathrm{d}_{p,q}t = \begin{cases} \lambda\mu(1-\mu) - \frac{q\mu^2}{p+q}, & (\lambda+q)\mu \le \lambda, \\ \frac{2(\lambda-\lambda\mu)^2}{q}\left(1 - \frac{1}{p+q}\right) \\ \quad + \frac{q\mu^2}{p+q} - \lambda\mu(1-\mu), & (\lambda+q)\mu > \lambda, \end{cases} \tag{3.5}$$

$$\Phi_5(\lambda,\mu;p,q) = \int_0^{1} \Big|qt+\lambda\mu-1\Big|{}_0\mathrm{d}_{p,q}t = \begin{cases} \frac{p}{p+q} - \lambda\mu, & \lambda\mu+q \le 1, \\ \frac{2(1-\lambda\mu)^2}{q}\left(1 - \frac{1}{p+q}\right) \\ \quad + \lambda\mu - \frac{p}{p+q}, & \lambda\mu+q > 1, \end{cases} \tag{3.6}$$

and

$$\Phi_6(\lambda,\mu;p,q) = \int_0^{\mu} \left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t$$
$$= \begin{cases} \mu(1-\lambda\mu) - \frac{q\mu^2}{p+q}, & (\lambda+q)\mu \le 1, \\ \frac{2(1-\lambda\mu)^2}{q}\left(1-\frac{1}{p+q}\right) \\ \quad +\frac{q\mu^2}{p+q} - \mu(1-\lambda\mu), & (\lambda+q)\mu > 1. \end{cases} \tag{3.7}$$

Proof. Utilizing Lemma 1 and the convexity of $|{}_aD_{p,q}f|$, we have

$$\begin{aligned}
&\left|\Lambda(\lambda,\mu;a,b)\right| \\
&\le (b-a)\Big\{\int_0^{\mu} \left|qt+\lambda\mu-\lambda\right|\left|{}_aD_{p,q}f(tb+(1-t)a)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad + \int_{\mu}^{1} \left|qt+\lambda\mu-1\right|\left|{}_aD_{p,q}f(tb+(1-t)a)\right|{}_0\mathrm{d}_{p,q}t\Big\} \\
&\le (b-a)\Big\{\int_0^{\mu} \left|qt+\lambda\mu-\lambda\right|\Big[t\left|{}_aD_{p,q}f(b)\right| + (1-t)\left|{}_aD_{p,q}f(a)\right|\Big]{}_0\mathrm{d}_{p,q}t \\
&\quad + \int_{\mu}^{1} \left|qt+\lambda\mu-1\right|\Big[t\left|{}_aD_{p,q}f(b)\right| + (1-t)\left|{}_aD_{p,q}f(a)\right|\Big]{}_0\mathrm{d}_{p,q}t\Big\} \\
&= (b-a)\Big\{\Big[\int_0^{\mu} t\left|qt+\lambda\mu-\lambda\right|{}_0\mathrm{d}_{p,q}t + \int_0^{1} t\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t \\
&\quad - \int_0^{\mu} t\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t\Big]\left|{}_aD_{p,q}f(b)\right| \\
&\quad + \Big[\int_0^{\mu} \left|qt+\lambda\mu-\lambda\right|{}_0\mathrm{d}_{p,q}t + \int_0^{1} \left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t \\
&\quad - \int_0^{\mu} \left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t - \int_0^{\mu} t\left|qt+\lambda\mu-\lambda\right|{}_0\mathrm{d}_{p,q}t \\
&\quad - \int_0^{1} t\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t + \int_0^{\mu} t\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t\Big]\left|{}_aD_{p,q}f(a)\right|\Big\}.
\end{aligned}$$

The proof is completed. □

Remark 3. Consider $\mu = \frac{p}{p+q}$ in Theorem 3.
(i) For $\lambda = 0$, we obtain the midpoint-like integral inequality presented by Kunt et al. in [4, Theorem 7]. Specially, taking $p = 1$, we get Theorem 13 established by Alp et al. in [2].

