



Research article

Integral type contractions of soft set-valued maps with application to neutral differential equations

Mohammed Shehu Shagari* and Akbar Azam

Department of Mathematics, COMSATS University, Chak Shahzad, Islamabad, 44000, Pakistan

* **Correspondence:** Email: ssmohammed@abu.edu.ng.

Abstract: We establish e -soft fixed point results for soft set-valued maps under some integral contractive conditions on a complete metric space. Starting from Branciari integral contraction, the presented results are soft set extensions of many existing results on point-to-point and point-to-set mappings. In particular, the established idea herein improves the recently introduced concepts of soft set-valued maps. Moreover, examples and applications to nonlinear neutral differential equations are provided to support the usability of the derived results.

Keywords: fuzzy set; soft set; soft set-valued map; E -soft fixed point; neutral delay differential equation

Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Fixed point theory is derived from the investigation of the solution of some types of differential equations using the method of successive approximation. Indeed, Banach contraction principle [7] is a reformulation of the successive approximation techniques originally used by some earlier mathematicians, namely Cauchy, Liouville, Picard, Lipschitz and so on. In recent time, fixed point techniques have been used extensively to find solutions of both linear and nonlinear equations arising in many areas of sciences. The original idea of fixed point theorem due to Banach has been refined in different directions. See, for example, Rhoades [25] or Taskovic [27].

In 2002, Branciari [11] analyzed the existence of fixed point of a single-valued mapping defined on a complete metric space satisfying general contractive condition of integral type. The condition is analogous to Banach-Caccioppoli's Principle. With different method of proof, however, the result in [11] reduces to the Banach contraction theorem if the integrand of the contractive condition in [11] is set identically to one. The first extension of Branciari's result was presented by Rhoades [25]. Thereafter, more than a handful of researchers extended the result of Branciari and obtained fixed

point theorems for single-valued and multi-valued mappings involving several other contractive conditions of integral type in various spaces, such as metric spaces, modular spaces, fuzzy metric spaces, cone metric spaces, symmetric spaces, Hausdorff topological spaces and so on (see, for example, [1, 2, 9, 18, 23, 25]). On this development, Jachymski [18] argued that most available contractive conditions of integral type are direct analogues of classical ones and hence provided a new contractive condition which does not coincide with the earlier results. By using the notion of modular spaces, Beygmohammadi and Razani [9] proved two fixed point theorems for mappings satisfying a general contractive condition of integral type in modular spaces, thereby generalizing fixed point theorem for a quasicontraction mapping given by Khamsi [20]. Murthy and Kumam [23] introduced the concept of fuzzy metric spaces and presented common fixed point of self maps satisfying an integral type contractive condition which improved the results of Branciari [11] and Rhoades [25]. Altun and Torkoglu [2] obtained fixed point theorem for weakly compatible mappings satisfying contractive condition of integral type in Hausdorff d -complete topological spaces by neglecting some possible assumptions in the result of Hicks and Rhoades [17]. On the other hand, Weiss [29] and Butnairu [12] originated the study of fixed point of fuzzy mappings. Thereafter, the idea of fuzzy set-valued mappings was introduced by Heilpern [16], which is a fuzzy generalization of Banach fixed point theorem and Nadler's [24] fixed point theorem for multivalued mappings. After wards, several authors [3–6, 10] studied fixed points of fuzzy mappings satisfying various contractive conditions.

Along the lane, the applications of mathematics witnessed tremendous developments as a result of the introduction of soft set theory by Molodstov [22]. The method of handling problems in classical mathematics is in the opposite of the technique of soft set theory. In conventional mathematics, to describe any system or object, we first construct its mathematical model and then attempt to obtain the exact solution. If the exact solution is too complicated, then we define the notion of approximate solution. On the other hand, in soft set theory, the initial description of an object takes an approximate nature with no restriction, and the notion of exact solution is not essential. In other words, to describe an object in soft set theory, any convenient parametrization tools which may be words/sentences, numbers, mappings/functions, to mention a few, may be used. Thereby, making the theory more easier and flexible in terms of applications in every day life. In [22], Moldstov highlighted several directions for the applications of soft set, such as smoothness of functions, game theory, Riemann-integration, operation research, probability and so on. At present, researchers have been extending the concepts of soft set into different fields of studies. For example, see [13, 15, 26] and references therein.

Not long ago, Shagari and Azam [28] studied the concept of soft set-valued maps and introduced the notions of e -soft fixed points and E -soft fixed point of maps whose range set is a family of soft sets. Consequently, they presented analogues of fixed point theorems of Nadler's and Edelstein's type. Motivated by the ideas of [28], the main aim of this paper is to establish e -soft fixed point theorem of soft set-valued map defined on a complete metric space under some generalized contractive conditions of integral type. In particular, our result extends the main result of Shagari and Azam [28] into integral contractive condition. Consequently, several other well-known results are deduced as corollaries. Moreover, as an application, our main result is used to investigate existence conditions for a solution of nonlinear neutral differential equation.

2. Preliminary

In this section, we present preliminary concepts which are relevant to the aims of this paper. These basic concepts are recalled from [4, 16, 22, 28].

Throughout, \mathbb{N} and \mathbb{R} represent the set of natural and real numbers, respectively. Let (\mathfrak{X}, ϱ) be a metric space and \mathfrak{X}^* be the set of all nonempty closed and bounded subsets of \mathfrak{X} . For $\Theta, \Phi \in \mathfrak{X}^*$, the Hausdorff metric H on \mathfrak{X}^* induced by ϱ is defined as:

$$H(\Theta, \Phi) = \max\{\sup_{l \in \Theta} \varrho(l, \Phi), \sup_{k \in \Phi} \varrho(\Theta, k)\},$$

where $\varrho(l, \Theta) = \inf_{k \in \Theta} \varrho(l, k)$.

The theory of fuzzy set was launched by Zadeh [30]. The notion provides suitable framework for representing and analyzing vague ideas by allowing partial memberships. A fuzzy set Θ in a universal set \mathfrak{X} is a function with domain \mathfrak{X} and values in $[0, 1] = I$. For $l \in \mathfrak{X}$, the function value $\Theta(l)$ is called the membership value of $l \in \mathfrak{X}$ in the fuzzy set Θ . The α -level set of Θ , denoted by $[\Theta]_\alpha$, is defined as follows:

$$[\Theta]_\alpha = \{l : \Theta(l) \geq \alpha\}, \text{ if } \alpha \in (0, 1],$$

and $[\Theta]_0 = \{l : \Theta(l) > \alpha\}$, is called the strong α -level set.

