AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 557–567.
DOI:10.3934/math.2020037
Received: 10 October 2019
Accepted: 06 December 2019
Published: 12 December 2019

Research article

Solvability for boundary value problems of nonlinear fractional differential equations with mixed perturbations of the second type

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Abstract: In this paper, we consider the solvability for boundary value problems of nonlinear fractional differential equations with mixed perturbations of the second type. The expression of the solution for the boundary value problem of nonlinear fractional differential equations with mixed perturbations of the second type is discussed based on the definition and the property of the Caputo differential operators. By the fixed point theorem in Banach algebra due to Dhage, an existence theorem for the boundary value problem of nonlinear fractional differential equations with mixed perturbations of the second type is given under mixed Lipschitz and Carathéodory conditions. As an application, an example is presented to illustrate the main results. Our results in this paper extend and improve some well-known results. To some extent, our work fills the gap on some basic theory for the boundary value problems of fractional differential equations with mixed perturbations of the second type involving Caputo differential operator.

Keywords: existence; boundary value problem; fractional differential equation; mixed perturbations
Mathematics Subject Classification: 34N05, 34A12

1. Introduction

In this paper, we discuss the following boundary value problem of nonlinear fractional differential equations with mixed perturbations of the second type

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right] = g(t,u(t)), \quad t \in J = [0,T], \\ a\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right]_{t=0} + b\left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right]_{t=T} = c, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, ${}^{C}D^{\alpha}_{0^+}$ is Caputo fractional derivative, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $k \in C(J \times \mathbb{R}, \mathbb{R})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$, and $a,\ b,\ c$ are real constants with $a + b \neq 0$.

Let $J = [0, T]$ be a bounded interval in $\mathbb{R}$ with $T > 0$. Let $C(J \times \mathbb{R}, \mathbb{R})$ denote the class of all continuous functions $k : J \times \mathbb{R} \to \mathbb{R}$. Let $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto \frac{u-k(t,u)}{f(t,u)}$ is measurable for each $u \in \mathbb{R}$, and
(ii) the map $u \mapsto \frac{u-k(t,u)}{f(t,u)}$ is continuous for each $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

By a solution of the boundary value problem (1.1), we mean a function u such that

(i) the function $t \mapsto \frac{u-k(t,u)}{f(t,u)}$ is continuous for each $u \in \mathbb{R}$, and
(ii) u satisfies the equations in (1.1).

Fractional calculus has been drawn people's attention extensively. This is because of its extensive development by the theory and by its applications in various fields, such as physics, engineering, chemistry and biology; see [1]. Compared with integer derivatives, fractional derivatives are used for a better description of considered material properties, and the design of mathematical models by the differential equations of fractional order can be more accurately illustrated the characteristics of the real-world phenomena, such as the exothermic reactions model having constant heat source [2], the fractional SIRS-SI model describing the transmission of malaria disease [3], the fractional model of nonlinear wave-like equations [4], the fractional Biswas-Milovic model having Kerr and parabolic law nonlinearities [5] and the fractional-order chaotic and hyperchaotic systems [6–8]. Many papers about the solvability for fractional equations and systems have appeared; see [9–20].

Benchohra et al. [17] investigated the existence of solutions for first order boundary value problems for fractional order differential equations

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}x(t) = f(t, x(t)), \quad t \in [0, T], \\ ax(0) + bx(T) = c, \end{cases}$$

where $0 < \alpha < 1$, ${}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $a,\ b,\ c$ are real constants with $a + b \neq 0$.

In recent years, the theory of nonlinear differential equations with perturbations has been a hot research topic; see [18–22]. Dhage [22] discussed the following first order hybrid differential equation with mixed perturbations of the second type

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] = g(t, x(t)), \quad t \in [t_0, t_0 + a], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $[t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some $t_0,\ a \in \mathbb{R}$ with $a > 0$, $f \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $k,\ g \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R})$. They developed the theory of hybrid differential equations with mixed perturbations of the second type, and gave some original and interesting results.