(*ii*) For $\lambda = \frac{1}{3}$, we obtain the Simpson-like integral inequality

$$\begin{aligned}
&\left|\frac{1}{3}\left[\frac{qf(a)+pf(b)}{p+q}+2f\left(\frac{qa+pb}{p+q}\right)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq (b-a)\left\{\left[\Phi_1\left(\frac{1}{3},\frac{p}{p+q};p,q\right)+\Phi_2\left(\frac{1}{3},\frac{p}{p+q};p,q\right)\right.\right. \\
&\quad\quad\left.-\Phi_3\left(\frac{1}{3},\frac{p}{p+q};p,q\right)\right]\left|{}_aD_{p,q}f(b)\right|+\left[\Phi_4\left(\frac{1}{3},\frac{p}{p+q};p,q\right)\right. \\
&\quad\quad+\Phi_5\left(\frac{1}{3},\frac{p}{p+q};p,q\right)-\Phi_6\left(\frac{1}{3},\frac{p}{p+q};p,q\right)-\Phi_1\left(\frac{1}{3},\frac{p}{p+q};p,q\right) \\
&\quad\quad\left.\left.-\Phi_2\left(\frac{1}{3},\frac{p}{p+q};p,q\right)+\Phi_3\left(\frac{1}{3},\frac{p}{p+q};p,q\right)\right]\left|{}_aD_{p,q}f(a)\right|\right\}.
\end{aligned}$$

Specially, taking $p = 1$ and let $q \to 1^-$, we get Corollary 1 established by Alomari et al. in [1].
(*iii*) For $\lambda = \frac{1}{2}$, we obtain the averaged midpoint-trapezoid-like integral inequality

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{qf(a)+pf(b)}{p+q}+f\left(\frac{qa+pb}{p+q}\right)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq (b-a)\left\{\left[\Phi_1\left(\frac{1}{2},\frac{p}{p+q};p,q\right)+\Phi_2\left(\frac{1}{2},\frac{p}{p+q};p,q\right)\right.\right. \\
&\quad\quad\left.-\Phi_3\left(\frac{1}{2},\frac{p}{p+q};p,q\right)\right]\left|{}_aD_{p,q}f(b)\right|+\left[\Phi_4\left(\frac{1}{2},\frac{p}{p+q};p,q\right)\right. \\
&\quad\quad+\Phi_5\left(\frac{1}{2},\frac{p}{p+q};p,q\right)-\Phi_6\left(\frac{1}{2},\frac{p}{p+q};p,q\right)-\Phi_1\left(\frac{1}{2},\frac{p}{p+q};p,q\right) \\
&\quad\quad\left.\left.-\Phi_2\left(\frac{1}{2},\frac{p}{p+q};p,q\right)+\Phi_3\left(\frac{1}{2},\frac{p}{p+q};p,q\right)\right]\left|{}_aD_{p,q}f(a)\right|\right\}.
\end{aligned}$$

Specially, taking $p = 1$ and let $q \to 1^-$, we get Corollary 3.4 established by Xi and Qi in [15].
(*iv*) For $\lambda = 1$, we obtain the trapezoid-like integral inequality

$$\begin{aligned}
&\left|\frac{qf(a)+pf(b)}{p+q}-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x\right| \\
&\quad\leq (b-a)\left\{\Phi_2\left(1,\frac{p}{p+q};p,q\right)\left|{}_aD_{p,q}f(b)\right|\right. \\
&\quad\quad\left.+\left[\Phi_5\left(1,\frac{p}{p+q};p,q\right)-\Phi_2\left(1,\frac{p}{p+q};p,q\right)\right]\left|{}_aD_{p,q}f(a)\right|\right\}.
\end{aligned}$$

Specially, taking $p = 1$, we get Theorem 4.1 given by Sudsutad et al. in [11].

If $|{}_aD_{p,q}f|^r$ for $r > 1$ is convex, then we have the following theorem.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be continuous and (p,q)-differentiable on (a,b), and let ${}_aD_{p,q}f$ be integrable on $[a,b]$. Then the following inequality*

$$\begin{aligned}
&\left|\Lambda(\lambda,\mu;a,b)\right|\\
&\le (b-a)\Bigg\{\Phi_5^{1-\frac{1}{r}}(\lambda,\mu;p,q)\Big[\Phi_2(\lambda,\mu;p,q)\left|{}_aD_{p,q}f(b)\right|^r\\
&\quad+\Big(\Phi_5(\lambda,\mu;p,q)-\Phi_2(\lambda,\mu;p,q)\Big)\left|{}_aD_{p,q}f(a)\right|^r\Big]^{\frac{1}{r}}\\
&\quad+(1-\lambda)\mu^{1-\frac{1}{r}}\left[\frac{\mu^2}{p+q}\left|{}_aD_{p,q}f(b)\right|^r+\left(\mu-\frac{\mu^2}{p+q}\right)\left|{}_aD_{p,q}f(a)\right|^r\right]^{\frac{1}{r}}\Bigg\}
\end{aligned}$$

holds for all $\lambda,\mu\in[0,1]$ if $|{}_aD_{p,q}f|^r$ for $r>1$ is convex on $[a,b]$, where $\Phi_2(\lambda,\mu;p,q)$ and $\Phi_5(\lambda,\mu;p,q)$ are defined by (3.3) and (3.6), respectively.

