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Research article

Geometry of configurations in tangent groups

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Abstract: This article relates the Grassmannian complexes of geometric configurations to the tangent to the Bloch-Suslin complex and to the tangent to Goncharov's motivic complex. By means of morphisms, we bring the geometry of configurations in tangent groups, $T\mathcal{B}_2(F)$ and $T\mathcal{B}_3(F)$ that produce commutative diagrams. To show the commutativity of diagrams, we use combinatorial techniques that include permutations in symmetric group S_6. We also create analogues of the Siegel's cross-ratio identity for the truncated polynomial ring $F[\varepsilon]_\nu$.

Keywords: affine spaces; cross-ratio; tangent complex; odd permutation; symmetric group
Mathematics Subject Classification: 11G55, 18G, 19D

1. Introduction

Goncharov was the first to find relations between the Grassmannian complex of projective configurations and Bloch-Suslin complex for weight $n = 2$, and to the Goncharov's motivic complex for weight $n = 3$ (see [3]). This idea leads to the remarkable proof of Zagier's conjecture for weights $n = 2, 3$ (see [4]). On the other hand, Cathelineau introduced the tangent form of the Bloch-Suslin complex and provided some suggestions about the tangent form of Goncharov's complex (see [1]).

The main idea of this article is to view geometric features of tangent groups, $T\mathcal{B}_2(F)$ and $T\mathcal{B}_3(F)$, where $T\mathcal{B}_2(F)$ is the tangent form of Bloch group $\mathcal{B}_2(F)$ (see [1]), and $T\mathcal{B}_3(F)$ is the tangent form of Goncharov's group $\mathcal{B}_3(F)$ (see §3.2) for any field F. To accomplish this task, we define morphisms $\tau^2_{0,\varepsilon}$, $\tau^2_{1,\varepsilon}$, (between the Grassamannian complex of geometric configurations and tangent to the Bloch-Suslin complex) and $\tau^3_{0,\varepsilon}$, $\tau^3_{1,\varepsilon}$, $\tau^3_{2,\varepsilon}$ (between the Grassamannian complex of geometric configurations and tangent to the Goncharov's complex) for weights $n = 2, 3$. Due to these morphisms, we get diagrams which are shown to be commutative (main result Theorem 3.7). The major techniques for showing our main result, are to invoke combinatorics in the symmetric group S_6 and to rewrite triple ratios in the product of two projected cross-ratios. Here, we use permutations of symmetric group S_6 in the alternation sums. The alternation sum Alt_6 in our map $\tau^3_{2,\varepsilon}$ has 6! terms, but due to inversion and cyclic symmetry, it reduces to $6!/(3!) = 120$ terms.

The cross ratio identity over a field F was first defined by Siegel (see [6]). To view the geometry of configurations in tangent groups, it is required to introduce an analogue to the Siegel cross-ratio identity for the determinants of matrices of order 2×2 (see Lemma 2.1) that can also be extended to 3×3 determinants of matrices. These analogues and Lemma 2.1 enabled us to produce the analogues of cross-ratios and triple ratios.

On the basis of these analogues, we find morphisms between the Grassmannian subcomplex $C_*(\mathbb{A}^n_{F[\varepsilon]_2}, d)$ and tangent to the Bloch-Suslin and to Goncharov complexes (see §3.1 and §3.2). The proof of the main result requires projected five term relation in $T\mathcal{B}_2(F)$. To serve this purpose, we prove the existence of the projected five term relation in $T\mathcal{B}_2(F)$ (see Lemma 3.4). This relation is also an analogue of Goncharov's projected five term relation in $\mathcal{B}_2(F)$.

In §3.2, we define the tangent group $T\mathcal{B}_3(F)$ which was first hypothetically defined in §9 of [1]. On the basis of our definition, we mimic construction of $T\mathcal{B}_3(F)$ with the F-vector space $\beta_3^D(F)$ ([5]) and reproduce Cathelineau's 22-term functional equation for $T\mathcal{B}_3(F)$.

2. Materials and method

Let F be a field of characteristic 0. For $\nu \geq 1$, we denote the νth truncated polynomial ring over F by $F[\varepsilon]_\nu := F[\varepsilon]/\varepsilon^\nu$. Further define $C_m\left(\mathbb{A}^n_{F[\varepsilon]_\nu}\right)$ as a free abelian group generated by m generic points in $\mathbb{A}^n_{F[\varepsilon]_\nu}$ (an n dimensional affine space over $F[\varepsilon]_\nu$). Here, we are not considering degenerate points and are also assuming that no two points coincide and no three points lie on a line. Now for $n = 2$ and $\nu = 2$, any $\eta_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in \mathbb{A}^2_F \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$ and $\eta_{i,\varepsilon} := \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \end{pmatrix} \in \mathbb{A}^2_F$, we put $\eta_i^* = \begin{pmatrix} a_i + a_{i,\varepsilon}\varepsilon \\ b_i + b_{i,\varepsilon}\varepsilon \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \end{pmatrix}\varepsilon = \eta_i + \eta_{i,\varepsilon}\varepsilon$ and define a boundary map

$$d : C_{m+1}\left(\mathbb{A}^2_{F[\varepsilon]_2}\right) \to C_m\left(\mathbb{A}^2_{F[\varepsilon]_2}\right)$$

$$d : (\eta_0^*, \ldots, \eta_m^*) \mapsto \sum_{i=0}^{m} (-1)^i (\eta_0^*, \ldots, \hat{\eta}_i^*, \ldots, \eta_m^*).$$

Let $\omega \in V_2^*$ be a volume element formed in $V_2 := \mathbb{A}^2_F$ and $\Delta(\eta_i, \eta_j) = \langle \omega, \eta_i \wedge \eta_j \rangle$, where $\eta_i, \eta_j \in \mathbb{A}^2_F$. Here we define

$$\Delta(\eta_i^*, \eta_j^*) = \Delta(\eta_i^*, \eta_j^*)_{\varepsilon^0} + \Delta(\eta_i^*, \eta_j^*)_{\varepsilon^1}\varepsilon$$

where

$$\Delta(\eta_i^*, \eta_j^*)_{\varepsilon^0} = \Delta(\eta_i, \eta_j) \quad \text{and} \quad \Delta(\eta_i^*, \eta_j^*)_{\varepsilon^1} = \Delta(\eta_i, \eta_{j,\varepsilon}) + \Delta(\eta_{i,\varepsilon}, \eta_j).$$

More generally for $\nu = n + 1$, we have

$$\eta_i^* = \eta_i + \eta_{i,\varepsilon}\varepsilon + \eta_{i,\varepsilon^2}\varepsilon^2 + \cdots + \eta_{i,\varepsilon^n}\varepsilon^n \quad \text{and} \quad \eta_{i,\varepsilon^0} = \eta_i$$

and we get

$$\Delta(\eta_i^*, \eta_j^*) = \Delta(\eta_i, \eta_j) + \Delta(\eta_i^*, \eta_j^*)_\varepsilon \varepsilon + \Delta(\eta_i^*, \eta_j^*)_{\varepsilon^2}\varepsilon^2 + \cdots + \Delta(\eta_i^*, \eta_j^*)_{\varepsilon^n}\varepsilon^n,$$

where

$$\Delta(\eta_i^*, \eta_j^*)_{\varepsilon^n} = \Delta(\eta_i, \eta_{j,\varepsilon^n}) + \Delta(\eta_{i,\varepsilon}, \eta_{j,\varepsilon^{n-1}}) + \cdots + \Delta(\eta_{i,\varepsilon^n}, \eta_j)$$

Consider the Siegel cross-ratio identity for the 2×2 determinants of four vectors in $C_4\left(\mathbb{A}^2_F\right)$ (see [3],[6])

$$\Delta(\eta_0,\eta_1)\Delta(\eta_2,\eta_3)=\Delta(\eta_0,\eta_2)\Delta(\eta_1,\eta_3)-\Delta(\eta_0,\eta_3)\Delta(\eta_1,\eta_2) \tag{2.1}$$

With the above notation, an analogue to the Siegel cross-ratio identity turns out to be true for $\mathbb{A}^2_{F[\varepsilon]_{n+1}}$, and we can extract further results which are essential for the proof of our main results. Throughout this section we will assume that $\Delta(\eta_i,\eta_j)\neq 0$ for $i\neq j$.

Lemma 2.1. *For* $(\eta_0^*,\eta_1^*,\eta_2^*,\eta_3^*)\in C_4\left(\mathbb{A}^2_{F[\varepsilon]_{n+1}}\right)$, *we have*

$$\Delta(\eta_0^*,\eta_1^*)\Delta(\eta_2^*,\eta_3^*)=\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)-\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*) \tag{2.2}$$

where

$$\eta_i^*=\eta_i+\eta_{i,\varepsilon}\varepsilon+\eta_{i,\varepsilon^2}\varepsilon^2+\cdots+\eta_{i,\varepsilon^n}\varepsilon^n \quad \text{and} \quad \eta_{i,\varepsilon^0}=\eta_i$$

$$\Delta(\eta_i^*,\eta_j^*)=\Delta(\eta_i,\eta_j)+\Delta(\eta_i^*,\eta_j^*)_\varepsilon\varepsilon+\Delta(\eta_i^*,\eta_j^*)_{\varepsilon^2}\varepsilon^2+\cdots+\Delta(\eta_i^*,\eta_j^*)_{\varepsilon^n}\varepsilon^n$$

for

$$\Delta(\eta_i^*,\eta_j^*)_{\varepsilon^n}=\Delta(\eta_i,\eta_{j,\varepsilon^n})+\Delta(\eta_{i,\varepsilon},\eta_{j,\varepsilon^{n-1}})+\cdots+\Delta(\eta_{i,\varepsilon^n},\eta_j)$$

Proof. For $r=0,\ldots,n$, we can write $\eta^*=\begin{pmatrix}\sum_{r\geq 0}\eta_r\varepsilon^r\\ \sum_{r\geq 0}\eta'_r\varepsilon^r\end{pmatrix}$ and $m^*=\begin{pmatrix}\sum_{r\geq 0}m_r\varepsilon^r\\ \sum_{r\geq 0}m'_r\varepsilon^r\end{pmatrix}$.

Now we have

$$\begin{aligned}\Delta(\eta^*,m^*)&=\begin{vmatrix}\sum_{r\geq 0}\eta_r\varepsilon^r & \sum_{r\geq 0}m_r\varepsilon^r\\ \sum_{r\geq 0}\eta'_r\varepsilon^r & \sum_{r\geq 0}m'_r\varepsilon^r\end{vmatrix}=\sum_{r\geq 0}\left(\sum_{k=0}^{r}\eta_k m'_{r-k}-\sum_{k=0}^{r}\eta'_k m_{r-k}\right)\varepsilon^r\\ &=\sum_{r\geq 0}\left(\sum_{k=0}^{r}\Delta\left(\eta_k,m_{r-k}\right)\right)\varepsilon^r\end{aligned}$$

Hence

$$\begin{aligned}\Delta(\eta_0^*,\eta_1^*)\Delta(\eta_2^*,\eta_3^*)&=\sum_{r\geq 0}\left(\sum_{k=0}^{r}\Delta(\eta_{0,k},\eta_{1,r-k})\right)\varepsilon^r\cdot\sum_{s\geq 0}\left(\sum_{j=0}^{r}\Delta(\eta_{0,j},\eta_{1,r-j})\right)\varepsilon^s\\ &=\sum_{t\geq 0}\varepsilon^t\left(\sum_{r=0}^{t}\left(\sum_{k=0}^{r}\Delta(\eta_{0,k},\eta_{1,r-k})\sum_{j=0}^{t-r}\Delta(\eta_{2,j},\eta_{3,t-r-j})\right)\right)\\ &=\sum_{t\geq 0}\varepsilon^t\left(\sum_{r=0}^{t}\left(\sum_{k=0}^{r}\sum_{j=0}^{t-r}\Delta(\eta_{0,k},\eta_{1,r-k})\Delta(\eta_{2,j},\eta_{3,t-r-j})\right)\right),\end{aligned}$$

and similarly for $\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)$ and $\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*)$. Hence we use the validity of (2.1) to deduce the analogue for $\Delta(\eta_i^*,\eta_j^*)$'s in place of $\Delta(\eta_i,\eta_j)$ passing from the ring $F[\![\varepsilon]\!]$ of power series to a truncated polynomial ring, say to $F[\varepsilon]_{n+1}$. □

For the special cases; we find the identity (2.1) for $n=0$, while for $n=1$ we have the following identity which will be used extensively below:

$$\Delta(\eta_0,\eta_1)\Delta(\eta_2^*,\eta_3^*)_\varepsilon+\Delta(\eta_2,\eta_3)\Delta(\eta_0^*,\eta_1^*)_\varepsilon$$

$$= \{\Delta(\eta_0,\eta_2)\Delta(\eta_1^*,\eta_3^*)_\varepsilon + \Delta(\eta_1,\eta_3)\Delta(\eta_0^*,\eta_2^*)_\varepsilon\} - \{\Delta(\eta_0,\eta_3)\Delta(\eta_1^*,\eta_2^*)_\varepsilon + \Delta(\eta_1,\eta_2)\Delta(\eta_0^*,\eta_3^*)_\varepsilon\}. \qquad (2.3)$$

if we write

$$(ab)_{\varepsilon^n} := a_{\varepsilon^n} b_{\varepsilon^0} + a_{\varepsilon^{n-1}} b_\varepsilon + \cdots + a_{\varepsilon^0} b_{\varepsilon^n}$$

then (2.3) can be more concisely written as

$$\{\Delta(\eta_0^*,\eta_1^*)\Delta(\eta_2^*,\eta_3^*)\}_\varepsilon = \{\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)\}_\varepsilon - \{\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*)\}_\varepsilon .$$