As far as we know, there are no results for the boundary value problem (1.1) of nonlinear fractional differential equations with mixed perturbations of the second type. From the above works, we consider the solvability of the boundary value problem (1.1). An existence theorem for the boundary value problem (1.1) is given under mixed Lipschitz and Carathéodory conditions. Our results in this paper extend and improve some well-known results.

The paper is organized as follows: Section 2 gives some definitions and lemmas to prove our main results. Section 3 establishes an existence theorem for the boundary value problem (1.1) under mixed Lipschitz and Carathéodory conditions by the fixed point theorem in Banach algebra due to Dhage. Section 4 presents an example to illustrate the main results, which is followed by the conclusion in Section 5.

2. Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of the boundary value problem (1.1). These materials can be found in the recent literature, see [23, 24].

Definition 2.1. *([24]) The Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* $f : (0, +\infty) \to \mathbb{R}$ *is given by*

$$^{C}D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. *([24]) The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $f : (0, +\infty) \to \mathbb{R}$ *is given by*

$$I^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Caputo derivative, we can obtain the following statement.

Lemma 2.1. *([24]) Let* $\alpha > 0$. *If we assume* $u \in C(0, 1) \cap L(0, 1)$, *then the fractional differential equation*

$$^{C}D^{\alpha}_{0^+} u(t) = 0$$

has $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},\ c_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, n-1,$ *as unique solutions, where n is the smallest integer greater than or equal to α.*

Lemma 2.2. *([24]) Assume that* $u \in C^n[0, 1]$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C^n[0, 1]$. *Then*

$$I^{\alpha}_{0^+} {}^{C}D^{\alpha}_{0^+} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \cdots, n-1,$ *where n is the smallest integer greater than or equal to α.*

The following fixed point theorem in Banach algebra due to Dhage [23] is useful in the proofs of our main results.

Lemma 2.3. *([23]) Let Q be a closed convex and bounded subset of the Banach space P and let $A,\ C:\ P\to P$ and $B:\ Q\to P$ be three operators such that*

(a) A and C are Lipschitz with the Lipschitz constants α and β respectively,
(b) B is compact and continuous,
(c) $u = AuBv + Cu$ for all $v\in Q \Rightarrow u\in Q$, and
(d) $\alpha M+\beta<1$, where $M=\|B(Q)\|=\sup\{\|B(u)\|:\ u\in Q\}$.

Then the operator equation $AuBu + Cu = u$ has a solution in Q.

3. Existence result

In this section, we discuss the existence results for boundary value problems (1.1).

We place the boundary value problem (1.1) in the space $C(J,\mathbb{R})$ of all continuous functions defined on J. $\|\cdot\|$ denotes a supremum norm in $C(J,\mathbb{R})$ by

$$\|u\| = \sup_{t\in J}|u(t)|,$$

and a multiplication "$\cdot$" in $C(J,\mathbb{R})$ by

$$(u\cdot v)(t) = (uv)(t) = u(t)v(t)$$

for $u,\ v\in C(J,\mathbb{R})$. Clearly $C(J,\mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it. $L^1(J,\mathbb{R})$ denotes the space of Lebesgue integrable functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|u\|_{L^1} = \int_0^T |u(s)|ds.$$

We present the following hypotheses.

(A_1) There exist constants $L_1>0$ and $L_2>0$ such that

$$|f(t,u)-f(t,v)|\le L_1|u-v|$$

and

$$|k(t,u)-k(t,v)|\le L_2|u-v|$$

for all $t\in J$ and $u,\ v\in\mathbb{R}$.

(A_2) There exists a function $h\in L^1(J,\mathbb{R})$ such that

$$|g(t,u)|\le h(t),\ t\in J$$

for all $u\in\mathbb{R}$.