Example 2.1. A dealer of certain domestic animals wants to know the different determinants of his customers' satisfaction. An indicated quality shown by a fraction of his customers is 'maturity'. If $\mathfrak{X} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is the set of all domestic animals, then the fuzzy set $\Theta = \{\text{some animals are matured}\}$ can be seen graphically in Figure 1.

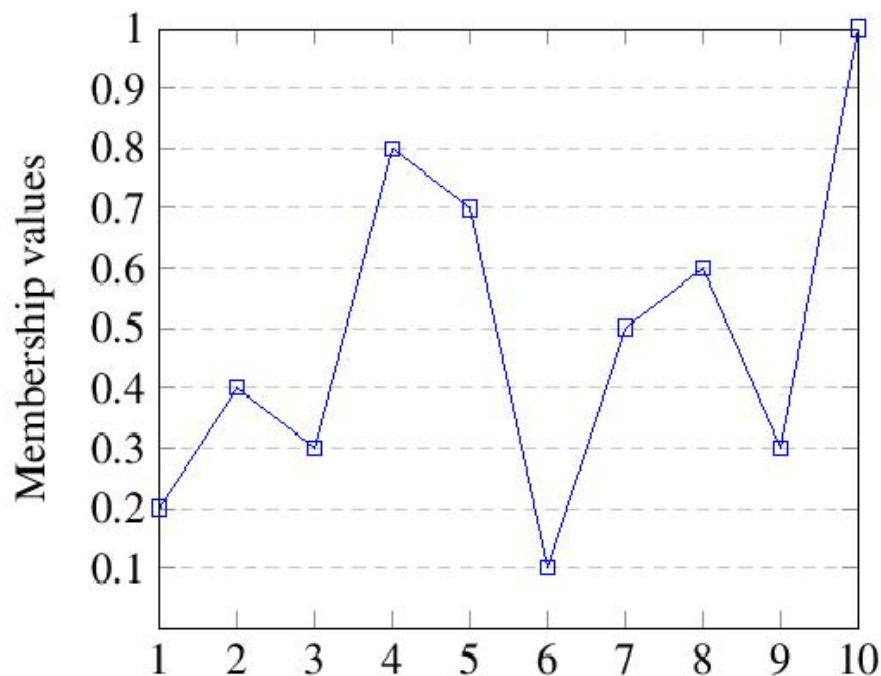


Figure 1. Graphical representation of the fuzzy set in Example 2.1.

Notice that from Figure 1, the α -level set and strong α -level set for $\alpha = 0.5$, are respectively

$$[\Theta]_\alpha = \{4, 5, 7, 8, 10\} \quad \text{and} \quad [\Theta]_\alpha^* = \{4, 5, 8, 10\}.$$

Let $F_{\mathfrak{X}}$ be the collection of all fuzzy sets in a reference set \mathfrak{X} .

Definition 2.1. Let \mathfrak{X} be an arbitrary set and \mathfrak{Y} a metric space. A mapping $\Upsilon : \mathfrak{X} \rightarrow F_{\mathfrak{Y}}$ is called a fuzzy mapping. A fuzzy mapping Υ is a fuzzy subset of $\mathfrak{X} \times \mathfrak{Y}$ with the grade of membership $\Upsilon(l)(k)$ of k in $\Upsilon(l)$. A point $u \in \mathfrak{X}$ is a fuzzy fixed point of Υ if $u \in [\Upsilon u]_\alpha$, for some $\alpha \in [0, 1]$.

Soft set theory was introduced by Molodstov [22]. The concept provides a parametrization point of view for modeling uncertainty and soft computing. Let \mathfrak{X} be the universal set and E be the set of all parameters related to the elements of \mathfrak{X} . In this case, each parameter is a word or sentence. Denote by $P(\mathfrak{X})$, the power set of \mathfrak{X} .

Definition 2.2. [22] A pair (F, Θ) is called a soft set over \mathfrak{X} , where $\Theta \subseteq E$ and F is a set-valued mapping $F : \Theta \rightarrow P(\mathfrak{X})$. In this way, a soft set over \mathfrak{X} is a parameterized family of subsets of \mathfrak{X} .

Example 2.2. Suppose a soft (F, E) gives the section(s) of research papers that is/are more interested to reviewers. Let the universal set of reviewers under consideration be given by

$$\mathfrak{X} = \{l_1, l_2, l_3, l_4, l_5\}$$

and the set of all parameters be represented by

$$E = \{e_1, e_2, e_3, e_4\} = \{\text{title, abstract, introduction, results}\}.$$

Define $F : E \rightarrow P(\mathfrak{X})$ by $F(e_1) = \{l_1, l_2\}$, $F(e_2) = \{l_2, l_4\}$, $F(e_3) = \{l_5\}$, $F(e_4) = \{l_1, l_2, l_3\}$. So, the soft set (F, E) is a parameterized family $\{F(e_i) : i = 1, 2, 3, 4\}$ of $P(\mathfrak{X})$.

Denote by $[P(\mathfrak{X})]^E$, the family of soft set over \mathfrak{X} under E . Consider two soft sets (F, Θ) and (G, Φ) , $(a, b) \in \Theta \times \Phi$. Assume that $F(a), G(b) \in \mathfrak{X}^*$. For $\epsilon > 0$, define $N^\epsilon(\epsilon, F(a))$, $S_{E\mathfrak{X}}^{(a,b)}(F, G)$ and $E_{(F_a, G_b)}^\epsilon$ as follows:

$$N^\epsilon(\epsilon, F(a)) = \{l \in \mathfrak{X} : \varrho(l, k) < \epsilon, \text{ for some } k \in F(a)\}$$

$$E_{(F_a, G_b)}^\epsilon = \{\epsilon > 0 : F(a) \subseteq N^\epsilon(\epsilon, G(b)), \quad G(b) \subseteq N^\epsilon(\epsilon, F(a))\},$$

$$S_{E\mathfrak{X}}^{(a,b)}(F, G) = \inf_{(F_a, G_b)} E_{(F_a, G_b)}^\epsilon.$$

Define a distance function $S_{E\mathfrak{X}}^\infty : [P(\mathfrak{X})]^E \times [P(\mathfrak{X})]^E \rightarrow \mathbb{R}$ by

$$S_{E\mathfrak{X}}^\infty(F, G) = \sup_{(a,b) \in \overline{\Theta} \times \overline{\Phi}} S_{E\mathfrak{X}}^{(a,b)}(F, G), \text{ where}$$

$$\overline{\Theta} \times \overline{\Phi} = \{(a, b) \in \Theta \times \Phi : F(a), G(b) \in \mathfrak{X}^*\}.$$

Definition 2.3. [28] A mapping $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ is called a soft set-valued map. A point $u \in \mathfrak{X}$ is called an e -soft fixed point of Υ if $u \in (\Upsilon u)(a(u))$, for some $a(u) \in E$. This is also written as $u \in \Upsilon u$, for short. If $Dom \Upsilon l = E$, and $u \in (\Upsilon u)(e)$ for all $e \in E$, then u is said to be an E -soft fixed point of Υ . We shall denote the set of all E -soft fixed point of Υ by $E_{Fix(\Upsilon)}$.

