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*Research article*

# On k-type pseudo null slant helices due to the Bishop frame in Minkowski 3-space $E_1^3$

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**Abstract:** In this study, we examine *k*-type pseudo null slant helices due to the Bishop frame, where $k \in \{0,1,2\}$. There are two different cases of the Bishop frame of a pseudo null curve related to the Bishop curvatures. Based on these cases, we present that every pseudo null curve is a *k*-type pseudo null curve according to the Bishop frame in Minkowski 3-space $E_1^3$. Then we obtain the axes of *k*-type pseudo null slant helices, and determine their causal characters.

**Keywords:** Bishop Frame; pseudo null curve; *k*-type pseudo null slant helix
**Mathematics Subject Classification:** 53A99, 53B99

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## 1. Introduction

In the theory of curves, general helix is very important class of curves. One of the most elementary but widely researched topics is helix in the theory of curves. In his famous theorem, Lancret states that helices are characterized by the constant ratio of curvature and torsion. Slant helices as a special class of general helices were firstly defined by Izumiya and Takeuchi [1]. All helices (W-curves) have been completely classified in $E_1^3$ by Walrave [2]. Several authors introduced different types of helices and investigated their properties. Kula and Yayli studied spherical images of tangent and binormal indicatrices of slant helices and also showed that spherical images are spherical helices [3]. Kula et al. characterized slant helices in Euclidean 3-space [4]. Also the work [5] studies the physical applications of slant helices in the ordinary space.

The notion of *k*-type slant helices refers to the class of curves having a property that the scalar product of frame's vector field and a fixed axis is constant. The studies about *k-type slant helices* are as follows: Ergüt et al. studied non-null *k*-type slant helices in Minkowski 3-space [6]. Ali et al. examined *k*-type partially null and pseudo null curves in Minkowski 4-space $E_1^4$ [7]. Pseudo null Darboux helices, null Cartan Darboux helices, *k*-type pseudo null Darboux helices, and *k*-type null Cartan helices were discussed in [8–10]. Qian presented some results of *k*-type null slant helices in Minkowski space time [11]. Recently, Grbovic and Nešovic obtained some results of *k*-type null Cartan slant helices according to the generalized Bishop frame [12].

The vanishing of second derivative of a curve has led to the study of the new frame. First the behaviour of a curve was studied by a new adapted frame which is called Bishop frame or relatively parallel adapted frame [13]. This frame is composed of the vectors; the tangential vector field $T$, and two normal vector fields $N_1$ and $N_2$ which are obtained by rotating the Serret-Frenet vectors $N$ and $B$ in the normal plane $T^{\perp}$ of the curve, in such a way that they become relatively parallel [13]. Bishop frame have been defined for curves in different Euclidean ambient spaces [14–17]. There is also interesting study which points out the physical applications of Bishop frame, see [18].

In this paper, we study *k*-type pseudo null slant helices according to two possible forms of the Bishop frame given by Grbovic and Nesovic [15]. We show that every pseudo null curve is a *k*-type pseudo null curve according to the Bishop frame in Minkowski 3-space $E_1^3$. Then we find the axes of *k*-type pseudo null slant helices, and determine their causal characters.

## 2. Preliminaries

The three dimensional Minkowski space $E_1^3$ is a real vector space $R^3$ endowed with the standard indefinite flat metric $\langle , \rangle$ defined by

$$\langle , \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are any two vectors in $E_1^3$. Since this metric is an indefinite metric, an arbitrary vector $x \in E_1^3$ has one of three Lorentzian characters: it is a spacelike vector if $\langle x, x \rangle > 0$ or $x = 0$; timelike $\langle x, x \rangle < 0$ and null (lightlike) $\langle x, x \rangle = 0$ for $x \neq 0$. The pseudo-norm of the arbitrary vector $x \in E_1^3$ is given by $\|x\| = \sqrt{|\langle x, x \rangle|}$. Similarly, an arbitrary curve $\gamma = \gamma(s)$ in $E_1^3$ can locally be spacelike, timelike or null (lightlike) if its velocity vector $\gamma'$ is, respectively, spacelike, timelike or null (lightlike), for every $s \in I \subset E$. The curve $\gamma = \gamma(s)$ is called a unit speed curve if its velocity vector $\gamma'$ is unit one i.e, $\|\gamma'\| = 1$ [19,20].