2.1. Cross-ratio in $F[\varepsilon]_\nu$

Now we have enough tools to find the cross-ratios of four points over the truncated polynomial ring $F[\varepsilon]_\nu$. The identity (2.2) of Lemma 2.1 enables us to compute this ratio in $F[\varepsilon]_\nu$ for $\nu = n+1$. First we define the cross-ratio of four points $(\eta_0^*,\dots,\eta_3^*) \in C_4\left(\mathbb{A}^2_{F[\varepsilon]_{n+1}}\right)$ as

$$\mathbf{r}(\eta_0^*,\dots,\eta_3^*) = \frac{\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*)}{\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)}$$

We also expand $\mathbf{r}(\eta_0^*,\dots,\eta_3^*)$ as a truncated polynomial over $F[\varepsilon]_{n+1}$

$$\mathbf{r}(\eta_0^*,\dots,\eta_3^*) = \left(r_{\varepsilon^0} + r_\varepsilon \varepsilon + r_{\varepsilon^2}\varepsilon^2 + \cdots + r_{\varepsilon^n}\varepsilon^n\right)(\eta_0^*,\dots,\eta_3^*) \qquad (2.4)$$

After truncating this for $n = 0$, one gets

$$\mathbf{r}(\eta_0^*,\dots,\eta_3^*) = r_{\varepsilon^0}(\eta_0^*,\dots,\eta_3^*) = r(\eta_0,\dots,\eta_3) = \frac{\Delta(\eta_0,\eta_3)\Delta(\eta_1,\eta_2)}{\Delta(\eta_0,\eta_2)\Delta(\eta_1,\eta_3)} \qquad (2.5)$$

If we truncate (2.4) for $n = 1$ then the coefficient of ε^0 will remain the same as for $n = 0$, thus we only need to compute the coefficient of ε in the following way:

After considering $(\eta_0^*,\dots,\eta_3^*) \in C_4\left(\mathbb{A}^2_{F[\varepsilon]_2}\right)$ in a generic position, we get

$$\mathbf{r}(\eta_0^*,\dots,\eta_3^*) = \frac{\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*)}{\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)} = \frac{\{\Delta(\eta_0,\eta_3) + \Delta(\eta_0^*,\eta_3^*)_\varepsilon\varepsilon\}\{\Delta(\eta_1,\eta_2) + \Delta(\eta_1^*,\eta_2^*)_\varepsilon\varepsilon\}}{\{\Delta(\eta_0,\eta_2) + \Delta(\eta_0^*,\eta_2^*)_\varepsilon\varepsilon\}\{\Delta(\eta_1,\eta_3) + \Delta(\eta_1^*,\eta_3^*)_\varepsilon\varepsilon\}}$$

If $a \neq 0 \in F$ then $\frac{1}{a+b\varepsilon} = \frac{1}{a} - \frac{b}{a^2}\varepsilon \in F[\varepsilon]_2$ (this is the same as the inversion relation in $T\mathcal{B}_2(F)$ discussed later in §2.3).

Let us simplify the above obtained result by multiplying the inverses of denominators and separate the coefficients of ε^0 and ε. The coefficient of ε becomes

$$r_\varepsilon(\eta_0^*,\dots,\eta_3^*) = \frac{\{\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*)\}_\varepsilon}{\Delta(\eta_0,\eta_2)\Delta(\eta_1,\eta_3)} - r(\eta_0,\dots,\eta_3)\frac{\{\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)\}_\varepsilon}{\Delta(\eta_0,\eta_2)\Delta(\eta_1,\eta_3)} \qquad (2.6)$$

Let us trancate it for $n = 2$, i.e., $(\eta_0^*,\dots,\eta_3^*) \in C_4\left(\mathbb{A}^2_{F[\varepsilon]_3}\right)$. To make computations easy, we write (η_i,η_j) instead of $\Delta(\eta_i,\eta_j)$

$$\mathbf{r}(\eta_0^*,\dots,\eta_3^*) = \frac{\{(\eta_0,\eta_3) + (\eta_0^*,\eta_3^*)_\varepsilon\varepsilon + (\eta_0^*,\eta_3^*)_{\varepsilon^2}\varepsilon^2\}\{(\eta_1,\eta_2) + (\eta_1^*,\eta_2^*)_\varepsilon\varepsilon + (\eta_1^*,\eta_2^*)_{\varepsilon^2}\varepsilon^2\}}{\{(\eta_0,\eta_2) + (\eta_0^*,\eta_2^*)_\varepsilon\varepsilon + (\eta_0^*,\eta_3^*)_{\varepsilon^2}\varepsilon^2\}\{(\eta_1,\eta_3) + (\eta_1^*,\eta_3^*)_\varepsilon\varepsilon + (\eta_1^*,\eta_3^*)_{\varepsilon^2}\varepsilon^2\}}$$

simplify and then separate the coefficients of ε^0, ε^1 and ε^2. The coefficients of ε^0 and ε^1 are the same as we computed in (2.5) and (2.6) respectively, and the coefficient of ε^2 is

$$\begin{aligned} r_{\varepsilon^2}(\eta_0^*,\dots,\eta_3^*) =& \frac{\{(\eta_0^*,\eta_3^*)(l_1^*,l_2^*)\}_{\varepsilon^2}}{(\eta_0,\eta_2)(\eta_1,\eta_3)} - r_\varepsilon(\eta_0^*,\dots,\eta_3^*)\frac{\{(\eta_0^*,\eta_2^*)(\eta_1^*,\eta_3^*)\}_\varepsilon}{(\eta_0,\eta_2)(\eta_1,\eta_3)} \\ &- r(\eta_0,\dots,\eta_3)\frac{\{(\eta_0^*,l_2^*)(\eta_1^*,\eta_3^*)\}_{\varepsilon^2}}{(\eta_0,\eta_2)(\eta_1,\eta_3)} \end{aligned} \tag{2.7}$$

Remark 2.2. *The computation of coefficient of ε^n, which is $r_{\varepsilon^n}(\eta_0^*,\dots,\eta_3^*)$, in the truncated polynomial* (2.4) *will give us the following:*

$$\sum_{k=0}^{n}\left(\{\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)\}_{\varepsilon^k}\, r_{\varepsilon^{n-k}}(\eta_0^*,\dots,\eta_3^*)\right) = \{\Delta(\eta_0^*,\eta_3^*)\Delta(\eta_1^*,\eta_2^*)\}_{\varepsilon^n},$$

where $\Delta(\eta_i,\eta_j)\neq 0$ for $i\neq j$ and $(\eta_0^*,\dots,\eta_3^*)\in C_4\left(\mathbb{A}^2_{F[\varepsilon]_{n+1}}\right)$.

2.2. Triple-ratio in $F[\varepsilon]_\nu$

First, we define a triple-ratio $r_3 : C_6(\mathbb{A}^3_F)\to F$ as (see [4])

$$r_3(\eta_0,\dots,\eta_5) = \operatorname{Alt}_6\frac{\Delta(\eta_0,\eta_1,\eta_3)\Delta(\eta_1,\eta_2,\eta_4)\Delta(\eta_2,\eta_0,\eta_5)}{\Delta(\eta_0,\eta_1,\eta_4)\Delta(\eta_1,\eta_2,\eta_5)\Delta(\eta_2,\eta_0,\eta_3)}$$

where $C_6(\mathbb{A}^3_F)$ is a free abelian group generated by the configurations of six points in $\mathbb{A}^3_F$ and $\mathbb{A}^3_F$ is a three dimensional affine space over a field F. Here, we will discuss triple-ratio (generalized cross-ratio) of 6 points, i.e., $(\eta_0^*,\dots,\eta_5^*)\in C_6(\mathbb{A}^3_{F[\varepsilon]_\nu})$ for $\nu = n+1$. The calculations in triple-ratio are similar to the cross-ratio of 4 points $(\eta_0^*,\dots,\eta_3^*)\in C_4(\mathbb{A}^2_{F[\varepsilon]_\nu})$. Let's consider $\nu = 2$ since the other cases are not required.

We take $(\eta_0^*,\dots,\eta_5^*)\in C_6\left(\mathbb{A}^3_{F[\varepsilon]_2}\right)$, for any $\eta_i^*\in(\eta_0^*,\dots,\eta_5^*)$

$$l_i^* = \begin{pmatrix} a_i + a_{i,\varepsilon}\varepsilon \\ b_i + b_{i,\varepsilon}\varepsilon \\ c_i + c_{i,\varepsilon}\varepsilon \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} + \begin{pmatrix} a_{i,\varepsilon} \\ b_{i,\varepsilon} \\ c_{i,\varepsilon} \end{pmatrix}\varepsilon = \eta_i + \eta_{i,\varepsilon}\varepsilon$$

$$\Delta(\eta_i^*,\eta_j^*,\eta_k^*) = \Delta(\eta_i,\eta_j,\eta_k) + \Delta(\eta_i^*,\eta_j^*,\eta_k^*)_\varepsilon\varepsilon$$

where $\Delta(\eta_i,\eta_j,\eta_k)$ is a 3×3-determinant,

$$\Delta(\eta_i^*,\eta_j^*,\eta_k^*)_\varepsilon = \Delta(\eta_{i,\varepsilon},\eta_j,\eta_k) + \Delta(\eta_i,\eta_{j,\varepsilon},\eta_k) + \Delta(\eta_i,\eta_j,\eta_{k,\varepsilon})$$

and

$$\Delta(\eta_i^*,\eta_j^*,\eta_k^*)_{\varepsilon^0} = \Delta(\eta_i,\eta_j,\eta_k)$$

As we can expand, we also get the equalities.

$$\mathbf{r}_3(\eta_0^*,\dots,\eta_5^*) = r_3(\eta_0,\dots,\eta_5) + r_{3,\varepsilon}(\eta_0^*,\dots,\eta_5^*)\varepsilon$$

for $\Delta(\eta_i,\eta_j,\eta_j)\neq 0$, the multiplicative inverse of $\Delta(\eta_i^*,\eta_j^*,\eta_k^*)$ is $-\frac{1}{\Delta(\eta_i,\eta_j,\eta_j)}-\frac{\Delta(\eta_i^*,\eta_j^*,\eta_k^*)_\varepsilon}{\Delta(\eta_i,\eta_j,\eta_j)^2}\varepsilon$ and from now on, If simplify the previous equalities, we may use $(\eta_i^*\eta_j^*\eta_k^*)$ instead of $\Delta(\eta_i^*,\eta_j^*,\eta_k^*)$ unless specified.

$$\begin{aligned}\mathbf{r}(\eta_0^*,\dots,\eta_5^*) &= \mathrm{Alt}_6\frac{(\eta_0^*\eta_1^*\eta_3^*)(\eta_1^*\eta_2^*\eta_4^*)(\eta_2^*\eta_0^*\eta_5^*)}{(\eta_0^*\eta_1^*\eta_4^*)(\eta_1^*\eta_2^*\eta_5^*)(\eta_2^*\eta_0^*\eta_3^*)}\\ &= \mathrm{Alt}_6\left\{\frac{\{(\eta_0\eta_1\eta_3)+(\eta_0^*\eta_1^*\eta_3^*)_\varepsilon\varepsilon\}\{(\eta_1\eta_2\eta_4)+(\eta_1^*\eta_2^*\eta_4^*)_\varepsilon\varepsilon\}\{(\eta_2\eta_0\eta_5)+(\eta_2^*\eta_0^*\eta_5^*)_\varepsilon\varepsilon\}}{\{(\eta_0\eta_1\eta_4)+(\eta_0^*\eta_1^*\eta_4^*)_\varepsilon\varepsilon\}\{(\eta_1\eta_2\eta_5)+(\eta_1^*\eta_2^*\eta_5^*)_\varepsilon\varepsilon\}\{(\eta_2\eta_0\eta_3)+(\eta_2^*\eta_0^*\eta_3^*)_\varepsilon\varepsilon\}}\right\}\end{aligned}$$

Simplifying the above and separating the coefficients of ε^0 and ε^1, we see that the coefficient of ε^0 is the triple-ratio of six points $(\eta_0,\dots,\eta_5)\in C_6\left(\mathbb{A}_F^3\right)$ and the coefficient of ε is the following:

$$\begin{aligned}&r_{3,\varepsilon}(\eta_0^*,\dots,\eta_5^*)\\ &= \mathrm{Alt}_6\left\{\frac{\{(\eta_0^*\eta_1^*\eta_3^*)(\eta_1^*\eta_2^*\eta_4^*)(\eta_2^*\eta_0^*\eta_5^*)\}_\varepsilon}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}-\frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}\frac{\{(\eta_0^*\eta_1^*\eta_4^*)(\eta_1^*\eta_2^*\eta_5^*)(\eta_2^*\eta_0^*\eta_3^*)\}_\varepsilon}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}\right\}\end{aligned} \quad (2.8)$$