Notice that if $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ is a soft set-valued map, then $\Upsilon l : E \rightarrow P(\mathfrak{X})$ is a soft set, for all $l \in \mathfrak{X}$.

Example 2.3. Let $\mathfrak{X} = \mathbb{R}$ and $E = \mathbb{N}$. Define $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ by

$$(\Upsilon l)(e) = \left\{ \frac{l}{1+l}, \frac{l}{2+l} \right\},$$

for all $l \in \mathfrak{X}$ and $e \in E$. Then Υ is a soft set-valued map.

Example 2.4. Let $\mathfrak{X} = \mathbb{R}$ and $E = \{1, 3, 10\}$. Define $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ by

$$\Upsilon l := \left\{ (1, \{l\}), \left(3, \left\{ \frac{l}{2} \right\} \right), (10, \{l^2, -6l\}) \right\},$$

for all $l \in \mathfrak{X}$. Then Υ is a soft set-valued map.

Example 2.5. Let $\mathfrak{X} = \{a, b, c, p, q, r\}$ and $E = \{1, 2, 3\}$. Define $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ by

$$(\Upsilon l)(e) = \begin{cases} \{a, b, r\}, & \text{if } e = 1 \\ \{b, p\}, & \text{if } e = 2 \\ \{a, b, c, q\}, & \text{if } e = 3. \end{cases}$$

Then Υ is a soft set-valued map. Notice that $b \in (\Upsilon b)(e)$, for all $e \in E$ and $Dom \Upsilon b = E$. Hence, the set of all e -soft fixed point of Υ is given by $E_{Fix(\Upsilon)} = \{b\}$. The soft set-valued in Example 2 can also be seen graphically in Figure 2.

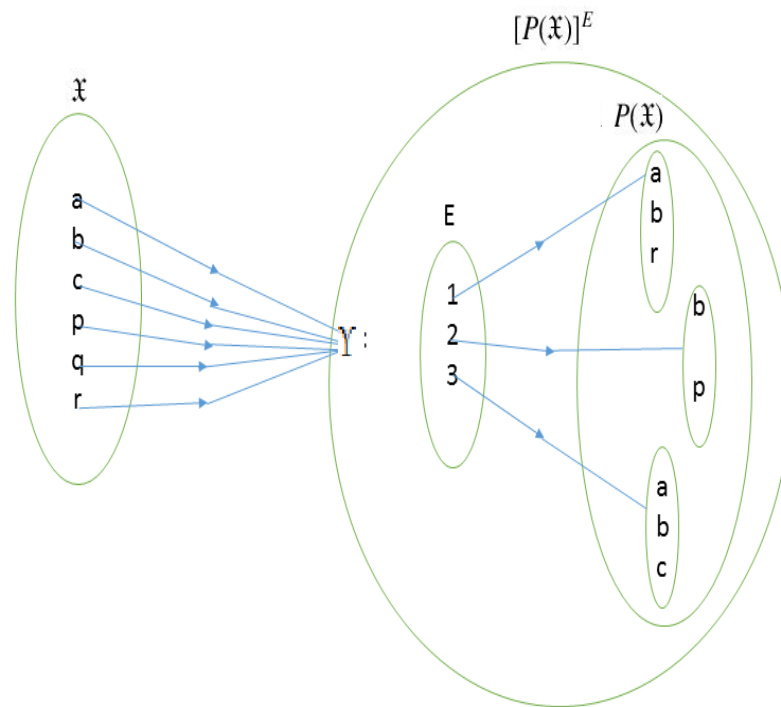


Figure 2. Graphical representation of the soft set-valued map in the Example 2.5.

Definition 2.4. Denote by Ψ , the class of functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following conditions:

- (i) φ is Lebesgue integrable, summable on each compact subset of $[0, \infty)$,
- (ii) φ is nonnegative and such that for each $\eta > 0$, $\int_0^\eta \varphi(t) dt > 0$.

3. Main results

In this section, we analyze the existence of e -soft fixed point of soft set-valued map defined on complete metric spaces satisfying some generalized contractive inequalities of integral type.

Theorem 3.1. Let (\mathfrak{X}, ϱ) be a complete metric space, E be the parameter set and $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ be a soft set-valued map. Let $\varphi \in \Psi$ and $\tau : [0, \infty) \rightarrow (0, 1)$ be a decreasing function. Suppose that for $l \in \mathfrak{X}$, there exists $a(l) \in \text{Dom} \Upsilon l$ such that $(\Upsilon l)(a(l)) \in \mathfrak{X}^*$. If for all $l, k \in \mathfrak{X}$,

$$\int_0^{S_{E\mathfrak{X}}^{(e(l), e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \leq \tau(\varrho(l, k)) \left(\int_0^{\varrho(l, k)} \varphi(t) \varrho t \right) \quad (3.1)$$

then there exists $u \in \mathfrak{X}$ such that $\{u\} \subset \Upsilon u$.

Proof. Let $l \in \mathfrak{X}$ be arbitrary and rename it as $l := l_0$, then by hypothesis, $(\Upsilon l_0)(a(l_0))$ is a nonempty closed and bounded subset of \mathfrak{X} . Hence, there exists some $l_1 \in \mathfrak{X}$ such that $l_1 \in (\Upsilon l_0)(e(l_0))$, $l_0 \neq l_1$. Setting $l = l_0$ and $k = l_1$ in (3.10), we have

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(e(l_0), e(l_1))}(\Upsilon l_0, \Upsilon l_1)} \varphi(t) \varrho t &\leq \tau(\varrho(l_0, l_1)) \left(\int_0^{\varrho(l_0, l_1)} \psi(t) dt \right) \\ &< \int_0^{\varrho(l_0, l_1)} \varphi(t) \varrho t. \end{aligned} \quad (3.2)$$