A spacelike curve $\gamma : I \to E_1^3$ is called a pseudo null curve, if its principal normal vector field $N$ and binormal vector field B are null vector fields satisfying the condition $\langle N, B \rangle = 1$. The Frenet formulae of a non-geodesic pseudo null curve $\gamma = \gamma(s)$ have the form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.2}$$

where the first Frenet curvature $\kappa(s) = 1$ and the second Frenet curvature (torsion) $\tau(s)$ is an arbitrary function of arc-length parameter $s$ of $\gamma$ [2]. Also the vector fields of Frenet frame holds the following relations:

$$\langle T,T\rangle = 1, \langle N,N\rangle = \langle B,B\rangle = 0, \langle T,N\rangle = \langle T,B\rangle = 0, \langle N,B\rangle = 1,$$

and

$$T \times N = N, \quad N \times B = T, \quad B \times T = B.$$

The Frenet frame $\{T,N,B\}$ is positively oriented, if $\det(T,N,B) = [T,N,B] = 1$.

**Definition 2.1.** The Bishop frame $\{T_1,N_1,N_2\}$ of a pseudo null curve $\gamma$ in $E_1^3$ is positively oriented pseudo-orthonormal frame consisting of the tangential vector field $T_1$ and two relatively parallel lightlike normal vector fields $N_1$ and $N_2$ [15].

The vector fields of The Bishop frame of a pseudo null curve $\gamma$ in $E_1^3$ satisfy the relations [15]

$$\langle T_1,T_1\rangle = 1, \langle N_2,N_2\rangle = \langle N_1,N_1\rangle = 0, \langle T_1,N_1\rangle = \langle T_1,N_2\rangle = 0, \langle N_1,N_2\rangle = 1, \tag{2.3}$$

and

$$T_1 \times N_1 = -T_1, \quad N_1 \times N_2 = -N_2, \quad N_2 \times T_1 = N_1. \tag{2.4}$$

**Theorem 2.1. (**[15]**)** Let $\gamma$ be a pseudo null curve in $E_1^3$ parameterized by the arc-length $s$ with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$:

**(i)** Then the Bishop frame $\{T_1,N_1,N_2\}$ and the Frenet frame $\{T,N,B\}$ of $\gamma$ are related by:

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\kappa_2} & 0 \\ 0 & 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.5}$$

and the Frenet equations of $\gamma$ according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.6}$$

where $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau(s)ds}, c_0 \in R_0^+$;

**(ii)** Then the Bishop frame $\{T_1,N_1,N_2\}$ and the Frenet frame $\{T,N,B\}$ of $\gamma$ are related by:

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\kappa_1 \\ 0 & -\dfrac{1}{\kappa_1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.7}$$

and the Frenet equations of $\gamma$ according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.8}$$

where $\kappa_1(s) = c_0 e^{\int \tau(s)ds}, c_0 \in R_0^-$ and $\kappa_2(s) = 0.$

## 3. On k-type pseudo null slant helices due to the Bishop frame in Minkowski 3-space

In this section, we study $k$-type pseudo null slant helices framed by the Bishop frame in Minkowski 3-space $E_1^3$. From Equations (2.7) and (2.8), there are two cases arising from the Bishop curvatures. In the first case, the first Bishop curvature $\kappa_1$ vanishes, and the vector field $N_1'$ is zero vector. In the second case, the second Bishop curvature $\kappa_2$ vanishes, and the vector field $N_2'$ is zero vector. We will examine these cases separately in this section.

**Definition 3.1.** A pseudo null curve $\gamma$ in $E_1^3$ given by the Bishop frame $\{T_1, N_1, N_2\}$ is called a 0-type pseudo null slant helix if there exists a non zero fixed direction $V \in E_1^3$ such that satisfies

$$\langle T_1, V \rangle = c,\ c \in R,$$

and a $k$-type pseudo null slant helices for $k \in \{1,2\}$ if there exists a non zero fixed direction $V \in E_1^3$ such that hold

$$\langle N_k, V \rangle = c,\ c \in R. \tag{3.1}$$

The fixed direction $V$ is called axis of the helix.

*3.1. The first case*

**Theorem 3.1.** Every pseudo null curve $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is a $k$-type $\{k = 0,1,2\}$ pseudo null slant helix.