Lemma 3.1. *Suppose that $a,\ b,\ c$ are real constants with $a+b\neq 0$. Then for any $v\in L^1(J,\mathbb{R})$, the function u is a solution of the boundary value problem*

$$^{C}D^{\alpha}_{0^+}\Big[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\Big] = v(t),\ 0<\alpha\le 1,\ t\in J, \tag{3.1}$$

and

$$a\Big[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\Big]_{t=0}+b\Big[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\Big]_{t=T}=c, \tag{3.2}$$

if and only if u satisfies the integral equation

$$\begin{aligned} u(t) &= f(t,u(t))\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds \\ &\quad +\frac{1}{a+b}\Big(c-\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds\Big)\Big)+k(t,u(t)), \quad t\in J. \end{aligned} \tag{3.3}$$

Proof. Let u be a solution of the problem (3.1) and (3.2). Applying the Riemann-Liouville fractional integral $I_{0^+}^{\alpha}$ on both sides of (3.1), by Lemma 2.2, then we obtain

$$\frac{u(t)-k(t,u(t))}{f(t,u(t))}=I_{0^+}^{\alpha}v(t)+\tilde{c},$$

for some $\tilde{c}\in\mathbb{R}$. Consequently, the general solution of (3.1) is

$$u(t)=k(t,u(t))+f(t,u(t))\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds+\tilde{c}\right).$$

Substituting $t=0$ and $t=T$ in the above equality implies

$$\frac{u(0)-k(0,u(0))}{f(0,u(0))}=\tilde{c},$$

$$\frac{u(T)-k(T,u(T))}{f(T,u(T))}=\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds+\tilde{c}.$$

By (3.2), then we have

$$a\tilde{c}+\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds+b\tilde{c}=c,$$

that is

$$\tilde{c}=\frac{1}{a+b}\left(c-\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds\right).$$

Therefore, (3.3) holds.

Conversely, suppose that u satisfies the equation (3.3). Applying the Caputo fractional operator of the order α on both sides of (3.3), then (3.1) is satisfied. Thus, substituting $t=0$ and $t=T$ in (3.1) implies

$$\frac{u(0)-k(0,u(0))}{f(0,u(0))}=\frac{1}{a+b}\left(c-\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds\right),$$

$$\frac{u(T)-k(T,u(T))}{f(T,u(T))}=\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds+\frac{1}{a+b}\left(c-\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds\right).$$

Then,

$$a\Big[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\Big]_{t=0}+b\Big[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\Big]_{t=T}$$

$$= \frac{a}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds\right) + \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds$$
$$+ \frac{b}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}v(s)ds\right) = c.$$

Hence, (3.2) also holds. □

Now we will give the following existence theorem for the boundary value problem (1.1).

Theorem 3.1. *Suppose that* (A_1) *and* (A_2) *hold. Furthermore, if*

$$L_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)+L_2<1, \tag{3.4}$$

then the boundary value problem (1.1) *has a solution defined on* J.

Proof. Set $U = C(J,\mathbb{R})$ and define a subset S of U by

$$S = \{u \in U \mid \|u\| \le N\},$$

where

$$N = \frac{F_0\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)+K_0}{1-L_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)-L_2},$$

$F_0 = \sup_{t\in J}|f(t,0)|$ and $K_0 = \sup_{t\in J}|k(t,0)|$.

Clearly, S is a closed, convex and bounded subset of the Banach space U. By Lemma 3.1, the boundary value problem (1.1) is equivalent to the nonlinear integral equation

$$\begin{aligned} u(t) &= f(t,u(t))\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,u(s))ds \\ &\quad +\frac{1}{a+b}\Big(c-\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s,u(s))ds\Big)\Big)+k(t,u(t)), \quad t\in J. \end{aligned} \tag{3.5}$$

Define three operators $A,\ C:\ U \to U$ and $B:\ S \to U$ by

$$Au(t) = f(t,u(t)), \quad t \in J, \tag{3.6}$$

$$\begin{aligned} Bu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,u(s))ds \\ &\quad +\frac{1}{a+b}\left(c-\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s,u(s))ds\right), \quad t\in J, \end{aligned} \tag{3.7}$$

and

$$Cu(t) = k(t,u(t)), \quad t \in J. \tag{3.8}$$

Then the equation (3.5) is transformed into the operator equation as

$$u(t) = Au(t)Bu(t) + Cu(t), \quad t \in J.$$

Next, we prove the operators $A,\ B$ and C satisfy all the conditions of Lemma 2.3.