For this $l_1 \in \mathfrak{X}$, by hypothesis, $(\Upsilon l_1)(a(l_1)) \in \mathfrak{X}^*$. Therefore, we can find some $l_2 \in \mathfrak{X}$ such that $l_2 \in (\Upsilon l_1)(a(l_1))$, $l_1 \neq l_2$. Notice that from (3.10) and (3.2), we have

$$\begin{aligned} \int_0^{\varrho(l_1, l_2)} \varphi(t) \varrho t &\leq \int_0^{S_{E\mathfrak{X}}^{(e(l_0), e(l_1))}(\Upsilon l_0, \Upsilon l_1)} \varphi(t) \varrho t \\ &< \int_0^{\varrho(l_0, l_1)} \varphi(t) \varrho t, \end{aligned}$$

which implies

$$\varrho(l_1, l_2) < \varrho(l_0, l_1).$$

Continuing in this fashion, we obtain a sequence $\{l_j\}_{j \in \mathbb{N}}$ such that $l_{j+1} \in (\Upsilon l_j)(a(l_j))$ and

$$0 < \varrho(l_j, l_{j+1}) < \varrho(l_{j-1}, l_j) < \cdots < \varrho(l_0, l_1).$$

This shows that the sequence $\{\varrho(l_j, l_{j+1})\}_{j \in \mathbb{N}}$ is decreasing and bounded below, and thus converges to its infimum, say m . Hence, $m \leq \varrho(l_j, l_{j+1})$, for all $j \in \mathbb{N}$. Suppose $m > 0$, then from (3.10), and the decreasingness of τ , we have

$$\begin{aligned} \int_0^{\varrho(l_j, l_{j+1})} \varphi(t) \varrho t &\leq \int_0^{S_{E\mathfrak{X}}^{(e(l_{j-1}), e(l_j))}(\Upsilon l_{j-1}, \Upsilon l_j)} \varphi(t) \varrho t \\ &\leq \tau(\varrho(l_{j-1}, l_j)) \left(\int_0^{\varrho(l_{j-1}, l_j)} \varphi(t) \varrho t \right) \end{aligned}$$

$$\leq \tau(m) \left(\int_0^{\varrho(l_{j-1}, l_j)} \varphi(t) \varrho t \right). \quad (3.3)$$

Taking limit in (3.3) as $j \rightarrow \infty$, yields

$$\int_0^m \varphi(t) \varrho t \leq \tau(m) \left(\int_0^m \varphi(t) \varrho t \right) < \int_0^m \varphi(t) \varrho t,$$

which is a contradiction. So $m = 0$. Next, we shall show that $\{l_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence. Assume contrary, therefore, there exists $\eta > 0$ and infinitely many pairs (l_β, l_k) with $\varrho(l_\beta, l_k) \geq \eta$. The subsequence of pairs $\{(l_{\beta_p}, l_{k_p})\}_{p \in \mathbb{N}}$, where $l_p < k_p$ for all $p \in \mathbb{N}$ is chosen to satisfy the condition:

$$\varrho(l_{\beta_p}, l_{k_p}) \geq \eta, \quad \varrho(l_{\beta_p}, l_t) < \eta, \quad \text{for all } t \in \{l_p + 2, \dots, k_p - 1\}.$$

It follows that

$$\begin{aligned} \eta \leq \varrho(l_{\beta_p}, l_{k_p}) &\leq \varrho(l_{\beta_p}, l_{\beta_{p-1}}) + \varrho(l_{\beta_{p-1}}, l_{k_p}) \\ &< \eta + \varrho(l_{\beta_p}, l_{\beta_{p-1}}). \end{aligned} \quad (3.4)$$

As $p \rightarrow \infty$ in (3.4), we have

$$\eta \leq \lim_{p \rightarrow \infty} \varrho(l_{\beta_p}, l_{k_p}) \leq \eta,$$

which implies $\varrho(l_{\beta_p}, l_{k_p}) \rightarrow \eta$ as $p \rightarrow \infty$. Now, considering

$$\begin{aligned} \varrho(l_{\beta_p}, l_{k_p}) &\leq \varrho(l_{\beta_p}, l_{\beta_{p-1}}) \\ &\quad + \varrho(l_{\beta_{p-1}}, l_{k_{p-1}}) + \varrho(l_{k_{p-1}}, l_{k_p}), \end{aligned}$$

we conclude that

$$\lim_{p \rightarrow \infty} \varrho(l_{\beta_p}, l_{k_p}) = \lim_{p \rightarrow \infty} \varrho(l_{\beta_{p-1}}, l_{k_{p-1}}) = \eta.$$

Therefore, there exists $p_0 \in \mathbb{N}$ such that for all $p > p_0$,

$$\varrho(l_{\beta_{p-1}}, l_{k_{p-1}}) \geq \frac{\eta}{3}, \quad \text{which implies}$$

$\tau(\varrho(l_{\beta_{p-1}}, l_{k_{p-1}})) \leq \tau\left(\frac{\eta}{3}\right)$ for $p > p_0$. Now, from (3.10), we have

$$\begin{aligned} \int_0^{\varrho(l_{\beta_p}, l_{k_p})} \varphi(t) \varrho t &\leq \int_0^{S_{EX}^{(\varrho(l_{\beta_{p-1}}, l_{k_{p-1}}))}(\Upsilon l_{p-1}, \Upsilon l_{k_{p-1}})} \varphi(t) \varrho t \\ &\leq \tau(\varrho(l_{\beta_{p-1}}, l_{k_{p-1}})) \left(\int_0^{\varrho(l_{\beta_{p-1}}, l_{k_{p-1}})} \varphi(t) \varrho t \right) \\ &\leq \tau\left(\frac{\eta}{3}\right) \left(\int_0^{\varrho(l_{\beta_{p-1}}, l_{k_{p-1}})} \varphi(t) \varrho t \right). \end{aligned} \quad (3.5)$$

Letting $p \rightarrow \infty$ in (3.5), yields

$$\int_0^m \varphi(t) \varrho t \leq \tau\left(\frac{\eta}{3}\right) \left(\int_0^m \varphi(t) \varrho t \right) < \int_0^m \varphi(t) \varrho t,$$

which is a contradiction . Hence, the sequence $\{l_j\}_{j \in \mathbb{N}}$ is Cauchy. The completeness of \mathfrak{X} implies that there exists $u \in \mathfrak{X}$ such that $l_j \rightarrow u$ as $j \rightarrow \infty$. Let $\delta_j = \varrho(l_j, \{u\})$. Then by (3.10),

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(e(l_j), e(u))}(\Upsilon l_j, \Upsilon u)} \varphi(t) \varrho t &\leq \tau(\varrho(l_j, u)) \left(\int_0^{\varrho(l_j, \{u\})} \varphi(t) \varrho t \right) \\ &< \int_0^{\varrho(l_j, \{u\})} \varphi(t) \varrho t = \int_0^{\delta_j} \varphi(t) dt, \end{aligned}$$

that is,

$$\int_0^{S_{E\mathfrak{X}}^{(e(l_{j+1}), e(u))}(l_{j+1}, \Upsilon u)} \varphi(t) \varrho t < \int_0^{\delta_j} \varphi(t) \varrho t. \tag{3.6}$$