2.3. Tangent to Bloch group([1])

Let F be an algebraically closed field of characteristic 0. Let $F[\varepsilon]_2 = F[\varepsilon]/\varepsilon^2$ be the truncated polynomial ring (or a ring of dual numbers) for an arbitrary field F. We can define an $F^\times$-action in $F[\varepsilon]_2$ as follows. For $\lambda\in F^\times$,

$$\lambda : F[\varepsilon]_2 \to F[\varepsilon]_2, \phi+\phi'\varepsilon \mapsto \phi+\lambda\phi'\varepsilon$$

we denote this action by $\star$, so we use $\lambda\star(\phi+\phi'\varepsilon)=\phi+\lambda\phi'\varepsilon.$

The *tangent group* $T\mathcal{B}_2(F)$ is defined as a $\mathbb{Z}$-module generated by the combinations $[\phi+\phi'\varepsilon]-[\phi]\in\mathbb{Z}[F[\varepsilon]_2]$, $(\phi,\phi'\in F)$: For which we put shorthand $\langle\phi;\phi'] := [\phi+\phi'\varepsilon]-[\phi]$ and quotient by the subgroup generated by the following relation

$$\begin{aligned}\langle\phi;\phi'] - \langle\psi;\psi'] + \left\langle\frac{\psi}{\phi};\left(\frac{\psi}{\phi}\right)'\right] - \left\langle\frac{1-\psi}{1-\phi};\left(\frac{1-\psi}{1-\phi}\right)'\right]\\ + \left\langle\frac{\phi(1-\psi)}{\psi(1-\phi)};\left(\frac{\phi(1-\psi)}{\psi(1-\phi)}\right)'\right], \quad \phi,\psi\neq 0,1,\phi\neq\psi\end{aligned} \quad (2.9)$$

where

$$\left(\frac{\psi}{a}\right)' = \frac{\phi\psi'-\phi'\psi}{\phi^2},$$

$$\left(\frac{1-\psi}{1-\phi}\right)' = \frac{(1-\psi)\phi'-(1-\phi)\psi'}{(1-\phi)^2}$$

and

$$\left(\frac{\phi(1-\psi)}{\psi(1-\phi)}\right)' = \frac{\psi(1-\psi)\phi'-\phi(1-\phi)\psi'}{(\psi(1-\phi))^2}$$

Remark 2.3. *See [1] for a discussion of $T\mathcal{B}_2(F)$, where the definition of $T\mathcal{B}_2(F)$ was justified using Lemma 3.1 of [1]*

We give a list of relations in $T\mathcal{B}_2(F)$ from [1].

(1) The two-term relation:

$$\langle\phi;\psi]_2 = -\langle 1-\phi; -\psi]_2$$

(2) The inversion relation:

$$\langle\phi;\psi]_2 = \left\langle\frac{1}{\phi}; -\frac{\psi}{\phi^2}\right]_2$$

(3) Four-term relation:

If we use $\phi' = \phi(1-\phi)$ and $\psi' = \psi(1-\psi)$ then (2.9) becomes four-term relation (see [1]).

$$\langle\phi;\phi(1-\phi)]_2 - \langle\psi;\psi(1-\psi)]_2 + \phi\star\left\langle\frac{\psi}{\phi};\frac{\psi}{\phi}\left(1-\frac{\psi}{\phi}\right)\right]_2 + (1-\phi)\star\left\langle\frac{1-\psi}{1-\phi};\frac{1-\psi}{1-\phi}\left(1-\frac{1-\psi}{1-\phi}\right)\right]_2 = 0,$$

where $\phi,\psi \neq 0,1, \phi\neq\psi$.

The following map is an infinitesimal analogue of δ (defined in [4]) and ∂ (defined in [1] and [5]), Cathelineau called it the *tangential map*.

$$T\mathcal{B}_2(F) \xrightarrow{\partial_\varepsilon} (F\otimes F^\times)\oplus\left(\bigwedge{}^2 F\right)$$

with

$$\partial_\varepsilon\left(\langle\phi;\psi]_2\right) = \left(\frac{\psi}{\phi}\otimes(1-\phi) + \frac{\psi}{1-\phi}\otimes\phi\right) + \left(\frac{\psi}{1-\phi}\wedge\frac{\psi}{\phi}\right)$$

First term of the complex is in degree one and ∂_ε has a degree $+1$.

Note that we get the direct sum of two spaces on the right side.

3. Main results and discussion

3.1. Dilogarithmic bicomplexes

In this section, we will connect the Grassmannian bicomplex to the tangent to the Bloch-Suslin complex.

We will use the following notations throughout this section

$$\Delta(\eta_i^*,\eta_j^*)_\varepsilon = \Delta(\eta_{i,\varepsilon},\eta_j) + \Delta(\eta_i,\eta_{j,\varepsilon}) \qquad \text{and} \qquad \Delta(\eta_i^*,\eta_j^*)_{\varepsilon^0} = \Delta(\eta_i,\eta_j)$$

and we will assume that $\Delta(\eta_i,\eta_j)\neq 0$ (as we often want to divide by such determinants).

Let $C_m(\mathbb{A}^2_{F[\varepsilon]_2})$ be the free abelian group generated by the configuration of m points in $\mathbb{A}^2_{F[\varepsilon]_2}$, where $\mathbb{A}^2_{F[\varepsilon]_2}$ is defined as an affine space over $F[\varepsilon]_2$. The configurations of m points in $\mathbb{A}^2_{F[\varepsilon]_2}$ are 2-tuples of vectors over $F[\varepsilon]_2$ modulo $GL_2(F[\varepsilon]_2)$. In this case, one can write the Grassmannian complex as follows:

$$\cdots \xrightarrow{d} C_5(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{d} C_4(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{d} C_3(\mathbb{A}^2_{F[\varepsilon]_2})$$

$$d\colon (\eta_0^*,\ldots,\eta_{m-1}^*) \mapsto \sum_{i=0}^{m}(-1)^i(\eta_0^*,\ldots,\hat{\eta}_i^*,\ldots,\eta_{m-1}^*)$$

where $\eta_i^* = \begin{pmatrix} \phi_i + \phi_{i,\varepsilon}\varepsilon \\ \psi_i + \psi_{i,\varepsilon}\varepsilon \end{pmatrix} = \begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix} + \begin{pmatrix} \phi_{i,\varepsilon} \\ \psi_{i,\varepsilon} \end{pmatrix}\varepsilon = \eta_i + \eta_{i,\varepsilon}\varepsilon$ and $\phi_i, \psi_i, \phi_{i,\varepsilon}, \psi_{i,\varepsilon} \in F$, $\begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The diagram below gives the relation between Grassmannian complex and tangent to the Bloch-Suslin complex.

$$\begin{array}{ccccc} C_5(\mathbb{A}^2_{F[\varepsilon]_2}) & \xrightarrow{d} & C_4(\mathbb{A}^2_{F[\varepsilon]_2}) & \xrightarrow{d} & C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \\ & & \downarrow{\tau^2_{1,\varepsilon}} & & \downarrow{\tau^2_{0,\varepsilon}} \\ & & T\mathcal{B}_2(F) & \xrightarrow{\partial_\varepsilon} & F \otimes F^\times \oplus \bigwedge^2 F \end{array} \tag{4.2a}$$

where

$$\partial_\varepsilon : \langle \phi; \psi]_2 \mapsto \left(\frac{\psi}{\phi} \otimes (1-\phi) + \frac{\psi}{1-\phi} \otimes \phi\right) + \left(\frac{\psi}{1-\phi} \wedge \frac{\psi}{\phi}\right)$$

$\tau^2_{0,\varepsilon}$ can be written as a sum of two morphisms

$$\tau^{(1)} : C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \to F \otimes F^\times$$

and

$$\tau^{(2)} : C_3(\mathbb{A}^2_{F[\varepsilon]_2}) \to \bigwedge{}^2 F$$

where

$$\begin{aligned} &\tau^{(1)}(\eta_0^*, \eta_1^*, \eta_2^*) \\ =&\frac{\Delta(\eta_1^*, \eta_2^*)_\varepsilon}{\Delta(\eta_1, \eta_2)} \otimes \frac{\Delta(\eta_0, \eta_2)}{\Delta(\eta_0, \eta_1)} - \frac{\Delta(\eta_0^*, \eta_2^*)_\varepsilon}{\Delta(\eta_0, \eta_2)} \otimes \frac{\Delta(\eta_1, \eta_2)}{\Delta(\eta_1, \eta_0)} + \frac{\Delta(\eta_0^*, \eta_1^*)_\varepsilon}{\Delta(\eta_0, \eta_1)} \otimes \frac{\Delta(\eta_2, \eta_1)}{\Delta(\eta_2, \eta_0)} \end{aligned}$$

and

$$\begin{aligned} &\tau^{(2)}(\eta_0^*, \eta_1^*, \eta_2^*) \\ =&\frac{\Delta(\eta_0^*, \eta_1^*)_\varepsilon}{\Delta(\eta_0, \eta_1)} \wedge \frac{\Delta(\eta_1^*, \eta_2^*)_\varepsilon}{\Delta(\eta_1, \eta_2)} - \frac{\Delta(\eta_0^*, \eta_1^*)_\varepsilon}{\Delta(\eta_0, \eta_1)} \wedge \frac{\Delta(\eta_0^*, \eta_2^*)_\varepsilon}{\Delta(\eta_0, \eta_2)} + \frac{\Delta(\eta_1^*, \eta_2^*)_\varepsilon}{\Delta(\eta_1, \eta_2)} \wedge \frac{\Delta(\eta_0^*, \eta_2^*)_\varepsilon}{\Delta(\eta_0, \eta_2)} \end{aligned}$$

Furthermore, we put

$$\tau^2_{1,\varepsilon}(\eta_0^*, \dots, \eta_3^*) = \langle r(\eta_0, \dots, \eta_3); r_\varepsilon(\eta_0^*, \dots, \eta_3^*)]$$

where $r(\eta_0, \dots, \eta_3)$ and $r_\varepsilon(\eta_0^*, \dots, \eta_3^*)$ are the coefficients of ε^0 and ε^1 respectively.

Our maps $\tau^2_{0,\varepsilon}$ and $\tau^2_{1,\varepsilon}$ are based on ratios of determinants and cross-ratios respectively, so there is enough evidence that they are well-defined. This independence can be seen directly through the definition of maps.

We will also use shorthand $(\eta_i\eta_j)$ instead of $\Delta(\eta_i, \eta_j)$ wherever we find less space to accommodate long expressions.

Now we calculate,

$$1-\mathbf{r}(\eta_0^*,\ldots,\eta_3^*)=\frac{\Delta(\eta_0^*,\eta_1^*)\Delta(\eta_2^*,\eta_3^*)}{\Delta(\eta_0^*,\eta_2^*)\Delta(\eta_1^*,\eta_3^*)}$$
$$=\frac{(\eta_0\eta_1)(\eta_2\eta_3)}{(\eta_0\eta_2)(\eta_1\eta_3)}+\frac{y}{(\eta_0\eta_2)^2(\eta_1\eta_3)^2}\varepsilon \tag{3.1}$$

where

$$\begin{aligned}y=&+(\eta_0\eta_2)(\eta_1\eta_3)(\eta_0\eta_1)(\eta_2\eta_{3,\varepsilon})+(\eta_0\eta_2)(\eta_1\eta_3)(\eta_0\eta_1)(\eta_{2,\varepsilon}\eta_3)\\&+(\eta_0\eta_2)(\eta_1\eta_3)(\eta_2\eta_3)(\eta_0\eta_{1,\varepsilon})+(\eta_0\eta_2)(\eta_1\eta_3)(\eta_2\eta_3)(\eta_{0,\varepsilon}\eta_1)\\&-(\eta_0\eta_1)(\eta_2\eta_3)(\eta_0\eta_2)(\eta_1\eta_{3,\varepsilon})-(\eta_0\eta_1)(\eta_2\eta_3)(\eta_0\eta_2)(\eta_{1,\varepsilon}\eta_3)\\&-(\eta_0\eta_1)(\eta_2\eta_3)(\eta_1\eta_3)(\eta_0\eta_{2,\varepsilon})-(\eta_0\eta_1)(\eta_2\eta_3)(\eta_1\eta_3)(\eta_{0,\varepsilon}\eta_2)\end{aligned}$$

Remark 3.1. *The $F^\times$-action of $T\mathcal{B}_2(F)$ lifts to an $F^\times$-action on $C_4(\mathbb{A}^2_{F[\varepsilon]_2})$ in the obvious way:*

The $F^\times$-action is defined above for $F[\varepsilon]_2$ induces an $F^\times$-action in $\mathbb{A}^2_{F[\varepsilon]_2}$ diagonally as

$$\lambda\star\begin{pmatrix}a+a_\varepsilon\varepsilon\\ b+b_\varepsilon\varepsilon\end{pmatrix}=\begin{pmatrix}a+\lambda a_\varepsilon\varepsilon\\ b+\lambda b_\varepsilon\varepsilon\end{pmatrix}\in\mathbb{A}^2_{F[\varepsilon]_2},\lambda\in F^\times$$

Lemma 3.2. *The diagram (4.2a) is commutative.*

Proof. The proof follows directly from calculation. □

In the remainder of this section we prove that the following diagram is a bicomplex.

$$\begin{array}{ccc} C_5(\mathbb{A}^3_{F[\varepsilon]_2}) & \xrightarrow{d} & C_4(\mathbb{A}^3_{F[\varepsilon]_2})\\ \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'}\\ C_4(\mathbb{A}^2_{F[\varepsilon]_2}) & \xrightarrow{d} & C_3(\mathbb{A}^2_{F[\varepsilon]_2})\\ \downarrow{\scriptstyle \tau^2_{1,\varepsilon}} & & \downarrow{\scriptstyle \tau^2_{0,\varepsilon}}\\ T\mathcal{B}_2(F) & \xrightarrow{\partial_\varepsilon} & F\otimes F^\times\oplus\bigwedge^2F \end{array} \tag{4.2b}$$

To prove that the above diagram is bicomplex, we will give the next results.