Proof. Using Lemma 1 and the power mean inequality, we have

$$\begin{aligned}
&\left|\Lambda(\lambda,\mu;a,b)\right|\\
&\le (b-a)\Bigg\{\left(\int_0^1\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}}\\
&\quad\times\left(\int_0^1\left|qt+\lambda\mu-1\right|\left|{}_aD_{p,q}f(tb+(1-t)a)\right|^r{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}\\
&\quad+(1-\lambda)\left(\int_0^\mu 1\;{}_0\mathrm{d}_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_0^\mu\left|{}_aD_{p,q}f(tb+(1-t)a)\right|^r{}_0\mathrm{d}_{p,q}t\right)^{\frac{1}{r}}\Bigg\}.
\end{aligned}\tag{3.8}$$

Utilizing the convexity of $|{}_aD_{p,q}f|^r$, we get

$$\begin{aligned}
&\int_0^1\left|qt+\lambda\mu-1\right|\left|{}_aD_{p,q}f(tb+(1-t)a)\right|^r{}_0\mathrm{d}_{p,q}t\\
&\le\int_0^1\left|qt+\lambda\mu-1\right|\Big[t\left|{}_aD_{p,q}f(b)\right|^r+(1-t)\left|{}_aD_{p,q}f(a)\right|^r\Big]{}_0\mathrm{d}_{p,q}t\\
&=\left(\int_0^1 t\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t\right)\left|{}_aD_{p,q}f(b)\right|^r\\
&\quad+\left(\int_0^1\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t-\int_0^1 t\left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t\right)\left|{}_aD_{p,q}f(a)\right|^r
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\int_0^\mu\left|{}_aD_{p,q}f(tb+(1-t)a)\right|^r{}_0\mathrm{d}_{p,q}t\\
&\le\int_0^\mu\Big[t\left|{}_aD_{p,q}f(b)\right|^r+(1-t)\left|{}_aD_{p,q}f(a)\right|^r\Big]{}_0\mathrm{d}_{p,q}t\\
&=\frac{\mu^2}{p+q}\left|{}_aD_qf(b)\right|^r+\left(\mu-\frac{\mu^2}{p+q}\right)\left|{}_aD_qf(a)\right|^r.
\end{aligned}\tag{3.10}$$

Using (3.9) and (3.10) in (3.8), we deduce the desired result. The proof is completed. □

A similar result is embodied in the following theorem.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be continuous and (p,q)-differentiable on (a,b), and let ${}_aD_{p,q}f$ be integrable on $[a,b]$. Then the following inequality*

$$\begin{aligned}&\Big|\Lambda(\lambda,\mu;a,b)\Big|\\&\le(b-a)\Bigg\{\Psi^{\frac{1}{s}}(\lambda,\mu;p,q)\Big[\frac{1}{p+q}\Big|{}_aD_{p,q}f(b)\Big|^r+\Big(1-\frac{1}{p+q}\Big)\Big|{}_aD_{p,q}f(a)\Big|^r\Big]^{\frac{1}{r}}\\&\quad+(1-\lambda)\mu^{\frac{1}{s}}\Big[\frac{\mu^2}{p+q}\Big|{}_aD_{p,q}f(b)\Big|^r+\Big(\mu-\frac{\mu^2}{p+q}\Big)\Big|{}_aD_{p,q}f(a)\Big|^r\Big]^{\frac{1}{r}}\Bigg\}\end{aligned}$$

holds for all $\lambda,\mu\in[0,1]$ if $|{}_aD_{p,q}f|^r$ for $r>1$ with $r^{-1}+s^{-1}=1$ is convex on $[a,b]$, where

$$\begin{aligned}&\Psi(\lambda,\mu;p,q)\\&=\int_0^1\Big|qt+\lambda\mu-1\Big|^s{}_0\mathrm{d}_{p,q}t\\&=\begin{cases}(p-q)\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Big(1-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\Big)^s, & 0\le\lambda\mu\le1-q,\\ \left[\begin{array}{l}(p-q)(1-\lambda\mu)^{s+1}\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\Big(1-\frac{q^n}{p^{n+1}}\Big)^s\\+(p-q)\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Big(\frac{q^{n+1}}{p^{n+1}}-1+\lambda\mu\Big)^s\\-(p-q)(1-\lambda\mu)^{s+1}\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\Big(\frac{q^n}{p^{n+1}}-1\Big)^s\end{array}\right], & 1-q<\lambda\mu\le1.\end{cases}\end{aligned}$$