Since $\delta_j \rightarrow 0$ as $j \rightarrow \infty$, therefore, from (3.6) $S_{E\mathfrak{X}}^{(e(l_{j+1}), e(u))}(l_{j+1}, \Upsilon u) = 0$ as $j \rightarrow \infty$. Now, using

$$\begin{aligned} S_{E\mathfrak{X}}^{(e(u), e(u))}(\{u\}, \Upsilon u) &\leq \varrho(\{u\}, l_{j+1}) + S_{E\mathfrak{X}}^{(e(l_{j+1}), e(u))}(l_{j+1}, \Upsilon u) \\ &= \delta_j + S_{E\mathfrak{X}}^{(e(l_{j+1}), e(u))}(l_{j+1}, \Upsilon u). \end{aligned} \tag{3.7}$$

As $j \rightarrow \infty$ in (3.7), $S_{E\mathfrak{X}}^{(e(u), e(u))}(\{u\}, \Upsilon u) = 0$, which implies $\{u\} \subset \Upsilon u$. □

Remark 3.1. Theorem 3.1. will fail if we admit zero value almost everywhere near zero for the mapping φ . We illustrate this observation with an example as follows.

Example 3.1. Let $\mathfrak{X} = E = \{1, 2, 3\}$ and

$$\varphi(t) = \begin{cases} \frac{1+\ln(t)}{4}, & \text{if } t > 0 \\ 0, & \text{if } t \in [0, 2]. \end{cases}$$

Define a soft set-valued map $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ by

$$(\Upsilon l)(e) = \begin{cases} \{1\}, & \text{if } l \neq 1, e \in E \\ \{2\}, & \text{if } l \notin \{1, 2\}, e \in E \\ \emptyset, & \text{if } l, e \in \mathfrak{X}. \end{cases}$$

Define $\tau : [0, \infty) \rightarrow (0, 1)$ by $\tau(t) = \lambda$. Let $\varrho : E^2 \rightarrow [0, \infty)$ be the Euclidean metric restricted to E so that (E, ϱ) is a complete metric space. We see that for each $l, k \in \mathfrak{X}$ and for all $e \in E$, $S_{\mathfrak{X}X}^{(e(l), e(k))}(\Upsilon l, \Upsilon k) \leq 2$. Consequently,

$$\int_0^{S_{E\mathfrak{X}}^{(e(l), e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \leq \int_0^2 \varphi(t) \varrho t = 0 \leq \lambda \int_0^{\varrho(l, k)} \varphi(t) \varrho t.$$

Thus, Inequality (3.10) is satisfied for all $\lambda \in (0, 1)$, but Υ has no e -soft fixed point in \mathfrak{X} . Similarly, one can construct an example to show that the mapping $\varphi \in \Psi$ in Theorem 3.1 cannot admit negative value.

Next, we provide an example to support the hypotheses of Theorem 3.1.

Example 3.2. Let $\mathfrak{X} = E = \{1, 2, 3\}$ and $\varrho : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{R}$ be defined by

$$\varrho(l, k) = \begin{cases} 0, & \text{if } l = k \\ 1, & \text{if } l \neq k \text{ and } l, k \in \{1, 3\} \\ 2, & \text{if } l \neq k \text{ and } l, k \in \{1, 2\} \\ 3, & \text{if } l \neq k \text{ and } l, k \in \{2, 3\}. \end{cases}$$

Define a soft set-valued map $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ by

$$(\Upsilon l)(e) = \begin{cases} \{1\}, & \text{if } e = 3 \text{ and } l = 1 \\ \{3\}, & \text{if } e = 2 \text{ and } l = 3 \\ \emptyset, & \text{if } e = 1 \text{ for all } l \in \mathfrak{X}. \end{cases}$$

Take $\varphi(t) = \frac{\ln(1+t)}{1+t}$, $\forall t \in [0, \infty)$. Clearly, $\varphi \in \Psi$. It can be seen that for all $l, k \in \mathfrak{X}$, $e \in E$, and $\tau(t) \in (0, 1)$, $t \geq 0$,

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(e(l), e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t &\leq \tau(t) \int_0^{\varrho(l, k)} \frac{\ln(1+t)}{1+t} \varrho t \\ &\leq \frac{\tau(t)}{2} \left[\ln^2(1+t) \right]_0^{\varrho(l, k)}. \end{aligned}$$

Hence, all the conditions of Theorem 3.1. are satisfied. In this case, $\{1, 3\}$ is the set of all e -soft fixed points of Υ in \mathfrak{X} for $e = 2, 3$.

The following Corollary is the main result of [28, Theorem 3.1] with $f = I_{\mathfrak{X}}$, the identity mapping on \mathfrak{X} .

Corollary 3.1. [28] Let (\mathfrak{X}, ϱ) be a complete metric space, E be the parameter set and $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ be a soft set-valued map. Suppose that for $l \in \mathfrak{X}$, there exists $a(l) \in \text{Dom} \Upsilon l$ such that $(\Upsilon l)(a(l)) \in \mathfrak{X}^*$. If there exists $\lambda \in (0, 1)$ such that for all $l, k \in \mathfrak{X}$,

$$S_{E\mathfrak{X}}^{(e(l), e(k))}(\Upsilon l, \Upsilon k) \leq \lambda \varrho(l, k), \quad (3.8)$$

then there exists $u \in \mathfrak{X}$ such that $u \in \Upsilon u$.

Proof. Define $\tau : [0, \infty) \rightarrow (0, 1)$ by $\tau(t) = \lambda$ and take $\varphi := 1$ in Theorem 3.1. □

Corollary 3.2. Let (\mathfrak{X}, ϱ) be a complete metric space, E be the parameter set and $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ be a soft set-valued map; let $\varphi \in \Psi$ and $\tau : [0, \infty) \rightarrow (0, 1)$ be a decreasing function. Suppose that for $l \in \mathfrak{X}$, there exists $a(l) \in \text{Dom} \Upsilon l$ such that $(\Upsilon l)(a(l)) \in \mathfrak{X}^*$. If for all $l, k \in \mathfrak{X}$,

$$\int_0^{S_{E\mathfrak{X}}^{\infty}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \leq \tau(\varrho(l, k)) \left(\int_0^{\varrho(l, k)} \varphi(t) \varrho t \right) \quad (3.9)$$

then there exists $u \in \mathfrak{X}$ such that $\{u\} \subset \Upsilon u$.