Proposition 3.3. *The map $C_4(\mathbb{A}^3_{F[\varepsilon]_2})\xrightarrow{d'}C_3(\mathbb{A}^2_{F[\varepsilon]_2})\xrightarrow{\tau^2_{0,\varepsilon}}(F\otimes F^\times)\oplus\left(\bigwedge^2F\right)$ is zero.*

Proof. Let $\omega\in\det V_3^*$ be the volume form in three-dimensional vector space V_3, i.e., $\Delta(\eta_i,\eta_j,\eta_k)=\langle\omega,\eta_i\wedge\eta_j\wedge\eta_k\rangle$. Then $\Delta(\eta_i,\cdot,\cdot)$ is a volume form in $V_3/\langle\eta_i\rangle$. Use

$$\Delta(\eta_i^*,\eta_j^*,\eta_k^*)=\Delta(\eta_i,\eta_j,\eta_k)+\left\{\Delta(\eta_i^*,\eta_j^*,\eta_k^*)_\varepsilon\right\}\varepsilon$$

where

$$\Delta(\eta_i^*,\eta_j^*,\eta_k^*)_\varepsilon=\Delta(\eta_{i,\varepsilon},\eta_j,\eta_k)+\Delta(\eta_i,\eta_{j,\varepsilon},\eta_k)+\Delta(\eta_i,\eta_j,\eta_{k,\varepsilon})$$

We can directly compute $\tau^2_{0,\varepsilon}\circ d'$ which gives zero. □

The following result is very important for proving Theorem 3.7. Through this result we are able to see the projected-five term relation for $T\mathcal{B}_2(F)$.

Lemma 3.4. *Let $x_0^*, \dots, x_4^* \in \mathbb{P}^2_{F[\varepsilon]_2}$ be 5 points in generic position. Then*

$$\sum_{i=0}^{4}(-1)^i \left\langle r(x_i|x_0,\dots,\hat{x}_i,\dots,x_4); r_\varepsilon(x_i^*|x_0^*,\dots,\hat{x}_i^*,\dots,x_4^*)\right] = 0 \in T\mathcal{B}_2(F), \tag{3.2}$$

where $x_i^* = x_i + x_i'\varepsilon$ and $x_i, x_i' \in \mathbb{P}^2_F$

$$\boldsymbol{r}(x_i^*|x_0^*,\dots,\hat{x}_i^*,\dots,x_4^*) = r(x_i|x_0,\dots,\hat{x}_i,\dots,x_4) + r_\varepsilon(x_i^*|x_0^*,\dots,\hat{x}_i^*,\dots,x_4^*)\varepsilon,$$

Proof. Consider five points $y_0,\dots,y_4 \in \mathbb{P}^1_F$ in generic position. We can write the five-term relation in terms of cross-ratios in $\mathcal{B}_2(F)$ as (see Proposition 4.5 (2)b in [2]):

$$\sum_{i=0}^{4}(-1)^i[r(y_0,\dots,\hat{y}_i,\dots,y_4)]_2 = 0$$

These five points depend on 2 parameters modulo the action of $PGL_2(F)$, whose action on $\mathbb{P}^1_F$ is 3-fold transitive. So we can express these five points with two variables modulo this action as, we can put

$$(y_0,\dots,y_4) = \left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}\frac{1}{\phi}\\1\end{pmatrix},\begin{pmatrix}\frac{1}{\psi}\\1\end{pmatrix}\right).$$

Then we can get the five-term relation in two variables (by using inversion relation in the last two terms).

$$[\phi]_2 - [\psi]_2 + \left[\frac{\psi}{\phi}\right]_2 + \left[\frac{1-\phi}{1-\psi}\right]_2 - \left[\frac{1-\frac{1}{\phi}}{1-\frac{1}{b}}\right]_2 = 0.$$

Now we consider five points $y_0^*,\dots,y_4^* \in \mathbb{P}^1_{F[\varepsilon]_2}$, in generic position, where $y_i^* = y_i + y_i'\varepsilon$ for $y_i, y_i' \in \mathbb{P}^1_F$. A generic 2×2 matrix in $PGL_2(F[\varepsilon]_2)$ depends on $6 = 2(2\times 2) - 2(1)$ parameters, while each point in $\mathbb{P}^1_{F[\varepsilon]_2}$ depends on 2 parameters, so these five points in $\mathbb{P}^1_{F[\varepsilon]_2}$ modulo the action of $PGL_2(F[\varepsilon]_2)$ have 4 parameters. Now we can express them by using four variables we choose:

$$(y_0^*,\dots,y_4^*) = \left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}\frac{1}{\phi}-\frac{\phi'}{\phi^2}\varepsilon\\1\end{pmatrix},\begin{pmatrix}\frac{1}{\psi}-\frac{\psi'}{\psi^2}\varepsilon\\1\end{pmatrix}\right).$$

We calculate all possible determinants which are the following:

$$\Delta(y_0,y_1) = \Delta(y_0,y_2) = \Delta(y_0,y_3) = \Delta(y_0,y_4) = 1, \Delta(y_1,y_2) = -1,$$
$$\Delta(y_1,y_3) = -\frac{1}{\phi}, \Delta(y_1,y_4) = -\frac{1}{\psi}, \Delta(y_2,y_3) = 1-\frac{1}{\phi}, \Delta(y_2,y_4) = 1-\frac{1}{\psi}$$
$$\Delta(y_0^*,y_1^*)_\varepsilon = \Delta(y_0^*,y_2^*)_\varepsilon = \Delta(y_0^*,y_3^*)_\varepsilon = \Delta(y_0^*,y_4^*)_\varepsilon = \Delta(y_1^*,y_2^*)_\varepsilon = 0$$
$$\Delta(y_1^*,y_3^*)_\varepsilon = \Delta(y_2^*,y_3^*)_\varepsilon = \frac{\phi'}{\phi^2}, \Delta(y_1^*,y_4^*)_\varepsilon = \Delta(y_2^*,y_4^*)_\varepsilon = \frac{\psi'}{\psi^2}$$

For $y_0^*,\dots,y_4^* \in \mathbb{P}^1_{F[\varepsilon]_2}$, we can write the following expression in $T\mathcal{B}_2(F)$

$$\sum_{i=0}^{4}(-1)^i \left\langle r(y_0,\dots,\hat{y}_i,\dots,y_4); r_\varepsilon(y_0^*,\dots,\hat{y}_i^*,\dots,y_4^*)\right]_2$$

If we expand the above expression and substitute all the determinants in it, we will get the following expression in two variables.

$$\langle\phi;\phi']_2-\langle\psi;\psi']_2+\left\langle\frac{\psi}{\phi};\frac{\phi\psi'-\phi'\psi}{\phi^2}\right]_2-\left\langle\frac{1-\psi}{1-\phi};\frac{(1-\psi)\phi'-(1-\phi)\psi'}{(1-\phi)^2}\right]_2$$
$$+\left\langle\frac{\phi(1-\psi)}{\psi(1-\phi)};\frac{\psi(1-\psi)\phi'-\phi(1-\phi)\psi'}{(\psi(1-\phi))^2}\right]_2$$

From (3.2) it is clear that the above is the LHS of the five-term relation in $T\mathcal{B}_2(F)$. We first show underneath, that this claim is valid, then later we reduce it to the five-term relation.

Consider $x_0,\dots,x_4\in\mathbb{P}^2_F$ in generic position. These five points also depend on 2 parameters modulo the action of $PGL_2(F)$, so we can express these five points in terms of two variables by the following choice:

$$(x_0,\dots,x_4)=\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}\frac{1}{\psi}\\\frac{1}{\phi}\\1\end{pmatrix}\right)$$

We compute all possible 3×3 determinants of the above and put them in the expansion of the following:

$$\sum_{i=0}^{4}(-1)^i\,[r(x_i|x_0,\dots,\hat{x}_i,\dots,x_4)]_2\in\mathcal{B}_2(F),$$

we get the following expression in two variables

$$[\phi]_2-\psi]_2+\left[\frac{\psi}{\phi}\right]_2+\left[\frac{1-\phi}{1-\psi}\right]_2-\left[\frac{1-\frac{1}{\phi}}{1-\frac{1}{\psi}}\right]_2,$$

clearly the above is the LHS of one version of five-term relation in $\mathcal{B}_2(F)$.

Since by assumption $x_0^*,\dots,x_4^*\in\mathbb{P}^2_{F[\varepsilon]_2}$ are 5 points in generic position, we can express them as modulo the action of $PGL_3(F[\varepsilon]_2)$ into 4 parameters, then we can choose these points in terms of four variables in the following way:

$$(x_0^*,\dots,x_4^*)=\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}\frac{1}{b}-\frac{b'}{b^2}\varepsilon\\\frac{1}{\phi}-\frac{\phi'}{\phi^2}\varepsilon\\1\end{pmatrix}\right)$$

We compute all possible 3×3 determinants and substitute them in an expansion of the following:

$$\sum_{i=0}^{4}(-1)^i\,\langle r(x_i|x_0,\dots,\hat{x}_i,\dots,x_4);r_\varepsilon(x_i^*|x_0^*,\dots,\hat{x}_i^*,\dots,x_4^*)]_2\in T\mathcal{B}_2(F),$$

we get

$$\langle\phi;\phi']_2-\langle\psi;\psi']_2+\left\langle\frac{\psi}{\phi};\frac{\phi\psi'-\phi'\psi}{\phi^2}\right]_2-\left\langle\frac{1-\psi}{1-\phi};\frac{(1-\psi)\phi'-(1-\phi)\psi'}{(1-\phi)^2}\right]_2$$
$$+\left\langle\frac{\phi(1-\psi)}{\psi(1-\phi)};\frac{\psi(1-\psi)\phi'-\phi(1-\phi)\psi'}{(\psi(1-\phi))^2}\right]_2$$

which is the five-term expression in $T\mathcal{B}_2(F)$ up to invoking the inversion relation for the last two terms, which also holds in $T\mathcal{B}_2(F)$. □

Lemma 3.4 indicates that we now have the projected five-term relation for $T\mathcal{B}_2(F)$ and this relation will help us to prove the commutative diagram for weight $n = 3$ in the tangential case.

Proposition 3.5. *The map* $C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{d'} C_4(\mathbb{A}^2_{F[\varepsilon]_2}) \xrightarrow{\tau^2_{1,\varepsilon}} T\mathcal{B}_2(F)$ *is zero.*

Proof. We can directly calculate $\tau^2_{1\varepsilon} \circ d'$.

$$\tau^2_{1,\varepsilon} \circ d'(\eta^*_0, \ldots, \eta^*_4) = \tau^2_{1,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i\,(\eta^*_i|\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_4)\right)$$
$$= \sum_{i=0}^{4}(-1)^i\,\langle r\,(\eta_i|\eta_0, \ldots, \hat{\eta}_i \ldots, \eta_4)\,; r_\varepsilon\,(\eta^*_i|\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_4)]_2 \tag{3.3}$$

The above is the projected five term relation in $T\mathcal{B}_2(F)$ by Lemma 3.4. □

Theorem 3.2 shows that the diagram (4.2a) is commutative and Propositions 3.3 and 3.5 shows that we have formed a bicomplex between the Grassmannian complex and Cathelineau's tangent complex.

3.2. Trilogarithmic complexes

We have already discussed the tangent group (or $\mathbb{Z}$-module) $T\mathcal{B}_2(F)$ over $F[\varepsilon]_2$ in §3.1. In this section we will discuss group $T\mathcal{B}_3(F)$ and its functional equations and will connect Grassmannian complex and tangential complex to Goncharov complex.

3.2.1. The Abelian group $T\mathcal{B}_3(F)$

The $\mathbb{Z}$-module $T\mathcal{B}_3(F)$ over $F[\varepsilon]_2$ is defined as the group generated by:

$$\langle a; b] = [a + b\varepsilon] - [a] \in \mathbb{Z}\,[F[\varepsilon]_2]\,, \quad a, b \in F, \quad a \neq 0, 1$$

and quotient by the kernel of the following map

$$\partial_{\varepsilon,3} : \mathbb{Z}\,[F[\varepsilon]_2] \to T\mathcal{B}_2(F) \otimes F^\times \oplus F \otimes \mathcal{B}_2(F), \langle a; b] \mapsto \langle a; b]_2 \otimes a + \frac{b}{a} \otimes [a]_2$$

Now we say that $\langle a; b]_3 \in T\mathcal{B}_3(F) \subset \mathbb{Z}[F[\varepsilon]_2]/\ker\partial_{\varepsilon,3}$.

We have the following relations which are satisfied in $T\mathcal{B}_3(F)$.