Proof. Using Lemma 1 and the Hölder inequality, one has

$$\begin{aligned}&\Big|\Lambda(\lambda,\mu;a,b)\Big|\\&\le(b-a)\Bigg\{\Big(\int_0^1\Big|qt+\lambda\mu-1\Big|^s{}_0\mathrm{d}_{p,q}t\Big)^{\frac{1}{s}}\Big(\int_0^1\Big|{}_aD_{p,q}f(tb+(1-t)a)\Big|^r{}_0\mathrm{d}_{p,q}t\Big)^{\frac{1}{r}}\\&\quad+(1-\lambda)\Big(\int_0^\mu1^s{}_0\mathrm{d}_{p,q}t\Big)^{\frac{1}{s}}\Big(\int_0^\mu\Big|{}_aD_{p,q}f(tb+(1-t)a)\Big|^r{}_0\mathrm{d}_{p,q}t\Big)^{\frac{1}{r}}\Bigg\}.\end{aligned}\tag{3.11}$$

Utilizing the convexity of $|{}_aD_{p,q}f|^r$, one gets

$$\begin{aligned}&\int_0^1\Big|{}_aD_{p,q}f(tb+(1-t)a)\Big|^r{}_0\mathrm{d}_{p,q}t\\&\le\int_0^1\Big[t\Big|{}_aD_{p,q}f(b)\Big|^r+(1-t)\Big|{}_aD_{p,q}f(a)\Big|^r\Big]{}_0\mathrm{d}_{p,q}t\\&=\frac{1}{p+q}\Big|{}_aD_{p,q}f(b)\Big|^r+\Big(1-\frac{1}{p+q}\Big)\Big|{}_aD_{p,q}f(a)\Big|^r\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
&\int_0^{\mu} \left|{}_aD_{p,q}f(tb+(1-t)a)\right|^r {}_0\mathrm{d}_{p,q}t \\
&\leq \int_0^{\mu} \left[t\left|{}_aD_{p,q}f(b)\right|^r + (1-t)\left|{}_aD_{p,q}f(a)\right|^r\right] {}_0\mathrm{d}_{p,q}t \\
&= \frac{\mu^2}{p+q}\left|{}_aD_{p,q}f(b)\right|^r + \left(\mu - \frac{\mu^2}{p+q}\right)\left|{}_aD_{p,q}f(a)\right|^r.
\end{aligned} \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), one has the desired result. The proof is completed. □

Remark 4. For $\mu = \frac{p}{p+q}$, if we take $\lambda = 0$, $\lambda = \frac{1}{3}$, $\lambda = \frac{1}{2}$ and $\lambda = 1$ in Theorem 4 and Theorem 5, respectively, then we obtain the midpoint-like integral inequality, the Simpson-like integral inequality, the averaged midpoint-trapezoid-like integral inequality and the trapezoid-like integral inequality, respectively.

The following result is a lower bound for (p,q)-integral inequality involving product of two convex functions.

Theorem 6. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous and non-negative on $[a,b]$. If f and g are convex functions on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_{p,q}x \\
&\leq \left(1 - \frac{1}{p+q} + \frac{1}{p^2+pq+q^2}\right)[f(a)g(b)+f(b)g(a)] \\
&\quad + \left(\frac{2}{p+q} - \frac{1}{p^2+pq+q^2}\right)f(a)g(a) + \left(1 - \frac{1}{p^2+pq+q^2}\right)f(b)g(b).
\end{aligned}$$