Proof. Since

$$\int_0^{S_{E\mathfrak{X}}^{(e(l),e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \leq \int_0^{S_{E\mathfrak{X}}^{\infty}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t,$$

therefore, using (3.10), we get

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(e(l),e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t &\leq \int_0^{S_{E\mathfrak{X}}^{\infty}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \\ &\leq \tau(\varrho(l, k)) \left(\int_0^{\varrho(l, k)} \varphi(t) \varrho t \right). \end{aligned}$$

Hence, Theorem 3.1. can be applied to find $u \in \mathfrak{X}$ such that $\{u\} \subset \Upsilon u$. \square

Corollary 3.3. Let (\mathfrak{X}, ϱ) be a complete metric space, E be the parameter set and $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ be a soft set-valued map; let $\tau : [0, \infty) \rightarrow (0, 1)$ be a decreasing function. Suppose that for $l \in \mathfrak{X}$, there exists $a(l) \in \text{Dom} \Upsilon l$ such that $(\Upsilon l)(a(l)) \in \mathfrak{X}^*$. If for all $l, k \in \mathfrak{X}$,

$$S_{E\mathfrak{X}}^{(e(l),e(k))}(\Upsilon l, \Upsilon k) \leq \tau(\varrho(l, k)) (\varrho(l, k)) \quad (3.10)$$

then, there exists $u \in X$ such that $\{u\} \subset \Upsilon u$.

Proof. Put $\varphi(t) := 1$ in Theorem 3. \square

Corollary 3.4. Let (\mathfrak{X}, ϱ) be a complete metric space, E be the parameter set and $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ be a soft set-valued mapping; let $\tau : [0, \infty) \rightarrow (0, 1)$ be a decreasing function. Suppose that for $l \in \mathfrak{X}$, there exists $a(l) \in \text{Dom} \Upsilon l$ such that $(\Upsilon l)(a(l)) \in \mathfrak{X}^*$. If for all $l, k \in \mathfrak{X}$,

$$S_{E\mathfrak{X}}^{\infty}(\Upsilon l, \Upsilon k) \leq \tau(\varrho(l, k)) (\varrho(l, k)) \quad (3.11)$$

then there exists $u \in \mathfrak{X}$ such that $\{u\} \subset \Upsilon u$.

Proof. Set $\varphi(t) := 1$ in Corollary 3. \square

In connection with Theorem 3.1., we have the next result.

Theorem 3.2. Let (\mathfrak{X}, ϱ) be a complete metric space, E be the parameter set and $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$ be a soft set-valued map; let $\varphi \in \Psi$ and $\tau : [0, \infty) \rightarrow (0, 1)$ be a decreasing function. Suppose that for $l \in \mathfrak{X}$, there exists $a(l) \in \text{Dom} \Upsilon l$ such that $(\Upsilon l)(a(l)) \in \mathfrak{X}^*$. If for all $l, k \in \mathfrak{X}$,

$$\int_0^{S_{E\mathfrak{X}}^{(e(l),e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \leq \tau(\varrho(l, k)) \max \left\{ \int_0^{\varrho(l, k)} \varphi(t) \varrho t, \int_0^{\varrho(l, \Upsilon l)} \varphi(t) \varrho t, \int_0^{\varrho(k, \Upsilon k)} \varphi(t) \varrho t \right\}, \quad (3.12)$$

then, there exists $u \in \mathfrak{X}$ such that $\{u\} \subset \Upsilon u$.

Proof. Let $l_0 \in \mathfrak{X}$ and $l_{n+1} \in (\Upsilon l_n)(a(l_n))$, for some $a(l_n) \in E$ and for all $n \in \mathbb{N}$. We shall prove that if $l_n = l_{n+1}$ for $n \in \mathbb{N}$, then there exists $u \in \mathfrak{X}$ such that $l_n = l_{n+1} = u \in (\Upsilon u)(a(u))$. Assume that $l_n \neq l_{n+1}$, for some $n \in \mathbb{N}$ so that $u \notin (\Upsilon u)(a(u))$, for all $a(u) \in E$. Applying Inequality (3.12) with $l = l_{n-1}$ and $k = l_n$, we have

$$\int_0^{S_{E\mathfrak{X}}^{(e(l_{n-1}),e(l_n))}(\Upsilon l_{n-1}, \Upsilon l_n)} \varphi(t) dt \leq \tau(\varrho(l_{n-1}, l_n))$$

$$\begin{aligned}
& \times \max \left\{ \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t, \int_0^{\varrho(l_{n-1}, \Upsilon l_{n-1})} \varphi(t) \varrho t, \int_0^{\varrho(l_n, \Upsilon l_n)} \varphi(t) \varrho t \right\} \\
& \leq \tau(\varrho(l_{n-1}, l_n)) \\
& \times \max \left\{ \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t, \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t, \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t \right\} \\
& \leq \tau(\varrho(l_{n-1}, l_n)) \\
& \times \max \left\{ \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t, \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t \right\}.
\end{aligned}$$

If

$$\max \left\{ \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t, \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t \right\} = \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t,$$

then

$$\int_0^{S_{E\mathfrak{X}}^{(e(l_{n-1}), e(l_n))(\Upsilon l_{n-1}, \Upsilon l_n)}} \varphi(t) \varrho t \leq \tau(\varrho(l_{n-1}, l_n)) \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t. \quad (3.13)$$

From (3.13), Theorem 3.1. can be applied to obtain the conclusion of Theorem 3.2. But suppose

$$\max \left\{ \int_0^{\varrho(l_{n-1}, l_n)} \varphi(t) \varrho t, \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t \right\} = \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t.$$

Then,

$$\begin{aligned}
\int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t & \leq \int_0^{S_{E\mathfrak{X}}^{(e(l_{n-1}), e(l_n))(\Upsilon l_{n-1}, \Upsilon l_n)}} \varphi(t) \varrho t \\
& \leq \tau(\varrho(l_{n-1}, l_n)) \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t \\
& < \int_0^{\varrho(l_n, l_{n+1})} \varphi(t) \varrho t,
\end{aligned}$$

gives a contradiction. Consequently, in either case, one obtains $u \in (\Upsilon u)(a(u))$, for some $a(u) \in E$. \square

4. Fixed points of fuzzy and multivalued mappings

As an application of e -soft fixed point theorem (Theorem 3.1) of the previous section, in this section, we obtain fixed points of fuzzy and multivalued mappings satisfying some integral type conditions of Heilpern and Nadler's type (see, [16, 24]).

Definition 4.1. [16] A fuzzy set Θ in \mathfrak{X} is called an approximate quantity if and only if its α -level set is a compact convex subset of \mathfrak{X} for each $\alpha \in [0, 1]$ and $\sup_{l \in \mathfrak{X}} \Theta(l) = 1$. The set of all approximate quantities in \mathfrak{X} is denoted by $W(\mathfrak{X})$.