*Proof.* Let us take the pseudo curve $\gamma$ framed by the Bishop frame. According to Definition 3.1, there exists a fixed direction $V \in E_1^3$ such that

$$\langle T_1, V\rangle = c, c \in R. \tag{3.2}$$

The fixed direction $V$ can be decomposed as

$$V = cT_1(s) + \lambda_1(s)N_1(s) + \lambda_2(s)N_2(s), \tag{3.3}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are some differential functions in terms of $s$. Differentiating the Eq. (3.3) with respect to $s$ and using (2.6), we have the following system of differential equations

$$\begin{cases} \lambda_2\kappa_2 = 0, \\ \lambda_1' + c\kappa_2 = 0, \\ \lambda_2' = 0. \end{cases} \tag{3.4}$$

From (3.4), we have

$$\begin{cases} \lambda_1(s) = c_1 - c\int \kappa_2(s)ds, \\ \lambda_2(s) = 0, \end{cases} \tag{3.5}$$

where $c_1 \in R.$ Using (3.5), we have the axis $V$ as

$$V = cT_1 + (c_1 - c\int \kappa_2(s)ds)N_1, \tag{3.6}$$

Differentiating (3.6) and using (2.6) gives $V'(s) = 0.$ Hence, $V$ is a fixed direction. Thus, $\gamma$ is a 0-type pseudo null slant helix.

Let us show that pseudo null curve is also a 1-type pseudo null slant helix. According to Definition 3.1, there exists a fixed direction $V \in E_1^3$ such that

$$\langle N_1, V\rangle = c,\ c \in R. \tag{3.7}$$

The fixed direction $V$ is decomposed as follows:

$$V = \lambda_1(s)T_1(s) + \lambda_2(s)N_1(s) + cN_2(s), \tag{3.8}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are some differential functions in terms of $s$. Differentiating the Eq. (3.8) with respect to $s$ and using (2.6), we have the following system of differential equations

$$\begin{cases} \lambda_1' - c\kappa_2 = 0, \\ \lambda_2' + \lambda_1\kappa_2 = 0. \end{cases} \tag{3.9}$$

From (3.9), we have

$$\begin{cases} \lambda_1(s) = -c\int \kappa_2(s)ds, \\ \lambda_2(s) = c\int \kappa_2(\int \kappa_2(s))ds, \end{cases} \tag{3.10}$$

where $c \in R$.
Using (3.10), then we have

$$V = -c\int \kappa_2(s)dsT_1 + c\int \kappa_2(\int \kappa_2(s))dsN_1 + cN_2(s). \tag{3.11}$$

Differentiating (3.11) and using (2.6), then we arrive at $V'(s) = 0$. Hence, $V$ is a fixed direction. Therefore, $\gamma$ is a 1-type pseudo null slant helix.

Let us show that pseudo null curve is also a 2-type pseudo null slant helix. Due to Definition 3.1, there exists a fixed direction $V \in E_1^3$ such that

$$\langle N_2, V \rangle = c,\ c \in R. \tag{3.12}$$

The fixed direction $V$ is written as

$$V = \lambda_1(s)T_1(s) + cN_1(s) + \lambda_2(s)N_2(s), \tag{3.13}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are some differential functions in terms of $s$. Differentiating the Eq. (3.13) with respect to $s$ and using (2.6), we have the following differential equation system

$$\begin{cases} \lambda_1' - \lambda_2\kappa_2 = 0, \\ \lambda_1\kappa_2 = 0, \\ \lambda_2' = 0. \end{cases} \tag{3.14}$$

From (3.14), we get

$$\lambda_1(s) = 0, \lambda_2(s) = 0. \tag{3.15}$$

Using (3.15), the axis $V$ is obtained as

$$V = cN_1, \tag{3.16}$$

From (3.16) and (2.6), we find $V'(s) = 0$. So, $V$ is a fixed direction.
As a result, every pseudo null curve according to the Bishop frame with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is a *k*-type pseudo null slant helix.

**Corollary 3.1.** An axis of the 0-type null Cartan slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is given by

$$V = cT_1 + (c_1 - c\int \kappa_2(s)ds)N_1,$$

where $c \in R$ and $c_1 \in R.$

**Corollary 3.2.** An axis of the 1-type null Cartan slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is given by

$$V = c\int \kappa_2(s)dsT_1 - c\int \kappa_2(\int \kappa_2(s)ds)dsN_1 + cN_2(s),$$

where $c \in R$ .