Proof. Since f and g are convex and non-negative, we have

$$\begin{aligned}
&4f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&= 4f\left(\frac{tb+(1-t)a}{2} + \frac{ta+(1-t)b}{2}\right)g\left(\frac{tb+(1-t)a}{2} + \frac{ta+(1-t)b}{2}\right) \\
&\leq \left[f(tb+(1-t)a) + f(ta+(1-t)b)\right]\left[g(tb+(1-t)a) + g(ta+(1-t)b)\right] \\
&= f(tb+(1-t)a)g(tb+(1-t)a) + f(tb+(1-t)a)g(ta+(1-t)b) \\
&\quad + f(ta+(1-t)b)g(tb+(1-t)a) + f(ta+(1-t)b)g(ta+(1-t)b) \\
&\leq f(tb+(1-t)a)g(tb+(1-t)a) \\
&\quad + [tf(b)+(1-t)f(a)][tg(a)+(1-t)g(b)] \\
&\quad + [tf(a)+(1-t)f(b)][tg(b)+(1-t)g(a)] \\
&\quad + [tf(a)+(1-t)f(b)][tg(a)+(1-t)g(b)] \\
&= f(tb+(1-t)a)g(tb+(1-t)a) + (1-t+t^2)[f(a)g(b)+f(b)g(a)] \\
&\quad + (2t-t^2)f(a)g(a) + (1-t^2)f(b)g(b).
\end{aligned}$$

Taking (p,q)-integral for the above inequality about t on $(0,1)$, we get

$$\begin{aligned}
&4f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)g(x)_a\mathrm{d}_{p,q}x\\
&\leq\Big[\int_0^1(1-t+t^2)_0\mathrm{d}_{p,q}t\Big][f(a)g(b)+f(b)g(a)]\\
&\quad+\Big[\int_0^1(2t-t^2)_0\mathrm{d}_{p,q}t\Big]f(a)g(a)+\Big[\int_0^1(1-t^2)_0\mathrm{d}_{p,q}t\Big]f(b)g(b).\\
&=\Big(1-\frac{1}{p+q}+\frac{1}{p^2+pq+q^2}\Big)[f(a)g(b)+f(b)g(a)]\\
&\quad+\Big(\frac{2}{p+q}-\frac{1}{p^2+pq+q^2}\Big)f(a)g(a)\\
&\quad+\Big(1-\frac{1}{p^2+pq+q^2}\Big)f(b)g(b).
\end{aligned}$$

This ends the proof. □

Our next result is an upper bound of (p,q)-integral inequality through product of two convex functions.

Theorem 7. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous and non-negative on $[a,b]$. If f and g are convex functions on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)_a\mathrm{d}_{p,q}x\\
&\leq\frac{1}{p^2+pq+q^2}f(b)g(b)+\Big(1-\frac{2}{p+q}+\frac{1}{p^2+pq+q^2}\Big)f(a)g(a)\\
&\quad+\Big(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\Big)[f(a)g(b)+f(b)g(a)].
\end{aligned}$$

Proof. Using the convexity of f and g, for all $t\in[0,1]$, we have

$$\begin{aligned}
&f(tb+(1-t)a)g(tb+(1-t)a)\\
&\leq[tf(b)+(1-t)f(a)][tg(b)+(1-t)g(a)]\\
&=t^2f(b)g(b)+(1-t)^2f(a)g(a)+t(1-t)[f(a)g(b)+f(b)g(a)].
\end{aligned}$$

Taking (p,q)-integral for the above inequality about t on $(0,1)$, we obtain

$$\begin{aligned}
&\int_0^1 f(tb+(1-t)a)g(tb+(1-t)a)_0\mathrm{d}_{p,q}t\\
&\leq\frac{1}{p^2+pq+q^2}f(b)g(b)+\Big(1-\frac{2}{p+q}+\frac{1}{p^2+pq+q^2}\Big)f(a)g(a)\\
&\quad+\Big(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\Big)[f(a)g(b)+f(b)g(a)].
\end{aligned}\tag{3.14}$$

A simple calculation shows that

$$\int_0^1 f(tb+(1-t)a)g(tb+(1-t)a)\,{}_0\mathrm{d}_{p,q}t = \frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_{p,q}x. \tag{3.15}$$

Combining (3.14) and (3.15), we deduce the desired result. This ends the proof. □

Corollary 1. *Putting $p = 1$ in Theorem 7, we get Theorem 4.3 established by Sudsutad et al. in [11].*

4. Further estimation results

If ${}_aD_{p,q}f$ is bounded, one gets the following theorem.