Theorem 4.1. Let (\mathfrak{X}, ϱ) be a complete metric space and $F : \mathfrak{X} \rightarrow W(\mathfrak{X})$ be a fuzzy mapping. If there exist $\lambda \in (0, 1)$ and $\varphi \in \Psi$ such that

$$\int_0^{\varrho_\infty(Fl, Fk)} \varphi(t) \varrho t \leq \lambda \int_0^{\varrho(l, k)} \varphi(t) \varrho t, \quad (4.1)$$

for each $l, k \in \mathfrak{X}$, then there exists $u \in \mathfrak{X}$ such that $\{u\} \subset F(u)$.

Proof. Let $\tau : [0, \infty) \rightarrow (0, 1)$ be a mapping defined by $\tau(t) = \lambda$, $t \geq 0$, and $E = [0, 1]$ be a parameter set. Consider a soft set-valued map

$$\Upsilon_F : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^{[0,1]}, \quad \text{defined by}$$

$$\Upsilon_F l(1) = \{t \in \mathfrak{X} : (Fl)(t) \geq 1\} = [Fl]_1.$$

Then

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(1,1)}(\Upsilon_F l, \Upsilon_F k)} \varphi(t) \varrho t &= \int_0^{\inf E_{((\tau l)(1), (\tau k)(1))}^{\varrho}} \varphi(t) \varrho t \\ &= \int_0^{H([Fl]_1, [Fk]_1)} \varphi(t) \varrho t \\ &\leq \int_0^{\varrho_{\infty}(Fl, Fk)} \varphi(t) \varrho t. \end{aligned}$$

By (4.1), we obtain

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(1,1)}(\Upsilon_F l, \Upsilon_F k)} \varphi(t) \varrho t &\leq \int_0^{\varrho_{\infty}(Fl, Fk)} \varphi(t) \varrho t \\ &\leq \lambda \int_0^{\varrho(l, k)} \varphi(t) \varrho t \\ &\leq \tau(t) \int_0^{\varrho(l, k)} \varphi(t) \varrho t. \end{aligned}$$

Consequently, by Theorem 3.1, the conclusion holds. \square

Corollary 4.1. (see [16]) Let (\mathfrak{X}, ϱ) be a complete metric space and $F : \mathfrak{X} \rightarrow W(\mathfrak{X})$ be a fuzzy mapping. If there exists $\lambda \in (0, 1)$ such that

$$\varrho_{\infty}(Fl, Fk) \leq \lambda \varrho(l, k), \quad (4.2)$$

for each $l, k \in \mathfrak{X}$, then there exists $u \in \mathfrak{X}$ such that $\{u\} \subset F(u)$.

Proof. Put $\varphi(t) := 1$ in Theorem 4. \square

Corollary 4.2. Let (\mathfrak{X}, ϱ) be a complete metric space and $\Upsilon : \mathfrak{X} \rightarrow \mathfrak{X}^*$ be a multivalued mapping satisfying the following conditions: There exist $\lambda \in (0, 1)$ and $\varphi \in \Psi$ such that

$$\int_0^{H(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \leq \lambda \int_0^{\varrho(l, k)}. \quad (4.3)$$

Then there exists $u \in \mathfrak{X}$ such that $u \in \Upsilon u$.

Proof. Let $\tau : [0, \infty) \rightarrow (0, 1)$ be a mapping defined by $\tau(t) = \lambda$, $t \geq 0$. Let $E = \{e_1, e_2\}$ and consider a soft set-valued map $\Upsilon : \mathfrak{X} \rightarrow [P(\mathfrak{X})]^E$, defined by

$$\Upsilon l(e) = \begin{cases} \mathfrak{X}, & \text{if } e = e_1 \\ \Upsilon l, & \text{if } e = e_2. \end{cases}$$

Then,

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(e_2(l), e_2(k))(\Upsilon l, \Upsilon k)}} \varphi(t) \varrho t &= \int_0^{\inf E_{((\Upsilon l)(e_2(l)), (\Upsilon k)(e_2(k)))}^{\varrho}} \varphi(t) \varrho t \\ &= \int_0^{\inf E_{(\Upsilon l, \Upsilon k)}^{\varrho}} \varphi(t) \varrho t \\ &\leq \int_0^{H(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t. \end{aligned}$$

Using (4.4), the last inequality can be written as

$$\begin{aligned} \int_0^{S_{E\mathfrak{X}}^{(e_2(l), e_2(k))}} \varphi(t) \varrho t &\leq \int_0^{H(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t \\ &\leq \lambda \int_0^{d(l, k)} \varphi(t) \varrho t \\ &\leq \tau(t) \int_0^{\varrho(l, k)} \varphi(t) \varrho t. \end{aligned}$$

Thus, the conclusion follows by Theorem 3. \square

Corollary 4.3. (see [24]) Let (\mathfrak{X}, ϱ) be a complete metric space and $\Upsilon : \mathfrak{X} \rightarrow \mathfrak{X}^*$ be a multivalued mapping satisfying the following conditions: there exist $\lambda \in (0, 1)$ such that

$$H(\Upsilon l, \Upsilon k) \leq \lambda \varrho(l, k). \quad (4.4)$$

Then, there exists $u \in \mathfrak{X}$ such that $u \in \Upsilon u$.

Proof. Set $\varphi(t) := 1$ in Theorem 4. \square

5. Application to neutral delay differential equations

Consider the nonlinear neutral delay differential equation with an unbounded delay:

$$l'(t) = -a(t)l(t) + b(t)l^2(t - r(t)) + c(t)l(t)(t - r(t))l'(t - r(t)), \quad (5.1)$$

where $a(t), b(t)$ are continuous, $c(t)$ is continuously differentiable and $r(t) > 0$ for all $t \in \mathbb{R}$ and is twice continuously differentiable. For details in this direction, the interested reader may consult [8, 14].

Lemma 5.1. [14] Suppose that $r'(t) \neq 1$ for all $t \in \mathbb{R}$. Then $l(t)$ is a solution of (5.1) if and only if

$$\begin{aligned} l(t) &= \left(l(0) - \frac{1}{2} \frac{c(0)}{1 - r'(0)} l^2(-r(0)) \right) e^{-\int_0^t a(s) \varrho s} \\ &\quad + \frac{1}{2} l^2(t - r(t)) \frac{c(t)}{1 - r'(t)} \\ &\quad - \frac{1}{2} \int_0^t (h(u) - 2b(u)) l^2(u - r(u)) e^{-\int_u^t a(s) \varrho s} \varrho u, \end{aligned} \quad (5.2)$$

where

$$h(u) = \frac{r''(u)c(u) + (c'(u) + c(u)a(u)(1 - r'(u)))}{(1 - r'(u))^2}.$$

Let $\rho : (-\infty, 0] \rightarrow \mathbb{R}$ be a given continuous function. We say that $l(t)$ is a solution of (5.1) if it satisfies (5.1) for $t \geq 0$ and $l(t) = \rho(t)$ for $t \leq 0$. Denote by $C(\mathbb{R}, \mathbb{R})$, the space of all continuous functions from \mathbb{R} to \mathbb{R} and define the set M_ρ by

$$M_\rho = \{l : \mathbb{R} \rightarrow \mathbb{R}, l(t) = \rho(t), t \leq 0, l \in C(\mathbb{R}, \mathbb{R})\}.$$

Then, equipped with the uniform norm $\| \cdot \|$, M_ρ is a Banach space.