(1) The three-term relation.

$$\langle 1 - a; (1 - a)_\varepsilon]_3 - \langle a; a_\varepsilon]_3 - \left\langle 1 - \frac{1}{a}; \left(1 - \frac{1}{a}\right)_\varepsilon\right]_3 = 0 \in T\mathcal{B}_3(F)$$

(2) The inversion relation

$$\langle a; a_\varepsilon]_3 = \left\langle \frac{1}{a}; \left(\frac{1}{a}\right)_\varepsilon\right]_3$$

(3) The Cathelineau 22-term relation ([2])
This relation $J(a,b,c)$ for the indeterminates a,b,c can be written in this way:

$$J(a,b,c) = [[a,c]] - [[b,c]] + a\left[\left[\frac{b}{a}, c\right]\right] + (1-a)\left[\left[\frac{1-b}{1-a}, c\right]\right], \tag{3.4}$$

where

$$[[a,b]] = (b-a)\tau(a,b) + \frac{1-b}{1-a}\sigma(a) + \frac{1-a}{1-b}\sigma(b),$$

while $\tau(a,b)$ is defined via five term relation and $\star$-action. We take $\langle x_i; x_{i,\varepsilon}]_3$ with coefficient $\frac{1}{1-x_i}$ which is handled by $\star$-action.

$$\begin{aligned}\tau(a,b) =& \left\langle a; a_\varepsilon \cdot \frac{1}{1-a}\right]_3 - \left\langle b; b_\varepsilon \cdot \frac{1}{1-b}\right]_3 + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)_\varepsilon \cdot \frac{1}{a-b}\right]_3 \\ &- \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)_\varepsilon \cdot \frac{1}{b-a}\right]_3 - \left\langle \frac{a(1-b)}{b(1-a)}; \left(\frac{a(1-b)}{b(1-a)}\right)_\varepsilon \cdot \frac{1}{b-a}\right]_3\end{aligned}$$

and

$$\sigma(a) = \langle a; a_\varepsilon \cdot a]_3 + \langle 1-a; (1-a)_\varepsilon \cdot (1-a)]_3.$$

Then we can calculate Cathelineau's 22-term expression by substituting all values in (3.4).

$$\begin{aligned}J(a,b,c) =& \langle a; a_\varepsilon c]_3 - \langle b; b_\varepsilon c]_3 + \langle c; c_\varepsilon(a-b+1)]_3 \\ &+ \langle 1-a; (1-a)_\varepsilon(1-c)]_3 - \langle 1-b; (1-b)_\varepsilon(1-c)]_3 \\ &+ \langle 1-c; (1-c)_\varepsilon(b-a)]_3 - \left\langle \frac{c}{a}; \left(\frac{c}{a}\right)_\varepsilon\right]_3 + \left\langle \frac{c}{b}; \left(\frac{c}{b}\right)_\varepsilon\right]_3 + \left\langle \frac{b}{a}; \left(\frac{b}{a}\right)_\varepsilon c\right]_3 \\ &- \left\langle \frac{1-c}{1-a}; \left(\frac{1-c}{1-a}\right)_\varepsilon\right]_3 + \left\langle \frac{1-c}{1-b}; \left(\frac{1-c}{1-b}\right)_\varepsilon\right]_3 + \left\langle \frac{1-b}{1-a}; \left(\frac{1-b}{1-a}\right)_\varepsilon c\right]_3 \\ &+ \left\langle \frac{a(1-c)}{c(1-a)}; \left(\frac{a(1-c)}{c(1-a)}\right)_\varepsilon\right]_3 - \left\langle \frac{ca}{b}; \left(\frac{ca}{b}\right)_\varepsilon\right]_3 - \left\langle \frac{b(1-c)}{c(1-b)}; \left(\frac{b(1-c)}{c(1-b)}\right)_\varepsilon\right]_3 \\ &+ \left\langle \frac{a-b}{a}; \left(\frac{a-b}{a}\right)_\varepsilon (1-c)\right]_3 + \left\langle \frac{b-a}{1-a}; \left(\frac{b-a}{1-a}\right)_\varepsilon (1-c)\right]_3 \\ &+ \left\langle \frac{c(1-a)}{1-b}; \left(\frac{c(1-a)}{1-b}\right)_\varepsilon\right]_3 - \left\langle \frac{(1-c)a}{a-b}; \left(\frac{(1-c)a}{a-b}\right)_\varepsilon\right]_3 \\ &- \left\langle \frac{(1-c)(1-a)}{b-a}; \left(\frac{(1-c)(1-a)}{b-a}\right)_\varepsilon\right]_3 \\ &+ \left\langle \frac{(1-c)b}{c(a-b)}; \left(\frac{(1-c)b}{c(a-b)}\right)_\varepsilon\right]_3 + \left\langle \frac{(1-c)(1-b)}{c(b-a)}; \left(\frac{(1-c)(1-b)}{c(b-a)}\right)_\varepsilon\right]_3\end{aligned} \tag{3.5}$$

For the special condition $a_\varepsilon = a(1-a)$,$b_\varepsilon = b(1-b)$ and $c_\varepsilon = c(1-c)$, this 22-term expression becomes zero in $T\mathcal{B}_3(F)$.

One can write the following complex for $T\mathcal{B}_3(F)$.

$$T\mathcal{B}_3(F) \xrightarrow{\partial_\varepsilon} \begin{smallmatrix} T\mathcal{B}_2(F)\otimes F^\times \\ \oplus \\ F\otimes\mathcal{B}_2(F)\end{smallmatrix} \xrightarrow{\partial_\varepsilon} \left(F \otimes \bigwedge{}^2 F^\times\right) \oplus \left(\bigwedge{}^3 F\right)$$

3.2.2. Morphisms between Grassmannian and tangent to Goncharov's complex in weight 3

In this section, we will introduce morphisms between the Grassmannian complex and the tangent to Goncharov's complex for weight $n = 3$. Consider the following diagram

$$\begin{array}{ccccc} C_6(\mathbb{A}^3_{F[\varepsilon]_2}) & \xrightarrow{d} & C_5(\mathbb{A}^3_{F[\varepsilon]_2}) & \xrightarrow{d} & C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \\ \downarrow \tau^3_{2,\varepsilon} & & \downarrow \tau^3_{1,\varepsilon} & & \downarrow \tau^3_{0,\varepsilon} \\ T\mathcal{B}_3(F) & \xrightarrow{\partial_\varepsilon} & (T\mathcal{B}_2(F)\otimes F^\times)\oplus(F\otimes\mathcal{B}_2(F)) & \xrightarrow{\partial_\varepsilon} & \left(F\otimes\bigwedge^2 F^\times\right)\oplus\left(\bigwedge^3 F\right) \end{array} \tag{4.3a}$$

Here we define the projected cross-ratio

$$\mathbf{r}(\eta_0^*|\eta_1^*,\eta_2^*,\eta_3^*,\eta_4^*) = \frac{\Delta(\eta_0^*,\eta_1^*,\eta_4^*)\Delta(\eta_0^*,\eta_2^*,\eta_3^*)}{\Delta(\eta_0^*,\eta_1^*,\eta_3^*)\Delta(\eta_0^*,\eta_2^*,\eta_4^*)}$$

which can be further simplifed to

$$\mathbf{r}(\eta_0^*|\eta_1^*,\eta_2^*,\eta_3^*,\eta_4^*) = r(\eta_0|\eta_1,\eta_2,\eta_3,\eta_4) + r_\varepsilon(\eta_0^*|\eta_1^*,\eta_2^*,\eta_3^*,\eta_4^*)\varepsilon$$

where

$$r(\eta_0|\eta_1,\eta_2,\eta_3,\eta_4) = \frac{\Delta(\eta_0,\eta_1,\eta_4)\Delta(\eta_0,\eta_2,\eta_3)}{\Delta(\eta_0,\eta_1,\eta_3)\Delta(\eta_0,\eta_2,\eta_4)}$$

$$r_\varepsilon(\eta_0^*|\eta_1^*,\eta_2^*,\eta_3^*,\eta_4^*) = \frac{u}{\Delta(\eta_0,\eta_1,\eta_3)^2\Delta(\eta_0,\eta_2,\eta_4)^2}$$

$$\begin{aligned} u = &-\Delta(\eta_0,\eta_1,\eta_4)\Delta(\eta_0,\eta_2,\eta_3)\{\Delta(\eta_0,\eta_1,\eta_3)\Delta(\eta_0^*,\eta_2^*,\eta_4^*)_\varepsilon + \Delta(\eta_0,\eta_2,\eta_4)\Delta(\eta_0^*,\eta_1^*,\eta_3^*)_\varepsilon\} \\ &+ \Delta(\eta_0,\eta_1,\eta_3)\Delta(\eta_0,\eta_2,\eta_4)\{\Delta(\eta_0,\eta_1,\eta_4)\Delta(\eta_0^*,\eta_2^*,\eta_3^*)_\varepsilon + \Delta(\eta_0,\eta_2,\eta_3)\Delta(\eta_0^*,\eta_1^*,\eta_4^*)_\varepsilon\} \end{aligned}$$

where the morphisms between the two complexes are defined as follows:

$$\begin{aligned} &\tau^3_{0,\varepsilon}(\eta_0^*,\ldots,\eta_3^*) \\ &= \sum_{i=0}^{3}(-1)^i\Bigg(\frac{\Delta(\eta_0^*,\ldots,\hat{\eta}_i^*,\ldots,\eta_3^*)_\varepsilon}{\Delta(\eta_0,\ldots,\hat{\eta}_i,\ldots,\eta_3)} \otimes \frac{\Delta(\eta_0,\ldots,\hat{\eta}_{i+1},\ldots,\eta_3)}{\Delta(\eta_0,\ldots,\hat{\eta}_{i+2},\ldots,\eta_3)} \\ &\qquad \wedge \frac{\Delta(\eta_0,\ldots,\hat{\eta}_{i+3},\ldots,\eta_3)}{\Delta(\eta_0,\ldots,\hat{\eta}_{i+2},\ldots,\eta_3)} + \bigwedge_{\substack{j=0\\ j\neq i}}^{3} \frac{\Delta(\eta_0^*,\ldots,\hat{\eta}_j^*,\ldots,\eta_3^*)_\varepsilon}{\Delta(\eta_0,\ldots,\hat{\eta}_j,\ldots,\eta_3)}\Bigg), \quad i \mod 4, \end{aligned}$$

$$\begin{aligned} &\tau^3_{1,\varepsilon}(\eta_0^*,\ldots,\eta_4^*) \\ &= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(\Big\langle r(\eta_i|\eta_0,\ldots,\hat{\eta}_i,\ldots,\eta_4); r_\varepsilon(\eta_i^*|\eta_0^*,\ldots,\hat{\eta^*}_i,\ldots,\eta_4^*)\Big]_2 \otimes \prod_{i\neq j}\Delta(\hat{\eta}_i,\hat{\eta}_j) \\ &\qquad + \sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{\Delta(\eta_0^*,\ldots,\hat{\eta}_i^*,\ldots,\hat{\eta}_j^*,\ldots,\eta_4^*)_\varepsilon}{\Delta(\eta_0,\ldots,\hat{\eta}_i,\ldots,\hat{\eta}_j,\ldots,\eta_4)}\right)\otimes[r(\eta_i|\eta_0,\ldots,\hat{\eta}_i,\ldots,\eta_4)]_2\Bigg) \end{aligned}$$

and

$$\tau^3_{2\varepsilon}(\eta^*_0, \ldots, \eta^*_5) = \frac{2}{45}\mathrm{Alt}_6 \left\langle r_3(\eta_0, \ldots, \eta_5); r_{3,\varepsilon}(\eta^*_0, \ldots, \eta^*_5)\right]_3$$

where

$$r_3(\eta_0, \ldots, \eta_5) = \frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}$$

and

$$\begin{aligned} &r_{3,\varepsilon}(\eta^*_0, \ldots, \eta^*_5) \\ &= \frac{\{(\eta^*_0\eta^*_1\eta^*_3)(\eta^*_1\eta^*_2\eta^*_4)(\eta^*_2\eta^*_0\eta^*_5)\}_\varepsilon}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)} - \frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)} \frac{\{(\eta^*_0\eta^*_1\eta^*_4)(\eta^*_1\eta^*_2\eta^*_5)(\eta^*_2\eta^*_0\eta^*_3)\}_\varepsilon}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)} \end{aligned} \quad (3.6)$$

the map ∂_ε is defined as

$$\begin{aligned} &\partial_\varepsilon \left(\langle a; b]_2 \otimes c + x \otimes [y]_2\right) \\ &= \left(-\frac{b}{1-a} \otimes a \wedge c - \frac{b}{a} \otimes (1-a) \wedge c + x \otimes (1-y) \wedge y\right) + \left(\frac{b}{1-a} \wedge \frac{b}{a} \wedge x\right) \end{aligned}$$

and

$$\partial_\varepsilon(\langle a; b]_3) = \langle a; b]_2 \otimes a + \frac{b}{a} \otimes [a]_2$$

Theorem 3.6. *Diagram (4.3a), i.e.,*

$$\begin{array}{ccc} C_5(\mathbb{A}^3_{F[\varepsilon]_2}) & \xrightarrow{\quad d \quad} & C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \\ \downarrow \tau^3_{1,\varepsilon} & & \downarrow \tau^3_{0,\varepsilon} \\ (T\mathcal{B}_2(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) & \xrightarrow{\quad \partial_\varepsilon \quad} & \left(F \otimes \bigwedge^2 F^\times\right) \oplus \left(\bigwedge^3 F\right) \end{array}$$

is commutative, i.e., $\tau^3_{0,\varepsilon} \circ d = \partial_\varepsilon \circ \tau^3_{1,\varepsilon}$