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be continuous and (p,q)-differentiable on (a,b), and let ${}_aD_{p,q}f$ be integrable on $[a,b]$. If there exists a constant M such that $\left|{}_aD_{p,q}f(x)\right| \le M < +\infty$ for all $x \in [a,b]$, then the following inequality*

$$\left|\Lambda(\lambda,\mu;a,b)\right| \le M(b-a)[\Phi_4(\lambda,\mu;p,q)+\Phi_5(\lambda,\mu;p,q)-\Phi_6(\lambda,\mu;p,q)] \tag{4.1}$$

holds together with $\lambda,\mu \in [0,1]$, where $\Phi_4(\lambda,\mu;p,q)$, $\Phi_5(\lambda,\mu;p,q)$ and $\Phi_6(\lambda,\mu;p,q)$ are defined by (3.5), (3.6) and (3.7), respectively.

Proof. From Lemma 1, utilizing the property of the modulus, we have

$$\begin{aligned}
&\left|\Lambda(\lambda,\mu;a,b)\right| \\
&\le (b-a)\Big\{\int_0^\mu \left|qt+\lambda\mu-\lambda\right|\left|{}_aD_{p,q}f(tb+(1-t)a)\right|{}_0\mathrm{d}_{p,q}t \\
&\quad + \int_\mu^1 \left|qt+\lambda\mu-1\right|\left|{}_aD_{p,q}f(tb+(1-t)a)\right|{}_0\mathrm{d}_{p,q}t\Big\} \\
&\le M(b-a)\Big\{\int_0^\mu \left|qt+\lambda\mu-\lambda\right|{}_0\mathrm{d}_{p,q}t + \int_0^1 \left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t \\
&\quad - \int_0^\mu \left|qt+\lambda\mu-1\right|{}_0\mathrm{d}_{p,q}t\Big\}.
\end{aligned}$$

Using (3.5), (3.6) and (3.7) in the above inequality, we deduce the desired result. The proof is completed. □

Corollary 2. *Consider Theorem 8.*
(i) Putting $\lambda = 0$, we get

$$\left|f(\mu b+(1-\mu)a) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\,{}_a\mathrm{d}_{p,q}x\right| \le M(b-a)\Big[\frac{2q\mu^2+p}{p+q}-\mu\Big].$$

(ii) Putting $\lambda = 1 = p$ and $\mu = \frac{1}{1+q}$, we get

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_a\mathrm{d}_q x\right| \le \frac{2q^2(b-a)M}{(1+q)^3}.$$

If ${}_aD_{p,q}f$ satisfies Lipschitz condition, one has the following theorem.

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be continuous and (p,q)-differentiable on (a,b), and let ${}_aD_{p,q}f$ be integrable on $[a,b]$. If ${}_aD_{p,q}f$ satisfies Lipschitz condition for some $L > 0$ on $[a,b]$, then the following inequality*

$$\begin{aligned}
&\Big|\Lambda(\lambda,\mu;a,b)\Big| \\
&\quad \le L(b-a)^2\Big[\Phi_1(\lambda,\mu;p,q) - \Phi_2(\lambda,\mu;p,q) + \Phi_3(\lambda,\mu;p,q) \\
&\qquad + \Phi_5(\lambda,\mu;p,q) - \Phi_6(\lambda,\mu;p,q)\Big] + (b-a)\Big[\Phi_4(\lambda,\mu;p,q)\big|{}_aD_{p,q}f(a)\big| \\
&\qquad + \big(\Phi_5(\lambda,\mu;p,q) - \Phi_6(\lambda,\mu;p,q)\big)\big|{}_aD_{p,q}f(b)\big|\Big]
\end{aligned}$$

holds together with $\lambda,\mu \in [0,1]$, where $\Phi_i(\lambda,\mu;p,q)$ $(i = 1,2,\cdots,6)$ are defined by (3.2)-(3.7), respectively.