Firstly, we prove that A is a Lipschitz operator on U with the Lipschitz constant L_1. Let $u,\ v \in U$. Then by (A_1),

$$|Au(t) - Av(t)| = |f(t, u(t)) - f(t, v(t))| \leq L_1 |u(t) - v(t)| \leq L_1 \|u - v\|,$$

for all $t \in J$. Taking supremum over t, then we have

$$\|Au - Av\| \leq L_1 \|u - v\|,$$

for all $u,\ v \in U$. This shows that A is a Lipschitz operator on U with the Lipschitz constant L_1. Similarly, it can be implied that C is also a Lipschitz operator on U with the Lipschitz constant L_2.

Next, we prove B is a compact and continuous operator on S into U. Firstly, we prove B is continuous on S. Let $\{u_n\}$ be a sequence in S converging to a point $u \in S$. Then by the Lebesgue dominated convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty} Bu_n(t) \\
&= \lim_{n\to\infty}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, u_n(s))ds + \frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s, u_n(s))ds\right)\right) \\
&= \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, u_n(s))ds + \lim_{n\to\infty}\frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s, u_n(s))ds\right) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, u(s))ds + \frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s, u(s))ds\right) \\
&= Bu(t)
\end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on S.

Next we prove B is a compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in U. On the one hand, let $u \in S$ be arbitrary. Then by (A_2),

$$\begin{aligned}
|Bu(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, u(s))ds + \frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s, u(s))ds\right)\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, u(s))ds\right| + \left|\frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s, u(s))ds\right)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |g(s, u(s))|\, ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} |g(s, u(s))| ds + \frac{|c|}{|a+b|} \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |h(s)|\, ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} |h(s)| ds + \frac{|c|}{|a+b|} \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1} + \frac{|b| T^\alpha}{|a+b|\Gamma(\alpha+1)}\|h\|_{L^1} + \frac{|c|}{|a+b|} \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|},
\end{aligned}$$

for all $t \in J$. Taking supremum over t,

$$\|Bu\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}$$

for all $u \in S$. This shows that B is uniformly bounded on S.

On the other hand, let $t_1,\ t_2 \in J$. Then for any $u \in S$, we get

$$\begin{aligned}|Bu(t_1) - Bu(t_2)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1}g(s,u(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2 - s)^{\alpha-1}g(s,u(s))ds\right| \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left|\int_{t_2}^{t_1}|g(s,u(s))|ds\right| \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\left|\int_{t_2}^{t_1}h(s)ds\right| \\ &= \frac{T^\alpha}{\Gamma(\alpha+1)}|p(t_1) - p(t_2)|,\end{aligned}$$

where $p(t) = \int_0^t h(s)ds$. Since the function p is continuous on compact J, it is uniformly continuous. Hence, for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Rightarrow |Bu(t_1) - Bu(t_2)| < \varepsilon,$$

for all $t_1, t_2 \in J$ and $u \in S$. This shows that $B(S)$ is an equicontinuous set in U. Now the set $B(S)$ is uniformly bounded and equicontinuous set in U, so it is compact by Arzela-Ascoli Theorem. Thus, B is a and compact operator on S.

Next, we show that (c) of Lemma 2.3 is satisfied. Let $u \in U$ and $v \in S$ be arbitrary such that $u = AuBv + Cu$. Then, by assumption (A_1), we have

$$\begin{aligned}|u(t)| &\leq |Au(t)||Bv(t)| + |Cu(t)| \\ &= |f(t,u(t))|\Big|\frac{1}{\Gamma(\alpha)}\int_0^t(t - s)^{\alpha-1}g(s,v(s))ds \\ &\quad + \frac{1}{a+b}\Big(c - \frac{b}{\Gamma(\alpha)}\int_0^T(T - s)^{\alpha-1}g(s,v(s))ds\Big)\Big| + |k(t,u(t))| \\ &\leq [|f(t,u(t)) - f(t,0)| + |f(t,0)|] \\ &\quad \cdot\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) \\ &\quad + |k(t,u(t)) - k(t,0)| + |k(t,0)| \\ &\leq [L_1|u(t)| + F_0]\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) + L_2|u(t)| + K_0.\end{aligned}$$

Thus, we get

$$|u(t)| \leq \frac{F_0\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) + K_0}{1 - L_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) - L_2}.$$

Taking supremum over t,

$$\|u\| \leq \frac{F_0\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) + K_0}{1 - L_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) - L_2} = N.$$

This shows that (c) of Lemma 2.3 is satisfied.