Proof. First we divide the map $\tau^3_{0,\varepsilon} = \tau^{(1)} + \tau^{(2)}$ then calculate $\tau^{(1)} \circ d(\eta^*_0, \ldots, \eta^*_4)$

$$\tau^{(1)} \circ d(\eta^*_0, \ldots, \eta^*_4) = \tau^3_{0,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i(\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_4)\right)$$

$$\begin{aligned} = \widetilde{\mathrm{Alt}}_{(01234)}\Bigg(\sum_{i=0}^{3}(-1)^i\Bigg(&\frac{\Delta(\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_3)_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_i, \ldots, \eta_3)} \otimes \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+1}, \ldots, \eta_3)}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+2}, \ldots, \eta_3)} \\ &\wedge \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+3}, \ldots, \eta_3)}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+2}, \ldots, \eta_3)}\Bigg), \quad i \mod 4\Bigg) \end{aligned} \quad (3.7)$$

Now, we expand the inner sum that contains 12 terms and pass them through the this alternation to the inner sum, gives us 60 different terms overall. We collect terms involving the same $\frac{\Delta(\eta^*_i,\eta^*_j,\eta^*_k)}{\Delta(\eta_i,\eta_j,\eta_k)} \otimes \cdots$ together for calculation purposes. On the other hand the second part of the map is:

$$\tau^{(1)} \circ d(\eta^*_0, \ldots, \eta^*_4) = \widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i \bigwedge_{\substack{j=0 \\ j\neq i}}^{3} \frac{\Delta(\eta^*_0, \ldots, \hat{\eta}^*_j, \ldots, \eta^*_3)_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_j, \ldots, \eta_3)}\right) \quad (3.8)$$

The other side of the proof requires tedious computations. For the calculation of $\partial_\varepsilon \circ \tau^3_{1,\varepsilon}$ we will use the short hand $(\eta^*_i\eta^*_j\eta^*_k)_\varepsilon$ for $\Delta(\eta^*_i, \eta^*_j, \eta^*_k)_\varepsilon$ and $(\eta_i\eta_j\eta_k)$ for $\Delta((\eta_i, \eta_j, \eta_k)$. First we write $\partial_\varepsilon \circ \tau^3_{1,\varepsilon}(\eta^*_0, \dots, \eta^*_4)$ by using the definitions above.

$$\begin{aligned}
&\partial_\varepsilon \circ \tau^3_{1,\varepsilon}(\eta^*_0, \dots, \eta^*_4)\\
&= \partial_\varepsilon\Bigg(- \frac{1}{3}\sum_{i=0}^{4}(-1)^i\Big(\big\langle r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4); r_\varepsilon(\eta^*_i|\eta^*_0, \dots, \hat{\eta}^*{}_i, \dots, \eta^*_4)\big]_2 \otimes \prod_{i\neq j}\Delta(\hat{\eta}_i, \hat{\eta}_j)\\
&\qquad + \sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{\Delta(\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \hat{\eta}^*_j, \dots, \eta^*_4)_\varepsilon}{\Delta(\eta_0, \dots, \hat{\eta}_i, \dots, \hat{\eta}_j, \dots, \eta_4)}\right) \otimes \left[r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)\right]_2 \Big)\Bigg)
\end{aligned}$$

then we divide $\partial_\varepsilon = \partial^{(1)} + \partial^{(2)}$. The first part $\partial^{(1)} \circ \tau^3_{1,\varepsilon}(\eta^*_0, \dots, \eta^*_4)$ is

$$\begin{aligned}
= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(&-\frac{r_\varepsilon(\eta^*_i|\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \eta^*_4)}{1 - r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)} \otimes r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4) \wedge \prod_{i\neq j}(\hat{\eta}_i, \hat{\eta}_j)\\
&-\frac{r_\varepsilon(\eta^*_i|\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \eta^*_4)}{r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)} \otimes (1 - r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)) \wedge \prod_{i\neq j}(\hat{\eta}_i, \hat{\eta}_j)\\
&+ \sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{\Delta(\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \hat{\eta}^*_j, \dots, \eta^*_4)_\varepsilon}{\Delta(\eta_0, \dots, \hat{\eta}_i, \dots, \hat{\eta}_j, \dots, \eta_4)}\right) \otimes (1 - r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4))\\
&\wedge r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)\Bigg)
\end{aligned} \tag{3.9}$$

The second part $\partial^{(2)} \circ \tau^3_{1,\varepsilon}(\eta^*_0, \dots, \eta^*_4)$ is

$$\begin{aligned}
= -\frac{1}{3}\sum_{i=0}^{4}(-1)^i\Bigg(&-\frac{r_\varepsilon(\eta^*_i|\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \eta^*_4)}{r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)} \wedge \frac{r_\varepsilon(\eta^*_i|\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \eta^*_4)}{1 - r(\eta_i|\eta_0, \dots, \hat{\eta}_i, \dots, \eta_4)}\\
&\wedge \sum_{\substack{j=0\\ j\neq i}}^{4}\left(\frac{\Delta(\eta^*_0, \dots, \hat{\eta}^*_i, \dots, \hat{\eta}^*_j, \dots, \eta^*_4)_\varepsilon}{\Delta(\eta_0, \dots, \hat{\eta}_i, \dots, \hat{\eta}_j, \dots, \eta_4)}\right)\Bigg)
\end{aligned} \tag{3.10}$$

then we calculate $\frac{b_\varepsilon}{a}$ and $\frac{b_\varepsilon}{1-a}$. i.e., all the values of the form $\frac{r_\varepsilon(\eta^*_0|\eta^*_1,\eta^*_2,\eta^*_3,\eta^*_4)}{r(\eta_0|\eta_1,\eta_2,\eta_3,\eta_4)}$ and $\frac{r_\varepsilon(\eta^*_0|\eta^*_1,\eta^*_2,\eta^*_3,\eta^*_4)}{1-r(\eta_0|\eta_1,\eta_2,\eta_3,\eta_4)}$. By using formula (2.6) we get

$$\frac{r_\varepsilon(\eta^*_0|\eta^*_1, \eta^*_2, \eta^*_3, \eta^*_4)}{r(\eta_0|\eta_1, \eta_2, \eta_3, \eta_4)} = \frac{(\eta^*_0\eta^*_1\eta^*_4)_\varepsilon}{(\eta_0\eta_1\eta_4)} + \frac{(\eta^*_0\eta^*_2\eta^*_3)_\varepsilon}{(\eta_0\eta_2\eta_3)} - \frac{(\eta^*_0\eta^*_2\eta^*_4)_\varepsilon}{(\eta_0\eta_2\eta_4)} - \frac{(\eta^*_0\eta^*_1\eta^*_3)_\varepsilon}{(\eta_0\eta_1\eta_3)}$$

Similarly, we can find this ratio for each value of $i = 0, \dots, 4$. Now use formula (2.6) with the identities (2.1) and (2.3);

$$\frac{r_\varepsilon(\eta^*_0|\eta^*_1, \eta^*_2, \eta^*_3, \eta^*_4)}{1 - r(\eta_0|\eta_1, \eta_2, \eta_3, \eta_4)} = \frac{(\eta^*_0\eta^*_2\eta^*_4)_\varepsilon}{(\eta_0\eta_2\eta_4)} + \frac{(\eta^*_0\eta^*_1\eta^*_3)_\varepsilon}{(\eta_0\eta_1\eta_3)} - \frac{(\eta^*_0\eta^*_3\eta^*_4)_\varepsilon}{(\eta_0\eta_3\eta_4)} - \frac{(\eta^*_0\eta^*_1\eta^*_2)_\varepsilon}{(\eta_0\eta_1\eta_2)}$$

After calculating all these values, expand the sums (3.9) and (3.10) and put all values that we have calculated above. For instance, let us calculate (3.9). In this sum we have a large number of terms, so we group them in a suitable way. First collect all the terms involving $\frac{(\eta_0^*\eta_1^*\eta_2^*)_\varepsilon}{(\eta_0\eta_1\eta_2)} \otimes \cdots$, we find that there are 6 different terms with coefficient -3 involving $\frac{(\eta_0^*\eta_1^*\eta_2^*)_\varepsilon}{(\eta_0\eta_1\eta_2)} \otimes \cdots$

$$-3\frac{(\eta_0^*\eta_1^*\eta_2^*)_\varepsilon}{(\eta_0\eta_1\eta_2)} \otimes \Big((\eta_0\eta_1\eta_3) \wedge (\eta_1\eta_2\eta_3) + (\eta_0\eta_2\eta_4) \wedge (\eta_1\eta_2\eta_3) + (\eta_0\eta_1\eta_4) \wedge (\eta_0\eta_2\eta_4)$$
$$-(\eta_0\eta_1\eta_3) \wedge (\eta_0\eta_2\eta_3) - (\eta_0\eta_1\eta_4) \wedge (\eta_1\eta_2\eta_4) - (\eta_0\eta_2\eta_3) \wedge (\eta_1\eta_2\eta_3)\Big)$$

There are exactly 10 possible terms of $\frac{(\eta_i^*\eta_j^*\eta_k^*)_\varepsilon}{(\eta_i\eta_j\eta_k)}$. Compute all of them individually. We will see that each will have the coefficient -3 that will be cancelled by $-\frac{1}{3}$ in (3.9) and then combine 60 different terms with 6 in a group of the same $\frac{(\eta_i^*\eta_j^*\eta_k^*)_\varepsilon}{(\eta_i\eta_j\eta_k)}$, write in the sum form then we will note that it will be the same as (3.7).

Computation for the second part is relatively easy and direct. We need to put all values of the form $\frac{r_\varepsilon(\eta_0^*|\eta_1^*,\eta_2^*,\eta_3^*,\eta_4^*)}{r(\eta_0|\eta_1,\eta_2,\eta_3,\eta_4)}$ and $\frac{r_\varepsilon(\eta_0^*|\eta_1^*,\eta_2^*,\eta_3^*,\eta_4^*)}{1-r(\eta_0|\eta_1,\eta_2,\eta_3,\eta_4)}$ in (3.10), expand the sums, use $a \wedge a = 0$ modulo 2 torsion. Here we will have simplified result which can be recombined in the sum notation which will be the same as (3.8). □

Theorem 3.7. *The following diagram (4.3a), i.e.,*

$$\begin{array}{ccc} C_6(\mathbb{A}^3_{F[\varepsilon]_2}) & \xrightarrow{\quad d \quad} & C_5(\mathbb{A}^3_{F[\varepsilon]_2}) \\ \downarrow \tau^3_{2,\varepsilon} & & \downarrow \tau^3_{1,\varepsilon} \\ T\mathcal{B}_3(F) & \xrightarrow{\quad \partial_\varepsilon \quad} & (T\mathcal{B}_2(F) \otimes F^\times) \oplus (F \otimes \mathcal{B}_2(F)) \end{array}$$

is commutative i.e., $\tau^3_{2,\varepsilon} \circ \partial_\varepsilon = d \circ \tau^3_{1,\varepsilon}$

Proof. The map $\tau^3_{2,\varepsilon}$ gives 720 terms and due to symmetry (cyclic and inverse) we find 120 different ones (up to the inverse). By definition, we have

$$\tau^3_{2,\varepsilon}(\eta_0^*, \ldots, \eta_5^*) = \frac{2}{45}\mathrm{Alt}_6 \left\langle r_3(\eta_0, \ldots, \eta_5); r_{3,\varepsilon}(\eta_0^*, \ldots, \eta_5^*)\right]_3$$

For convenience, and similar to our previous conventions, we will abbreviate our notation by dropping Δ and commas.

$$\partial_\varepsilon \circ \tau^3_{2\varepsilon}(\eta_0^* \ldots \eta_5^*)$$
$$= \frac{2}{45}\mathrm{Alt}_6 \left\{ \left\langle r_3(\eta_0 \ldots \eta_5); r_{3,\varepsilon}(\eta_0^* \ldots \eta_5^*)\right]_2 \otimes r_3(\eta_0 \ldots \eta_5) + \frac{r_{3,\varepsilon}(\eta_0^* \ldots \eta_5^*)}{r_3(\eta_0 \ldots \eta_5)} \otimes [r_3(\eta_0 \ldots \eta_5)]_2 \right\} \tag{3.11}$$

We need to compute the value of $\frac{r_{3,\varepsilon}(\eta_0^* \ldots \eta_5^*)}{r_3(\eta_0 \ldots \eta_5)}$ which is

$$= \frac{(\eta_0^*\eta_1^*\eta_3^*)_\varepsilon}{(\eta_0\eta_1\eta_3)} + \frac{(\eta_1^*\eta_2^*\eta_4^*)_\varepsilon}{(\eta_1\eta_2\eta_4)} + \frac{(\eta_2^*\eta_0^*\eta_5^*)_\varepsilon}{(\eta_2\eta_0\eta_5)} - \frac{(\eta_0^*\eta_1^*\eta_4^*)_\varepsilon}{(\eta_0\eta_1\eta_4)} - \frac{(\eta_1^*\eta_2^*\eta_5^*)_\varepsilon}{(\eta_1\eta_2\eta_5)} - \frac{(\eta_2^*\eta_0^*\eta_3^*)_\varepsilon}{(\eta_2\eta_0\eta_3)}$$