**Corollary 3.3.** An axis of the 2-type null Cartan slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is given by

$$V = cN_1,$$

where $c \in R$ .

**Corollary 3.4.** The causal character of the axis $V$ of the 0-type pseudo null slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is either spacelike or null.

**Corollary 3.5.** The causal character of the axis $V$ of the 1-type pseudo null slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is

**(i)** spacelike if

$$\{\int \kappa_2(s)ds\}^2 - 2\int \kappa_2(\int \kappa_2(s)ds)ds > 0,$$

**(ii)** timelike if

$$\{\int \kappa_2(s)ds\}^2 - 2\int \kappa_2(\int \kappa_2(s)ds)ds < 0,$$

**(iii)** null if

$$\{\int \kappa_2(s)ds\}^2 - 2\int \kappa_2(\int \kappa_2(s)ds)ds = 0.$$

**Corollary 3.6.** The causal character of the axis $V$ of the 2-type pseudo null slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau ds}$ is null.

**Example 3.1.** Let us consider a pseudo null curve in $E_1^3$ given by (Figure 1)

$$\gamma(s) = (e^s, e^s, s).$$

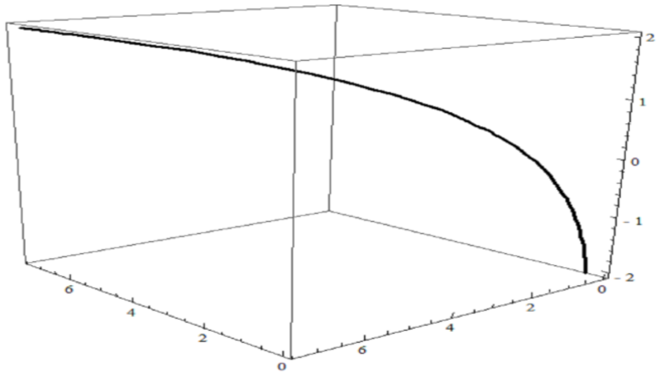


**Figure 1.** The k-type pseudo null slant helix.

According to the statement (ii) of Theorem 2.1, the Bishop curvatures of $\gamma$ are

$$\kappa_1(s) = 0, \quad \kappa_2(s) = 1.$$

The Bishop frame of $\gamma$ is computed as

$$\begin{aligned} T_1 &= (e^s, e^s, 1), \\ N_1 &= (\frac{1}{c_0}, \frac{1}{c_0}, \frac{1}{c_0 e^s}), \\ N_2 &= (-\frac{(e^{2s}+1)c_0}{2}, \frac{(1-e^{2s})c_0}{2}, -c_0 e^s). \end{aligned}$$

By Theorem 3.1, the pseudo null curve $\gamma$ holds 0,1,2-type slant helices whose axes are, respectively, calculated as follows:

$$\begin{aligned} V &= (ce^s + \frac{c_1 - cc_0 e^s}{c_0}, ce^s + \frac{c_1 - cc_0 e^s}{c_0}, cs + \frac{c_1 - cc_0 e^s}{c_0 e^s}), \\ V &= (-\frac{cc_0}{2}, \frac{cc_0}{2}, cc_0 se^s - \frac{3cc_0 e^s}{2}), \\ V &= (\frac{c}{c_0}, \frac{c}{c_0}, \frac{c}{c_0 e^s}), \end{aligned}$$

all of these axes satisfy the Eqs. (3.2), (3.7), and (3.12).

*3.2. The second case*

**Theorem 3.2.** Every pseudo null curve $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is a $k$-type $\{k = 0,1,2\}$ pseudo null slant helix.

*Proof.* Let us take the pseudo curve $\gamma$ framed by the Bishop frame. According to Definition 3.1, there exists a fixed direction $V \in E_1^3$ such that

$$\langle T_1, V \rangle = c,\ c \in R. \tag{3.17}$$

The fixed direction $V$ can be decomposed as

$$V = cT_1(s) + \lambda_1(s)N_1(s) + \lambda_2(s)N_2(s), \tag{3.18}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are some differential functions in terms of $s$. Differentiating the Eq. (3.18) with respect to $s$ and using (2.8), we have the following system of differential equations

$$\begin{cases} \lambda_1 \kappa_1 = 0, \\ \lambda_2' + c\kappa_1 = 0, \\ \lambda_1' = 0. \end{cases} \tag{3.19}$$

From (3.19), we have

$$\begin{cases} \lambda_1(s) = 0, \\ \lambda_2(s) = c_1 - c\int \kappa_1(s)ds, \end{cases} \tag{3.20}$$

where $c_1 \in R.$ Using (3.20), we arrive at

$$V = cT_1 + (c_1 - c\int \kappa_1(s)ds)N_2, \tag{3.21}$$

Using (2.8) in the differentiation of (3.21), then we find $V'(s) = 0.$ Hence, $V$ is a fixed direction. Thus, $\gamma$ is a 0-type pseudo null slant helix.