Theorem 5.1. Consider Eqs. (5.1) and (5.2). Suppose the following conditions hold:

- (i) there exists $L > 0$ such that $\|l\| \leq L$, for all $l \in M_\rho$;
- (ii) there exists $\lambda \in (0, 1)$ such that

$$\int_0^{\left(\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t |h(u) - 2b(u)| e^{-\int_u^t a(s) \varrho s} du\right)(L\|l-k\|)} \varphi(t) \varrho t \leq \lambda \int_0^{\|l-k\|} \varphi(t) \varrho t;$$

then problem (5.1) has a solution in M_ρ .

Proof. Let $E = (0, \infty)$ and $\mathfrak{X} = M_\rho$. Then under the Chebyshev metric ϱ induced by the uniform norm, (\mathfrak{X}, ϱ) is a complete metric space. Consider an arbitrary function $f_l : \mathfrak{X} \rightarrow \mathfrak{X}$ defined by

$$\begin{aligned} f_l(t) &= \left(l(0) - \frac{1}{2} \frac{c(0)}{1-r'(0)} l^2(-r(0)) \right) e^{-\int_0^t a(s) \varrho s} \\ &\quad + \frac{1}{2} l^2(t - r(t)) \frac{c(t)}{1-r'(t)} \\ &\quad - \frac{1}{2} \int_0^t (h(u) - 2b(u)) l^2(u - r(u)) e^{-\int_u^t a(s) \varrho s} du. \end{aligned} \quad (5.3)$$

Let $\tau : [0, \infty) \rightarrow (0, 1)$ be given by $\tau(t) = \lambda, t \geq 0$. Define a soft set-valued map $\Upsilon : \mathfrak{X} \rightarrow [\Upsilon(\mathfrak{X})]^E$ by

$$(\Upsilon l)(e) = \begin{cases} \{l \in \mathfrak{X} : l \neq f_l(t)\}, & \text{if } 0 < e < \frac{1}{100} \\ \{l \in \mathfrak{X} : l(t) = f_l(t)\}, & \text{if } e = \frac{1}{100} \\ \emptyset & \text{if } e > \frac{1}{100}. \end{cases}$$

Then for $l \in \mathfrak{X}$, there exists $e \in E$ such that

$$(\Upsilon l)(e) = \{f_l(t)\}.$$

Now, for $l, k \in \mathfrak{X}$ with $l \neq k$, we have

$$S_{E\mathfrak{X}}^{(e(l), e(k))}(\Upsilon l, \Upsilon k) = \inf_{((\Upsilon l)(e), (\Upsilon k)(e))} E_{((\Upsilon l)(e), (\Upsilon k)(e))}^{\varrho} = |f_l - f_k|.$$

Therefore,

$$\int_0^{S_{E\mathfrak{X}}^{(e(l), e(k))}(\Upsilon l, \Upsilon k)} \varphi(t) \varrho t = \int_0^{|f_l - f_k|} \varphi(t) \varrho t$$

$$\begin{aligned}
&\leq \int_0^1 \frac{1}{2} \left\{ \left| \frac{c(t)}{1-r'(t)} \right| + \int_0^t |h(u)-2b(u)| e^{-\int_u^t \rho^s \rho u} \right\} \|l^2 - k^2\| \varphi(t) \rho t \\
&\leq \int_0^1 \frac{1}{2} (2L) \left\{ \left| \frac{c(t)}{1-r'(t)} \right| + \int_0^t |h(u)-2b(u)| e^{-\int_u^t \rho^s \rho u} \right\} \|l-k\| \varphi(t) \rho t \\
&\leq \int_0^1 \left\{ \left| \frac{c(t)}{1-r'(t)} \right| + \int_0^t |h(u)-2b(u)| e^{-\int_u^t \rho^s \rho u} \right\} (L\|l-k\|) \varphi(t) \rho t \\
&\leq \lambda \int_0^{\|l-k\|} \varphi(t) \rho t.
\end{aligned}$$

Consequently, Theorem 3.1. implies that there exists $u \in \mathfrak{X}$ such that $\{u\} \subseteq \Upsilon u$, which solves problem (5.1). \square

6. Conclusion

Two classical theorems involving fixed points are Banach and Brouwer's Theorems. Banach fixed point theorem states that if \mathfrak{X} is a complete metric space and Υ is a contraction on \mathfrak{X} , then Υ has a unique fixed point in \mathfrak{X} . In Brouwer's fixed point theorem, \mathfrak{X} is required to be a closed unit ball in a Euclidean space. Then, any contraction Υ on \mathfrak{X} has a fixed point. But in this case, uniqueness of fixed point is not guaranteed. In Banach theorem, a metric on \mathfrak{X} is used with the assumption that Υ is a contraction. The unit ball in a Euclidean space is also a metric space and the metric topology determines the continuity of continuous functions, however, the main idea of Brouwer's theorem is a topological property of the unit ball, namely, the unit ball is compact and contractible. Banach theorem and Brouwer theorem tell us a difference between two main branches of fixed point theory, metric fixed point theory and topological fixed point theory. It is not easy to differentiate two fixed point theories in an exact way, or to determine a certain topics belonging to which branch. Generally, fixed point theory is regarded as a branch of topology. But due to deep influence on topics related to nonlinear analysis or dynamic systems, many areas of fixed point theory can be thought of as a branch of analysis.

In the setting of metric fixed point theory, here in this paper, fixed point theorems of soft-valued maps via some integral type contractive conditions are proved. As a consequence of our main result, some fixed point theorems in the framework of fuzzy set-valued and multivalued mappings are derived. Following the fundamental role of Banach fixed point theorem in the study of existence and uniqueness of solutions of equations, we investigated some conditions for the existence of solution of nonlinear neutral differential equations. We hope that the presented ideas herein will motivate the interested researcher(s) and hence, extend the concepts to other related areas such as topological fixed point theory. Moreover, the soft set component of this work can also be extended to other models such as N -soft set, fuzzy soft set, intuitionistic fuzzy soft set, intuitionistic neutrosophic soft set, rough sets, and so on.

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Conflict of interest

The authors declare that they have no competing interests.

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