Formula (3.11) can also be written as

$$=\frac{2}{45}\mathrm{Alt}_6\Big\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes\frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}$$
$$+\left(\frac{(\eta_0^*\eta_1^*\eta_3^*)_\varepsilon}{(\eta_0\eta_1\eta_3)}+\frac{(\eta_1^*\eta_2^*\eta_4^*)_\varepsilon}{(\eta_1\eta_2\eta_4)}+\frac{(\eta_2^*\eta_0^*\eta_5^*)_\varepsilon}{(\eta_2\eta_0\eta_5)}-\frac{(\eta_0^*\eta_1^*\eta_4^*)_\varepsilon}{(\eta_0\eta_1\eta_4)}-\frac{(\eta_1^*\eta_2^*\eta_5^*)_\varepsilon}{(\eta_1\eta_2\eta_5)}-\frac{(\eta_2^*\eta_0^*\eta_3^*)_\varepsilon}{(\eta_2\eta_0\eta_3)}\right)\otimes[r_3(\eta_0\ldots\eta_5)]_2\Big\}$$

We will consider here only the first part of the above relation.

$$\frac{2}{45}\mathrm{Alt}_6\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes\frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}$$

Further,

$$\begin{aligned}
&=\mathrm{Alt}_6\left\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_3)\right\}\\
&+\mathrm{Alt}_6\left\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes(\eta_1\eta_2\eta_4)\right\}\\
&+\mathrm{Alt}_6\left\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes(\eta_2\eta_0\eta_5)\right\}\\
&-\mathrm{Alt}_6\left\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_4)\right\}\\
&-\mathrm{Alt}_6\left\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes(\eta_1\eta_2\eta_5)\right\}\\
&-\mathrm{Alt}_6\left\{\langle r_3(\eta_0\ldots\eta_5);r_{3,\varepsilon}(\eta_0^*\ldots\eta_5^*)]_2\otimes(\eta_2\eta_0\eta_3)\right\}
\end{aligned}\tag{3.12}$$

We use the even cycle $(\eta_0\eta_1\eta_2)(\eta_3\eta_4\eta_5)$ (or $(\eta_0^*\eta_1^*\eta_2^*)(\eta_3^*\eta_4^*\eta_5^*)$) to obtain

$$\begin{aligned}
&\mathrm{Alt}_6\left\{\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5);r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_3)\right\}\\
=&\mathrm{Alt}_6\left\{\langle r_3(\eta_1\eta_2\eta_0\eta_4\eta_5\eta_3);r_{3,\varepsilon}(\eta_1^*\eta_2^*\eta_0^*\eta_4^*\eta_5^*\eta_3^*)]_2\otimes(\eta_1\eta_2\eta_4)\right\}
\end{aligned}$$

We can also use here the symmetry

$$\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5);r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)]_2=\langle r_3(\eta_1\eta_2\eta_0\eta_4\eta_5\eta_3);r_{3,\varepsilon}(\eta_1^*\eta_2^*\eta_0^*\eta_4^*\eta_5^*\eta_3^*)]_2$$

since

$$r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)=r_{3,\varepsilon}(\eta_1^*\eta_2^*\eta_0^*\eta_4^*\eta_5^*\eta_3^*)\quad\text{precisely both have the same factors}$$

and similar for the others as well so that (3.12) will be

$$\begin{aligned}
=\frac{2}{15}\mathrm{Alt}_6\Big\{&\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5);r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_3)\\
&-\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5);r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_4)\Big\}
\end{aligned}$$

If we apply the odd permutation $(\eta_3\eta_4)$ (or $(\eta_3^*\eta_4^*)$), then we have

$$=\frac{2}{15}\cdot 2\mathrm{Alt}_6\left\{\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5);r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_3)\right\}$$

Again apply the odd permutation $(\eta_0\eta_3)$ (or $(\eta_0^*\eta_3^*)$)

$$=\frac{2}{15}\mathrm{Alt}_6\Big\{\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5);r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)]_2\otimes(\eta_0\eta_1\eta_3)$$

$$-\left\langle r_3(\eta_3\eta_1\eta_2\eta_0\eta_4\eta_5); r_{3,\varepsilon}(\eta_3^*\eta_1^*\eta_2^*\eta_0^*\eta_4^*\eta_5^*)\right]_2 \otimes (\eta_3\eta_1\eta_0)\Big\}$$

but up to 2-torsion, which we ignore here, we have $(\eta_0\eta_1\eta_3) = (\eta_3\eta_1\eta_0)$ and then the above will become

$$= \frac{2}{15}\mathrm{Alt}_6\Big\{\Big(\left\langle r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5); r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)\right]_2 - \left\langle r_3(\eta_3\eta_1\eta_2\eta_0\eta_4\eta_5); r_{3,\varepsilon}(\eta_3^*\eta_1^*\eta_2^*\eta_0^*\eta_4^*\eta_5^*)\right]_2\Big) \otimes (\eta_0\eta_1\eta_3)\Big\} \tag{3.13}$$

Recall from the triple-ratio

$$r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5) = \frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}$$

can be expressed as the ratio of two projected cross-ratios.

We will see here that $r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)$ can also be converted into the ratio of two first order cross-ratios.

Let a and b be two projected cross-ratios whose ratio is the triple-ratio

$$r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5) = \frac{(\eta_0\eta_1\eta_3)(\eta_1\eta_2\eta_4)(\eta_2\eta_0\eta_5)}{(\eta_0\eta_1\eta_4)(\eta_1\eta_2\eta_5)(\eta_2\eta_0\eta_3)}$$

then $r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)$ will be written as $\left(\frac{a^*}{b^*}\right)_\varepsilon$. Since we can also write as

$$\mathbf{r}_3(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*) = r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5) + r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)\varepsilon$$

or

$$\mathbf{r}_3(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*) = r_3(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5) + (\mathbf{r}_3(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*))_\varepsilon\,\varepsilon$$

we get

$$r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*) = \left(\frac{(\eta_0^*\eta_1^*\eta_3^*)(\eta_1^*\eta_2^*\eta_4^*)(\eta_2^*\eta_0^*\eta_5^*)}{(\eta_0^*\eta_1^*\eta_4^*)(\eta_1^*\eta_2^*\eta_5^*)(\eta_2^*\eta_0^*\eta_3^*)}\right)_\varepsilon$$

Now it is clear that $r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*)$ can also be written as the ratio or product of two projected cross-ratios. There are exactly three ways to write it (projected by (η_0^* and η_1^*), (η_1^* and η_2^*) and (η_0^* and η_2^*)) but we will use here η_1^* and η_2^*. The last expression can be written as

$$r_{3,\varepsilon}(\eta_0^*\eta_1^*\eta_2^*\eta_3^*\eta_4^*\eta_5^*) = \left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)}{\mathbf{r}(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)}\right)_\varepsilon$$

and (3.13) can be written as

$$= \frac{2}{15}\mathrm{Alt}_6\left\{\left\langle \frac{r(\eta_2|\eta_1\eta_0\eta_5\eta_3)}{r(\eta_1|\eta_0\eta_2\eta_3\eta_4)}; \left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)}{\mathbf{r}(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)}\right)_\varepsilon\right]_2 \otimes (\eta_0\eta_1\eta_3) - \left\langle \frac{r(\eta_2|\eta_1\eta_3\eta_5\eta_0)}{r(\eta_1|\eta_3\eta_2\eta_0\eta_4)}; \left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_3^*\eta_5^*\eta_0^*)}{\mathbf{r}(\eta_1^*|\eta_3^*\eta_2^*\eta_0^*\eta_4^*)}\right)_\varepsilon\right]_2 \otimes (\eta_0\eta_1\eta_3)\right\}$$

Applying five-term relations in $T\mathcal{B}_2(F)$ which are analogous to the one in (2.9).

$$=\frac{2}{15}\mathrm{Alt}_6\Big\{\Big(\Big(\left\langle r(\eta_2|\eta_1\eta_0\eta_5\eta_3); r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)\right]_2 - \left\langle r(\eta_1|\eta_0\eta_2\eta_3\eta_4); r_\varepsilon(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)\right]_2$$

$$-\left\langle\frac{r(\eta_2|\eta_1\eta_5\eta_3\eta_0)}{r(\eta_1|\eta_0\eta_3\eta_4\eta_2)};\left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_5^*\eta_3^*\eta_0^*)}{\mathbf{r}(\eta_1^*|\eta_0^*\eta_3^*\eta_4^*\eta_2^*)}\right)_\varepsilon\right]_2\Bigg)\otimes(\eta_0\eta_1\eta_3)\Bigg\} \tag{3.14}$$

For each individual determinant, e.g., $(\eta_0\eta_1\eta_3)$ will have three terms. First consider the third term of (3.14)

$$\frac{2}{15}\mathrm{Alt}_6\left\{\left\langle\frac{r(\eta_2|\eta_1\eta_5\eta_3\eta_0)}{r(\eta_1|\eta_0\eta_3\eta_4\eta_2)};\left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_5^*\eta_3^*\eta_0^*)}{\mathbf{r}(\eta_1^*|\eta_0^*\eta_3^*\eta_4^*\eta_2^*)}\right)_\varepsilon\right]_2\otimes(\eta_0\eta_1\eta_3)\right\}$$
$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(\eta_0\eta_1\eta_3)(\eta_2\eta_4\eta_5)}\left(\left\langle\frac{\mathbf{r}(\eta_2|\eta_1\eta_5\eta_3\eta_0)}{r(\eta_1|\eta_0\eta_3\eta_4\eta_2)};\left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_5^*\eta_3^*\eta_0^*)}{\mathbf{r}(\eta_1^*|\eta_0^*\eta_3^*\eta_4^*\eta_2^*)}\right)_\varepsilon\right]_2\otimes(\eta_0\eta_1\eta_3)\right)\right\}$$

We need a subgroup in S_6 which fixes $(\eta_0\eta_1\eta_3)$ as a determinant i.e., $(\eta_0\eta_1\eta_3)\sim(\eta_3\eta_1\eta_0)\sim(\eta_3\eta_0\eta_1)\cdots$

Here in this case S_3 permuting $\{\eta_0,\eta_1,\eta_3\}$ and another one permuting $\{\eta_2,\eta_4,\eta_5\}$ i.e., $S_3\times S_3$. Now consider

$$\mathrm{Alt}_{(\eta_0\eta_1\eta_3)(\eta_2\eta_4\eta_5)}\left\{\left\langle\frac{r(\eta_2|\eta_1\eta_5\eta_3\eta_0)}{r(\eta_1|\eta_0\eta_3\eta_4\eta_2)};\left(\frac{\mathbf{r}(\eta_2^*|\eta_1^*\eta_5^*\eta_3^*\eta_0^*)}{\mathbf{r}(\eta_1^*|\eta_0^*\eta_3^*\eta_4^*\eta_2^*)}\right)_\varepsilon\right]_2\otimes(\eta_0\eta_1\eta_3)\right\}$$
$$=\mathrm{Alt}_{(\eta_0\eta_1\eta_3)(\eta_2\eta_4\eta_5)}\left\{\left\langle\frac{(\eta_2\eta_5\eta_3)(\eta_1\eta_0\eta_4)}{(\eta_2\eta_5\eta_0)(\eta_1\eta_3\eta_4)};\left(\frac{(\eta_2^*\eta_5^*\eta_3^*)(\eta_1^*\eta_0^*\eta_4^*)}{(\eta_2^*\eta_5^*\eta_0^*)(\eta_1^*\eta_3^*\eta_4^*)}\right)_\varepsilon\right]_2\otimes(\eta_0\eta_1\eta_3)\right\}$$

By using the odd permutation $(\eta_2\eta_5)$ the above becomes zero.

then (3.14) becomes

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\left(\langle r(\eta_2|\eta_1\eta_0\eta_5\eta_3);r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)]_2-\langle r(\eta_1|\eta_0\eta_2\eta_3\eta_4);r_\varepsilon(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)]_2\right)\otimes(\eta_0\eta_1\eta_3)\right\} \tag{3.15}$$

Consider the first term now,

$$\frac{2}{15}\mathrm{Alt}_6\left\{\langle r(\eta_2|\eta_1\eta_0\eta_5\eta_3);r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)]_2\otimes(\eta_0\eta_1\eta_3)\right\}$$
$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{36}\mathrm{Alt}_{(\eta_0\eta_1\eta_3)(\eta_2\eta_4\eta_5)}\left\{\langle r(\eta_2|\eta_1\eta_0\eta_5\eta_3);r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)]_2\otimes(\eta_0\eta_1\eta_3)\right\}\right\}$$

The permutation $(\eta_0\eta_2\eta_3)$ does not have any role because the ratio is projected by 2. So, it will be reduced to S_3.

$$=\frac{2}{15}\mathrm{Alt}_6\left\{\frac{1}{6}\mathrm{Alt}_{(\eta_2\eta_4\eta_5)}\left\{\langle r(\eta_2|\eta_1\eta_0\eta_5\eta_3);r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)]_2\otimes(\eta_0\eta_1\eta_3)\right\}\right\}$$

Write all possible inner alternation, then

$$=\frac{1}{45}\mathrm{Alt}_6\Bigg\{\Bigg(\langle r(\eta_4|\eta_1\eta_0\eta_2\eta_3);r_\varepsilon(\eta_4^*|\eta_1^*\eta_0^*\eta_2^*\eta_3^*)]_2-\langle r(\eta_2|\eta_1\eta_0\eta_4\eta_3);r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_4^*\eta_3^*)]_2$$
$$+\langle r(\eta_5|\eta_1\eta_0\eta_4\eta_3);r_\varepsilon(\eta_5^*|\eta_1^*\eta_0^*\eta_4^*\eta_3^*)]_2-\langle r(\eta_4|\eta_1\eta_0\eta_5\eta_3);r_\varepsilon(\eta_4^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)]_2$$
$$+\langle r(\eta_2|\eta_1\eta_0\eta_5\eta_3);r_\varepsilon(\eta_2^*|\eta_1^*\eta_0^*\eta_5^*\eta_3^*)]_2-\langle r(\eta_5|\eta_1\eta_0\eta_2\eta_3);r_\varepsilon(\eta_5^*|\eta_1^*\eta_0^*\eta_2^*\eta_3^*)]_2\Bigg)\otimes(\eta_0\eta_1\eta_3)\Bigg\}$$