Proof. From Lemma 1, utilizing the property of the modulus, we have

$$\begin{aligned}
&\Big|\Lambda(\lambda,\mu;a,b)\Big| \\
&\le (b-a)\bigg\{\int_0^\mu \big|qt+\lambda\mu-\lambda\big|\Big|{}_aD_{p,q}f(tb+(1-t)a) - {}_aD_{p,q}f(a)\Big|{}_0\mathrm{d}_{p,q}t \\
&\quad + \int_\mu^1 \big|qt+\lambda\mu-1\big|\Big|{}_aD_{p,q}f(tb+(1-t)a) - {}_aD_{p,q}f(b)\Big|{}_0\mathrm{d}_{p,q}t \\
&\quad + \big|{}_aD_{p,q}f(a)\big|\int_0^\mu \big|qt+\lambda\mu-\lambda\big|{}_0\mathrm{d}_{p,q}t + \big|{}_aD_{p,q}f(b)\big|\int_\mu^1 \big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t\bigg\} \\
&\le (b-a)\bigg\{\int_0^\mu L(b-a)t\big|qt+\lambda\mu-\lambda\big|{}_0\mathrm{d}_{p,q}t + \big|{}_aD_{p,q}f(a)\big|\int_0^\mu \big|qt+\lambda\mu-\lambda\big|{}_0\mathrm{d}_{p,q}t \\
&\quad + \int_\mu^1 L(b-a)(1-t)\big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t + \big|{}_aD_{p,q}f(b)\big|\int_\mu^1 \big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t\bigg\} \\
&= L(b-a)^2\bigg\{\int_0^\mu t\big|qt+\lambda\mu-\lambda\big|{}_0\mathrm{d}_{p,q}t + \int_0^1 \big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t - \int_0^\mu \big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t \\
&\quad - \int_0^1 t\big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t + \int_0^\mu t\big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t\bigg\} \\
&\quad + (b-a)\bigg\{\Big(\int_0^\mu \big|qt+\lambda\mu-\lambda\big|{}_0\mathrm{d}_{p,q}t\Big)\big|{}_aD_{p,q}f(a)\big| \\
&\quad + \Big(\int_0^1 \big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t - \int_0^\mu \big|qt+\lambda\mu-1\big|{}_0\mathrm{d}_{p,q}t\Big)\big|{}_aD_{p,q}f(b)\big|\bigg\}.
\end{aligned}$$

Using (3.2)-(3.7) in the above inequality, one get the desired result. This ends the proof. □

Corollary 3. *Consider Theorem 9.*

(i) Putting $\lambda = 0$, we get

$$\begin{aligned}&\left|f(\mu b + (1-\mu)a) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x\right| \\ &\leq L(b-a)^2\Big[\frac{(1+q)\mu^2+p-1}{p+q} + \frac{q}{p^2+pq+q^2} - \mu\Big] \\ &\quad + (b-a)\Big[\frac{q\mu^2}{p+q}\big|{}_aD_{p,q}f(a)\big| + \Big(\frac{q\mu^2+p}{p+q} - \mu\Big)\big|{}_aD_{p,q}f(b)\big|\Big].\end{aligned}$$

(ii) Putting $\lambda = 1 = p$ and $\mu = \frac{1}{1+q}$, we get

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a\mathrm{d}_q x\right| \\ &\leq L(b-a)^2\Big[\frac{q}{(1+q)^2} + \frac{q^2}{(1+q)^3} - \frac{q}{1+q+q^2}\Big] + \frac{q^2(b-a)}{(1+q)^3}\Big[\big|{}_aD_{p,q}f(a)\big| + \big|{}_aD_{p,q}f(b)\big|\Big].\end{aligned}$$

5. Examples

In this section, we give three examples to illustrate our main results.

Example 1. Let $f(x) = x^2$, for $x \in [1,3]$. Applying Theorem 2 with $a = 1$, $b = 3$, $q = \frac{1}{2}$ and $p = 1$, the left-hand side becomes:

$$\begin{aligned}&f\Big(\frac{qa+pb}{p+q}\Big) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x \\ &= \left(\frac{\frac{1}{2}+3}{1+\frac{1}{2}}\right)^2 - \frac{1}{3-1}\times\Big(1-\frac{1}{2}\Big)\times(3-1)\sum_{n=0}^{\infty}\frac{1}{2^n}\times\Big(\frac{1}{2^n}\times 3 + 1 - \frac{1}{2^n}\Big)^2 \\ &= \frac{49}{9} - \frac{250}{42} \approx -0.5079 < 0.\end{aligned}$$

For the right-hand side, one has:

$$\begin{aligned}&\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\mathrm{d}_{p,q}x - \frac{qf(a)+pf(b)}{p+q} \\ &= \Big(1-\frac{1}{2}\Big)\times(3-1)\sum_{n=0}^{\infty}\frac{1}{2^n}\times\Big(\frac{1}{2^n}\times 3 + 1 - \frac{1}{2^n}\Big)^2 - \frac{\frac{1}{2}\times 1 + 1\times 3^2}{1+\frac{1}{2}} \\ &= \frac{250}{42} - \frac{19}{3} \approx -0.3810 < 0.\end{aligned}$$