Let us show that pseudo null curve is also a 1-type pseudo null slant helix. According to Definition 3.1, there exists a fixed direction $V \in E_1^3$ such that

$$\langle N_1, V \rangle = c,\ c \in R. \tag{3.22}$$

The fixed direction $V$ is decomposed by

$$V = \lambda_1(s)T_1(s) + \lambda_2(s)N_1(s) + cN_2(s), \tag{3.23}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are some differential functions in terms of $s$. If we differentiate the Eq. (3.23) and use Eq. (2.8), then we obtain the following system

$$\begin{cases} \lambda_1' - \lambda_2 \kappa_1 = 0, \\ \lambda_1 \kappa_1 = 0, \\ \lambda_2' = 0. \end{cases} \tag{3.24}$$

By (3.24), we find

$$\begin{cases} \lambda_1(s) = 0, \\ \lambda_2(s) = 0. \end{cases} \tag{3.25}$$

Using (3.25), then the axis $V$ is as

$$V = cN_2(s). \tag{3.26}$$

Using (2.8) in the differentiation of (3.26) gives $V'(s) = 0$. From here, $\gamma$ is a 1-type pseudo null slant helix.

Consider pseudo null curve is also a 2-type pseudo null slant helix. According to Definition 3.1, there exists a fixed direction $V \in E_1^3$ such that

$$\langle N_2, V \rangle = c,\ c \in R. \tag{3.27}$$

The fixed direction $V$ is decomposed by

$$V = \lambda_1(s)T_1(s) + cN_1(s) + \lambda_2(s)N_2(s), \tag{3.28}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are some differential functions in terms of $s$. If we differentiate the Eq. (3.28) and use the Eq. (2.8), then we obtain following differential equation system

$$\begin{cases} \lambda_1' - c\kappa_1 = 0, \\ \lambda_2' + \lambda_1\kappa_1 = 0. \end{cases} \tag{3.29}$$

By (3.29), we get

$$\begin{cases} \lambda_1(s) = -c\int \kappa_2(s)ds, \\ \lambda_2(s) = -c\int \kappa_2(\int \kappa_2(s)ds)ds. \end{cases} \tag{3.30}$$

Using (3.30), then the axis $V$ is as

$$V = -c\int \kappa_2(s)dsT_1(s) + cN_1(s) - c\int \kappa_2(\int \kappa_2(s)ds)dsN_2(s), \tag{3.31}$$

where $c \in R$. Using (2.8) in the differentiation of (3.31) gives $V'(s) = 0$. Hence, $V$ is a fixed direction.

Consequently, every pseudo null curve according to the Bishop frame with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is a $k$-type $\{k = 0,1,2\}$ pseudo null slant helix.

**Corollary 3.7.** An axis of the 0-type null Cartan slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is given by

$$V = cT_1 + (c_1 - c\int \kappa_1(s)ds)N_2,$$

where $c, c_1 \in R$.

**Corollary 3.8.** An axis of the 1-type null Cartan slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is given by

$$V = cN_2(s),$$

where $c \in R$.

**Corollary 3.9.** An axis of the 2-type null Cartan slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is given by

$$V = -c\int \kappa_1(s)dsT_1(s) + cN_1(s) - c\int \kappa_1(\int \kappa_1(s)ds)dsN_2(s),$$

where $c \in R$.

**Corollary 3.10.** The causal character of the axis $V$ of the 0-type pseudo null slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is either spacelike or null.

**Corollary 3.11.** The causal character of the axis $V$ of the 1-type pseudo null slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is null.