Finally, we obtain

$$M = \|B(S)\| = \sup\{\|B(u)\| : \ u \in S\} \leq \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} \left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} \right),$$

and so,

$$L_1 M + L_2 \leq L_1 \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} \left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} \right) + L_2 < 1.$$

Thus, all the conditions of Lemma 2.3 are satisfied and hence the operator equation $AuBu + Cu = u$ has a solution in S. Therefore, the boundary value problem (1.1) has a solution defined on J. □

Remark 3.1. *Some existence results were given for the boundary value problem* (1.1)*:*

(I) *with* $k \equiv 0$, *and* $f \equiv 1$ *by Benchohra et al. in [17];*
(II) *with* $\alpha = 1$, $a = 1$, *and* $b = 0$ *by Dhage in [22];*
(III) *with* $\alpha = 1$, $k \equiv 0$, *and* $f \equiv 1$ *by Tisdell in [25].*

4. An example

In this section, we will present an example to illustrate the main results.

Example 4.1 Consider the following boundary value problem

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2}} \left[\frac{u(t) - \frac{1}{8}\sin u(t)}{\sqrt{u^2(t)+1}} \right] = \cos u(t), \quad t \in J = [0,1], \\ \left[\frac{u(t) - \frac{1}{8}\sin u(t)}{\sqrt{u^2(t)+1}} \right]_{t=0} + \left[\frac{u(t) - \frac{1}{8}\sin u(t)}{\sqrt{u^2(t)+1}} \right]_{t=1} = \frac{1}{4}, \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{2}$, $T = 1$, $k(t, u(t)) = \frac{1}{8}\sin u(t)$, $f(t, u(t)) = \sqrt{u^2(t)+1}$, $g(t, u(t)) = \cos u(t)$, $a = b = 1$ with $a + b \neq 0$, and $c = \frac{1}{4}$.

Let $L_1 = 1$, $L_2 = \frac{1}{8}$, $h(t) \equiv 1$. Then hypotheses (A_1) and (A_2) hold. Since

$$L_1 \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} \left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} \right) + L_2 = \frac{1}{\Gamma(\frac{1}{2}+1)} \left(1 + \frac{1}{2}\right) + \frac{1}{8} + \frac{1}{8} < 1.$$

Hence, (3.4) holds. Therefore, by Theorem 3.1, the boundary value problem (4.1) has a solution.

5. Conclusion

In this paper, we have studied the solvability for the boundary value problem (1.1) of nonlinear fractional differential equations with mixed perturbations of the second type. We have presented an existence theorem for the boundary value problem (1.1) of nonlinear fractional differential equations with mixed perturbations of the second type under mixed Lipschitz and Carathéodory conditions due to the fixed point theorem in Banach algebra due to Dhage. The main results have been well illustrated with the help of an example. Our results in this paper have been extended and improved some well-known results.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (61703180, 61803176, 61877028, 61807015, 61773010), the Natural Science Foundation of Shandong Province (ZR2019MF032, ZR2017BA010), the Project of Shandong Province Higher Educational Science and Technology Program (J18KA230, J17KA157), and the Scientific Research Foundation of University of Jinan (1008399, 160100101).

Conflict of interest

The authors declare that they have no competing interests in this paper.