Example 2. Let $f(x) = x^3$ and $g(x) = \frac{1}{x^3}$ on $[1,2]$. Applying Theorem 7 with $a = 1$, $b = 2$, $q = \frac{1}{2}$ and $p = 1$, the left-hand side becomes:

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)_a\mathrm{d}_{p,q}x &= \frac{1}{2-1}\times\Big(1-\frac{1}{2}\Big)\times(2-1)\sum_{n=0}^{\infty}\frac{1}{2^n} \\ &= 1.\end{aligned}$$

For the right-hand side, one has:

$$\begin{aligned}
&\frac{1}{p^2+pq+q^2}f(b)g(b)+\Big(1-\frac{2}{p+q}+\frac{1}{p^2+pq+q^2}\Big)f(a)g(a)\\
&\quad+\Big(\frac{1}{p+q}-\frac{1}{p^2+pq+q^2}\Big)[f(a)g(b)+f(b)g(a)]\\
&=\frac{1}{1+\frac{1}{2}+\frac{1}{4}}+\Big(1-\frac{2}{1+\frac{1}{2}}+\frac{1}{1+\frac{1}{2}+\frac{1}{4}}\Big)\\
&\quad+\Big(\frac{1}{1+\frac{1}{2}}-\frac{1}{1+\frac{1}{2}+\frac{1}{4}}\Big)\Big[1^3\times\frac{1}{2^3}+2^3\times\frac{1}{1^3}\Big]\\
&\approx 1.5833.
\end{aligned}$$

It is clear that $1 < 1.5833$, which demonstrates the result described in Theorem 7.

Example 3. Theorem 3, Theorem 4, Theorem 5, Theorem 8 and Theorem 9 provide an upper bound for the approximation of (p, q)-integral $\int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x$. There exist some (p, q)-integral functions that can not be easy to calculate. Therefore, the above theorems are of importance to deal with such (p, q)-integral mappings. For example, let $f(x) = \frac{1}{\ln x}$, for $x \in [2, \infty)$, if we take $\lambda = 0$, $\mu = \frac{1}{2}$, $a = 3$, $b = 5$, $q = \frac{1}{5}$ and $p = \frac{1}{2}$, then all assumptions in Theorem 8 are satisfied. The left-hand side term of (4.1) is:

$$\begin{aligned}
&\left|f(\mu b+(1-\mu)a)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a \mathrm{d}_{p,q}x\right|\\
&=\left|\frac{1}{\ln\left(\frac{3+5}{2}\right)}-\frac{1}{\frac{1}{2}\times(5-3)}\times\left(\frac{1}{2}-\frac{1}{5}\right)\times(4-3)\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}\times\frac{1}{\ln(\frac{2^{n+1}}{5^n}(4-3)+3)}\right|
\end{aligned} \tag{5.1}$$

Clearly, the term $\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}\times\frac{1}{\ln(\frac{2^{n+1}}{5^n}(4-3)+3)}$ can not easy solved directly. However, applying Theorem 8, we obtain an upper bound for (5.1), i.e.

$$\begin{aligned}
M(b-a)\Big[\frac{2q\mu^2+p}{p+q}-\mu\Big]&=\frac{1}{\ln 3}\times(5-3)\times\Big[\frac{2\times\frac{1}{5}\times\frac{1}{2^2}+\frac{1}{2}}{\frac{1}{2}+\frac{1}{5}}-\frac{1}{2}\Big]\\
&\approx 0.6502.
\end{aligned}$$

6. Conclusion

We have established a new (p, q)-integral identity with parameters and developed certain (p, q)-integral error estimations of different type inequalities, such as the midpoint-like inequalities, the Simpson-like inequalities, the averaged midpoint-trapezoid-like inequalities and the trapezoid-like inequalities. We also give the upper and lower bounds for (p, q)-integral inequalities via product of two convex functions. It is worthwhile to mention that some inequalities obtained in this article generalize certain results given by Alp, N. et al. (2018), Kunt, M. et al. (2018) and Sudsutad, W. et al. (2015). The (p, q)-integral inequalities deduced in the present research are very general and helpful in error estimations involved in various approximation processes. With these contributions, we hope to motivate the interested reader to explore this fascinating field of quantum integral inequalities based on these techniques and the ideas developed in this paper.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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