Now we can use projected five-term relation in $T\mathcal{B}_2(F)$ here,

$$= \frac{1}{45}\text{Alt}_6\Bigg\{\Bigg(\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 - \langle r(\eta_1|\eta_0\eta_2\eta_3\eta_4); r_\varepsilon(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)]_2$$
$$- \langle r(\eta_3|\eta_0\eta_1\eta_2\eta_4); r_\varepsilon(\eta_3^*|\eta_0^*\eta_1^*\eta_2^*\eta_4^*)]_2 + \langle r(\eta_0|\eta_1\eta_4\eta_3\eta_5); r_\varepsilon(\eta_0^*|\eta_1^*\eta_4^*\eta_3^*\eta_5^*)]_2$$
$$- \langle r(\eta_1|\eta_0\eta_4\eta_3\eta_5); r_\varepsilon(\eta_1^*|\eta_0^*\eta_4^*\eta_3^*\eta_5^*)]_2 - \langle r(\eta_3|\eta_0\eta_1\eta_4\eta_5); r_\varepsilon(\eta_3^*|\eta_0^*\eta_1^*\eta_4^*\eta_5^*)]_2$$
$$+ \langle r(\eta_0|\eta_1\eta_5\eta_3\eta_2); r_\varepsilon(\eta_0^*|\eta_1^*\eta_5^*\eta_3^*\eta_2^*)]_2 - \langle r(\eta_1|\eta_0\eta_5\eta_3\eta_2); r_\varepsilon(\eta_1^*|\eta_0^*\eta_5^*\eta_3^*\eta_2^*)]_2$$
$$- \langle r(\eta_3|\eta_0\eta_1\eta_5\eta_2); r_\varepsilon(\eta_3^*|\eta_0^*\eta_1^*\eta_5^*\eta_2^*)]_2\Bigg) \otimes (\eta_0\eta_1\eta_3)\Bigg\}$$

Use the cycle $(\eta_0\eta_1\eta_3)(\eta_2\eta_4\eta_5)$ then we get

$$= \frac{1}{45} \cdot 9\text{Alt}_6\left\{\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_3)\right\} \tag{3.16}$$

The second term of the relation (3.15) can also be written as

$$\frac{1}{45} \cdot -6\text{Alt}_6\left\{\langle r(\eta_1|\eta_0\eta_2\eta_3\eta_4); r_\varepsilon(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_3)\right\}$$

(3.16) can be combined with the above so we get

$$= \frac{1}{45}\text{Alt}_6\left\{\left(9\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 - 6\langle r(\eta_1|\eta_0\eta_2\eta_3\eta_4); r_\varepsilon(\eta_1^*|\eta_0^*\eta_2^*\eta_3^*\eta_4^*)]_2\right) \otimes (\eta_0\eta_1\eta_3)\right\} \tag{3.17}$$

Use the permutation $(\eta_0\eta_1\eta_3)(\eta_2\eta_4\eta_5)$ to get

$$= \frac{1}{3}\text{Alt}_6\left\{\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_3)\right\}$$

The Bloch group $\mathcal{B}_2(F)$ also holds the five-term relation, thus we write the following:

$$= \frac{1}{3}\text{Alt}_6\left\{\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_3) + \frac{(\eta_0^*\eta_1^*\eta_3^*)_\varepsilon}{(\eta_0\eta_1\eta_3)} \otimes [r(\eta_0|\eta_1\eta_2\eta_3\eta_4)]_2\right\} \tag{3.18}$$

Now go to the other side. Map $\tau_{1,\varepsilon}^3$ can also be written in the alternation sum form

$$\tau_{1,\varepsilon}^3(\eta_0^*\dots\eta_4^*) = \frac{1}{3}\text{Alt}\{\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_2)$$
$$+\frac{(\eta_0^*\eta_1^*\eta_2^*)_\varepsilon}{(\eta_0\eta_1\eta_2)} \otimes [r(\eta_0|\eta_1\eta_2\eta_3\eta_4)]_2\}$$

Compute $\tau_{1,\varepsilon}^3 \circ d(\eta_0^*\dots\eta_5^*)$ and apply cycle $(\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5)$ for d and then expand Alt_5 from the definition of $\tau_{1,\varepsilon}^3$;

$$\tau_{1\varepsilon}^3 \circ d(\eta_0^*\dots\eta_5^*) = \frac{1}{3}\text{Alt}_6\{\langle r(\eta_0|\eta_1\eta_2\eta_3\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_2^*\eta_3^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_2)$$
$$+\frac{(\eta_0^*\eta_1^*\eta_2^*)_\varepsilon}{(\eta_0\eta_1\eta_2)} \otimes [r(\eta_0|\eta_1\eta_2\eta_3\eta_4)]_2\}$$

Use the odd permutation $(\eta_2\eta_3)$, then

$$= -\frac{1}{3}\text{Alt}_6\{\langle r(\eta_0|\eta_1\eta_3\eta_2\eta_4); r_\varepsilon(\eta_0^*|\eta_1^*\eta_3^*\eta_2^*\eta_4^*)]_2 \otimes (\eta_0\eta_1\eta_3) + \frac{(\eta_0^*\eta_1^*\eta_3^*)_\varepsilon}{(\eta_0\eta_1\eta_3)} \otimes [r(\eta_0|\eta_1\eta_3\eta_2\eta_4)]_2\}$$

Finally use the two-term relation in $T\mathcal{B}_2(F)$ and the Bloch group $\mathcal{B}_2(F)$ to get the required sign. The final answer will be the same as (3.18) □

There are some more related results.

Proposition 3.8. *The map* $C_5(\mathbb{A}^4_{F[\varepsilon]_2}) \xrightarrow{d'} C_4(\mathbb{A}^3_{F[\varepsilon]_2}) \xrightarrow{\tau^3_{0,\varepsilon}} \left(F \otimes \bigwedge^2 F^\times\right) \oplus \left(\bigwedge^3 F\right)$ *is zero.*

Proof. The proof of this is obtained directly by calculation. Let $(\eta_0^*, \ldots, \eta_4^*) \in C_5(\mathbb{A}^4_{F[\varepsilon]_2})$ where

$$\eta_i^* = \begin{pmatrix} a + a_\varepsilon\varepsilon \\ b + b_\varepsilon\varepsilon \\ c + c_\varepsilon\varepsilon \\ d + d_\varepsilon\varepsilon \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} a_\varepsilon \\ b_\varepsilon \\ c_\varepsilon \\ d_\varepsilon\varepsilon \end{pmatrix} = \eta_i + \eta_{i\varepsilon}\varepsilon$$

Let ω be the volume formed in four-dimensional vector space, and $\Delta(\eta_i, \cdot, \cdot, \cdot)$ be the volume form in $V_4/\langle\eta_i\rangle$.

$$\tau^3_{0,\varepsilon} \circ d'(\eta_0^*, \ldots, \eta_4^*)$$
$$=\tau^3_{0,\varepsilon}\left(\sum_{i=0}^{4}(-1)^i(\eta_i^*|\eta_0^*, \ldots, \hat{\eta}_i^*, \ldots, \eta_4^*)\right)$$

Consider the first coordinate of the map first

$$=\widetilde{\text{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\Big(\frac{\Delta(\eta_0^*, \ldots, \hat{\eta}_i^*, \ldots, \eta_3^*, \eta_4^*)_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_i, \ldots, \eta_3, \eta_4)} \otimes \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+1}, \ldots, \eta_3, \eta_4)}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+2}, \ldots, \eta_3, \eta_4)} \wedge \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+3}, \ldots, \eta_3, \eta_4)}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+2}, \ldots, \eta_3, \eta_4)}\Big) \quad i \mod 4\right) \tag{3.19}$$

First, we expand inner sum that gives us 12 different terms after simplification. By applying alternation sum, we get 60 terms and there is direct cancellation which leads to zero. Now consider the second coordinate, which gives us

$$\widetilde{\text{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i \bigwedge_{\substack{j=0 \\ j\neq i}}^{3} \frac{\Delta(\eta_0^*, \ldots, \hat{\eta}_j^*, \ldots, \eta_3^*, \eta_4^*)_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_j, \ldots, \eta_3, \eta_4)}\right)$$

Again if we expand the inner sum, then we get only four different terms, but after the application of alternation we get zero. □

As an analogy of Proposition 3.8 in higher weight, we present the following result.

Proposition 3.9. *The map* $C_{n+2}(\mathbb{A}^{n+1}_{F[\varepsilon]_2}) \xrightarrow{d'} C_{n+1}(\mathbb{A}^{n}_{F[\varepsilon]_2}) \xrightarrow{\tau^n_{0,\varepsilon}} \left(F \otimes \bigwedge^{n-1} F^\times\right) \oplus (\bigwedge^n F)$ *is zero, where*

$$\begin{aligned}
&\tau^n_{0,\varepsilon}(\eta^*_0, \ldots, \eta^*_n)\\
&= \sum_{i=0}^{n}(-1)^i\Bigg(\frac{\Delta(\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_n)_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_i, \ldots, \eta_n)} \otimes \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+1}, \ldots, \eta_n)}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+2}, \ldots, \eta_n)}\\
&\qquad \wedge \cdots \wedge \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+(n-1)}, \ldots, \eta_n)}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+n}, \ldots, \eta_n)}\Bigg) + \Bigg(\bigwedge_{\substack{j=0\\ j\neq i}}^{n} \frac{\Delta(\eta^*_0, \ldots, \hat{\eta}^*_j, \ldots, \eta^*_n)_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_j, \ldots, \eta_n)}\Bigg),\\
&\qquad\qquad i \mod (n+1)
\end{aligned}$$

Proof. Let $(\eta^*_0, \ldots, \eta^*_{n+1}) \in C_{n+2}(\mathbb{A}^{n+1}_{F[\varepsilon]_2})$. We have

$$\tau^n_{0,\varepsilon} \circ d'(\eta^*_0, \ldots, \eta^*_{n+1}) = \tau^n_{0,\varepsilon}\left(\sum_{i=0}^{n}(-1)^i(\eta^*_i|\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_{n+1})\right)$$

Now use the definition of alternation to represent this sum then we have

$$\begin{aligned}
&\tau^n_{0,\varepsilon} \circ d'(\eta^*_0, \ldots, \eta^*_{n+1})\\
=&\widetilde{\mathrm{Alt}}_{(0\cdots n+1)}\Bigg\{\sum_{i=0}^{n}(-1)^i\Bigg(\Big(\frac{\Delta(\eta^*_0, \ldots, \hat{\eta}^*_i, \ldots, \eta^*_n, \eta^*_{n+1})_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_i, \ldots, \eta_n, \eta_{n+1})} \otimes \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+1}, \ldots, \eta_n, \eta_{n+1})}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+2}, \ldots, \eta_n, \eta_{n+1})}\\
&\wedge \cdots \wedge \frac{\Delta(\eta_0, \ldots, \hat{\eta}_{i+(n-1)}, \ldots, \eta_n, \eta_{n+1})}{\Delta(\eta_0, \ldots, \hat{\eta}_{i+n}, \ldots, \eta_n, \eta_{n+1})}\Big) + \Big(\bigwedge_{\substack{j=0\\ j\neq i}}^{n} \frac{\Delta(\eta^*_0, \ldots, \hat{\eta}^*_j, \ldots, \eta^*_n, \eta^*_{n+1})_\varepsilon}{\Delta(\eta_0, \ldots, \hat{\eta}_j, \ldots, \eta_n, \eta_{n+1})}\Big)\Bigg),\\
&\qquad\qquad i \mod n+1\Bigg\}
\end{aligned} \tag{3.20}$$

Expanding the inner sum gives us $n + 1$ number of terms. Expand again by using the properties of wedge that gives $n(n + 1)$ terms. Applying the alternation sum on that, gives us $n(n + 1)(n + 2)$ terms, so there are $n + 2$ sets each consisting $n(n + 1)$ terms and each term in $n(n + 1)$ term has $n + 1$ sets of n terms which cancel off set by set.

Now expand the inner sum in the second term of (3.20) that gives $n + 1$ terms and then apply the alternation sum which gives $n+2$ sets of $n+1$ terms, we now find cancellation in the expansion of sum accordingly, which gives a zero as well. □

4. Conclusion

Many studies have been done on Scissor's congruence and Bloch's groups. Bringing geometry of configurations in Bloch's and Goncharov's groups plays a vital role in proving Zagier's conjucutre for weights $n = 2, 3$. In this article, we introduced the tangent to Goncharov's complex and view them by means of geometric configurations. This leads to the idea at the higher orders of tangent groups.

Theorem 3.6 proves the commutativity of the right hand side square of the diagram (4.3a) and Theorem 3.7 shows the commutativity of the left hand square of the diagram (4.3a).

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Conflict of interest

The author declares no conflict of interest in this paper.

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