References

1. I. Podlubny, *Fractional differential equations, mathematics in science and engineering*, Academic Press, New York, 1999.
2. D. Kumar, J. Singh, K. Tanwar, et al. *A new fractional exothermic reactions model having constant heat source in porous media with power, exponential and Mittag-Leffler laws*, Int. J. Heat Mass Tran., **138** (2019), 1222–1227.
3. D. Kumar, J. Singh, M. Al Qurashi, et al. *A new fractional SIRS-SI malaria disease model with application of vaccines, antimalarial drugs, and spraying*, Adv. Differ. Equa., **2019** (2019), 278.
4. D. Kumar, J. Singh, S. D. Purohit, et al. *A hybrid analytical algorithm for nonlinear fractional wave-like equations*, Math. Model. Nat. Pheno., **14** (2019), 304.
5. J. Singh, D. Kumar, D. Baleanu, *New aspects of fractional Biswas-Milovic model with Mittag-Leffler law*, Math. Model. Nat. Pheno., **14** (2019), 303.
6. D. Peng, K. Sun, S. He, et al. *Numerical analysis of a simplest fractional-order hyperchaotic system*, Theoretical and Applied Mechanics Letters, **9** (2019), 220–228.
7. S. He, K. Sun, Y. Peng, *Detecting chaos in fractional-order nonlinear systems using the smaller alignment index*, Phys. Lett. A, **383** (2019), 2267–2271.
8. Y. Peng, K. Sun, D. Peng, et al. *Dynamics of a higher dimensional fractional-order chaotic map*, Physica A: Statistical Mechanics and its Applications, **525** (2019), 96–107.
9. D. Chergui, T. E. Oussaeif, M. Ahcene, *Existence and uniqueness of solutions for nonlinear fractional differential equations depending on lower-order derivative with non-separated type integral boundary conditions*, AIMS Mathematics, **4** (2019), 112–133.
10. M. Asaduzzaman, M. Z. Ali, *Existence of positive solution to the boundary value problems for coupled system of nonlinear fractional differential equations*, AIMS Mathematics, **4** (2019), 880–895.
11. Y. Zhao, X. Hou, Y. Sun, et al. *Solvability for some class of multi-order nonlinear fractional systems*, Adv. Differ. Equa., **2019** (2019), 23.
12. Q. Song, Z. Bai, *Positive solutions of fractional differential equations involving the Riemann-Stieltjes integral boundary condition*, Adv. Differ. Equ., **2018** (2018), 183.

13. K. Sheng, W. Zhang, Z. Bai, *Positive solutions to fractional boundary value problems with p-Laplacian on time scales*, Bound. Value Probl., **2018** (2018), 70.
14. Z. Bai, Y. Chen, H. Lian, et al. *On the existence of blow up solutions for a class of fractional differential equations*, Fract. Calc. Appl. Anal., **17** (2014), 1175–1187.
15. Z. Bai, Y. Zhang, *Solvability of fractional three-point boundary value problems with nonlinear growth*, Appl. Math. Comput., **218** (2011), 1719–1725.
16. Y. Zhao, S. Sun, Z. Han, et al. *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Appl. Math. Comput., **217** (2011), 6950–6958.
17. M. Benchohra, S. Hamani, S. K. Ntouyas, *Boundary value problems for differential equations with fractional order*, Surveys in Mathematics & its Applications, **3** (2008), 1–12.
18. H. Lu, S. Sun, D. Yang, et al. *Theory of fractional hybrid differential equations with linear perturbations of second type*, Bound. Value Probl., **2013** (2013), 23.
19. S. Sun, Y. Zhao, Z. Han, et al. *The existence of solutions for boundary value problem of fractional hybrid differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 4961–4967.
20. Y. Zhao, S. Sun, Z. Han, et al. *Theory of fractional hybrid differential equations*, Comput. Math. Appl., **62** (2011), 1312–1324.
21. Y. Zhao, Y. Sun, Z. Liu, et al. *Basic theory of differential equations with mixed perturbations of the second type on time scales*, Adv. Differ. Equa., **2019** (2019), 268.
22. B. C. Dhage, *Basic results in the theory of hybrid differential equations with mixed perturbations of second type*, Funct. Differ. Equ., **19** (2012), 87–106.
23. B. C. Dhage, *A fixed point theorem in Banach algebras with applications to functional integral equations*, Kyungpook Math. J., **44** (2004), 145–155.
24. A. A. Kilbas, H. H. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B. V., Amsterdam, 2006.
25. C. C. Tisdell, *On the solvability of nonlinear first-order boundary-value problems*, Electron. J. Differ. Equa., **2006** (2006), 1–8.