**Corollary 3.12.** The causal character of the axis $V$ of the 2-type pseudo null slant helix $\gamma$ in $E_1^3$ with the Bishop curvatures $\kappa_1(s) = c_0 e^{\int \tau ds}$ and $\kappa_2(s) = 0$ is

**(i)** spacelike if

$$\{\int \kappa_1(s)ds\}^2 - 2\int \kappa_1(\int \kappa_1(s)ds)ds > 0,$$

**(ii)** timelike if

$$\{\int \kappa_1(s)ds\}^2 - 2\int \kappa_1(\int \kappa_1(s)ds)ds < 0,$$

**(iii)** null if

$$\{\int \kappa_1(s)ds\}^2 - 2\int \kappa_1(\int \kappa_1(s)ds)ds = 0.$$

**Example 3.2.** Let us consider a pseudo null curve in $E_1^3$ given by (Figure 2)

$$\gamma(s) = (\frac{s^3}{3} + \frac{s^2}{2}, \frac{s^3}{3} + \frac{s^2}{2}, s).$$

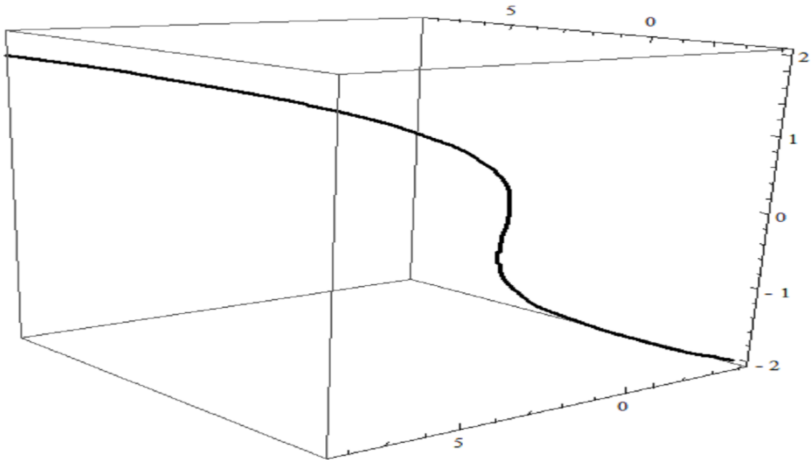


**Figure 2.** The *k*-type pseudo null slant helix.

According to the statement (ii) of Theorem 2.1, the Bishop curvatures of $\gamma$ are

$$\kappa_1(s) = c_0(2s+1), \quad \kappa_2(s) = 0.$$

The Bishop frame of $\gamma$ is computed as

$$\begin{aligned} T_1 &= (s^2+s, s^2+s, 1), \\ N_1 &= (\frac{1}{c_0}, \frac{1}{c_0}, 0), \\ N_2 &= c_0(-\frac{(s^2+s)^2+1}{2}, \frac{1-(s^2+s)^2}{2}, -s^2-s). \end{aligned}$$

By Theorem 3.2, the pseudo null curve $\gamma$ holds 0,1,2-type slant helices whose axes are, respectively, calculated as follows:

$$\begin{aligned} V &= c(s^2+s, s^2+s, 1) + (c_1c_0 - cc_0^2s^2 + cc_0^2s)(-\frac{(s^2+s)^2+1}{2}, \frac{1-(s^2+s)^2}{2}, -s^2-s), \\ V &= cc_0(-\frac{(s^2+s)^2+1}{2}, \frac{1-(s^2+s)^2}{2}, -s^2-s), \\ V &= -(cc_0s^2 + cc_0s)(s^2+s, s^2+s, 1) \\ &\quad + (\frac{c}{c_0}, \frac{c}{c_0}, 0) - (\frac{4cc_0^3s^3}{3} + cc_0^3s^2)(-\frac{(s^2+s)^2+1}{2}, \frac{1-(s^2+s)^2}{2}, -s^2-s), \end{aligned}$$

all of these axes satisfy the Eqs. (3.17), (3.22), and (3.26).

## 4. Conclusion

In this study, we examine *k*-type pseudo null slant helices due to the Bishop frame given by Grbovic and Nešovic [15] under two different cases. We show that every pseudo null curve framed by the Bishop frame is a $k-$type pseudo null slant helix. We find parameter equation of axis $V$ of all *k*-type pseudo null slant helices in terms of the Bishop frame's vector fields. Finally, we determine the causal characters of the axes in two possible cases.

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## Conflict of interest

All authors declare that there is no conflict of interest in this paper.

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