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Review

Hybrid fuzzy differential equations

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Abstract: In this paper we study the existence of the solution for a class of hybrid differential equations with fuzzy initial value. The some new results of generalized division are proposed and applied.

Keywords: hybrid differential equation; fuzzy initial value; generalized division; fuzzy solution
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1. Introduction

In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. We call such differential equations hybrid differential equations. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [6–8]. Dhage and Lakshmikantham [7] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right] = g(t,u(t)) \quad t \in J \\ u(t_0) = u_0 \in \mathbb{R} \end{cases}$$

where, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$ They established the existence results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison result.

From the above works, we develop the theory of hybrid differential equations with fuzzy initial condition [2–4] involving their compact and convex level-cuts and generated division.

As we can see, a key point in our investigation is played by the division concepts for fuzzy numbers. A recent very promising concept, the G-division proposed by [14] is studied here in detail. We observe that this division has a great advantage over peer concepts, namely that it always exists. We obtain relatively simple expressions, a minimality property and a characterization for the G-division.

It is well-known that the usual division between two fuzzy numbers exists only under very restrictive conditions [11]. The g-division (introduced in [14]) of two fuzzy numbers exists under much less restrictive conditions, however it does not always exist [14]. The G-division proposed in [14] overcomes these shortcomings of the above discussed concepts and the G-division of two fuzzy numbers always exists.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $\mathbb{R}_{\mathcal{F}} = \{u : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties [10]:

(i) u is normal i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex i.e., for $x, y \in \mathbb{R}$ and $0 < \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$$

(iii) u is upper semicontinuous i.e.,

(iv) $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}$, then from (i) to (iv) it follows that the α-cuts sets $[u]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which we denote by $[u]^\alpha = [u_1^\alpha, u_2^\alpha]$.

Where $P_K(\mathbb{R})$ denote the family of all nonempty compact convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual. The property of the fuzzy numbers is that the α-cuts $[u]^\alpha$ are closed sets for all $\alpha \in [0,1]$.

Definition 1. *[10, 12] We represent an arbitrary fuzzy number by an ordered pair of functions* $[u]^\alpha = [u_1^\alpha, u_2^\alpha], \quad \alpha \in [0,1]$ *which satisfy the following requirements:*

(a) u_1^α *is abounded monotonic nondecreasing left-continuous function* $\forall \alpha \in]0,1]$, *and right-continuous for* $\alpha = 0$

(b) u_2^α *is abounded monotonic nonincreasing left-continuous function* $\forall \alpha \in]0,1]$, *and right-continuous for* $\alpha = 0$

(c) $u_1^\alpha \leq u_2^\alpha$, $0 \leq \alpha \leq 1$

Theorem 1. *[10] Let* $u \in \mathbb{R}_{\mathcal{F}}$ *and denote* $C_\alpha = [u]^\alpha$ *for* $\alpha \in [0,1]$. *Then*

1. C_α *is a nonempty compact convex set in* $\mathbb{R}$ *for each* $\alpha \in [0,1]$
2. $C_\beta \subseteq C_\alpha$ *for* $0 < \alpha \leq \beta \leq 1$ *then*
3. $C_\alpha = \bigcap_{i=1}^\infty C_{\alpha_i}$, *for any nondecreasing sequence* $\alpha_i \to \alpha$ *on* $[0,1]$

A trapezoidal fuzzy number, denoted by $u = \langle a, b, c, d \rangle$, where $a \leq b \leq c \leq d$, has α-cuts $[u]^\alpha = [a + \alpha(b-a), d - \alpha(d-c)]$, $\alpha \in [0,1]$ obtaining a triangular fuzzy number if $b = c$.

The addition $u + v$ and the scalar multiplication ku are defined as having the level cuts $[u+v]^\alpha = [u_\alpha^1 + v_\alpha^1, u_\alpha^2 + v_\alpha^2]$

$$k[u]^\alpha = \begin{cases} [ku_1^\alpha, ku_2^\alpha] & k \geq 0 \\ [ku_2^\alpha, ku_1^\alpha] & k < 0 \end{cases}$$

$$[uv] = [\min\{u_1^\alpha.v_1^\alpha, u_1^\alpha.v_2^\alpha, u_2^\alpha.v_1^\alpha, u_2^\alpha.v_2^\alpha\}, \max\{u_1^\alpha.v_1^\alpha, u_1^\alpha.v_2^\alpha, u_2^\alpha.v_1^\alpha, u_2^\alpha.v_2^\alpha\}]$$

$$[u \div v]^\alpha = \Big[\min\Big\{\frac{u_1^\alpha}{v_1^\alpha}, \frac{u_1^\alpha}{v_2^\alpha}, \frac{u_2^\alpha}{v_1^\alpha}, \frac{u_2^\alpha}{v_2^\alpha}\Big\}, \max\Big\{\frac{u_1^\alpha}{v_1^\alpha}, \frac{u_1^\alpha}{v_2^\alpha}, \frac{u_2^\alpha}{v_1^\alpha}, \frac{u_2^\alpha}{v_2^\alpha}\Big\}\Big]$$

For a real interval $J = [0, T]$, a mapping $u : J \to \mathbb{R}_\mathcal{F}$ is called a fuzzy function. We denote $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$, for $t \in J$ and $\alpha \in [0, 1]$. the derivative $u'(t)$ of a fuzzy function u is defined by [13]

$$[u'(t)]^\alpha = [(u_1^\alpha)'(t), (u_2^\alpha)'(t)]$$

The integral $\int_a^b u(t)dt,\ a, b \in J$, is defined by [9]

$$\Big[\int_a^b u(t)dt\Big]^\alpha = \Big[\int_a^b u_1^\alpha(t)dt, \int_a^b u_2^\alpha(t)dt\Big]$$

provided that the Lebesgue integrals on the right exist.

3. Generalized division of fuzzy numbers

Definition 2. *Given two fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$ the generalized division (g-division for short) is the fuzzy number w, if it exists, such that*

$$[u]^\alpha \div_g [v]^\alpha = [w]^\alpha \Leftrightarrow \begin{cases} (i) & [u]^\alpha = [v]^\alpha [w]^\alpha \\ or\ (ii) & [v]^\alpha = [u]^\alpha ([w]^\alpha)^{-1} \end{cases} \tag{3.1}$$

here $([w]^\alpha)^{-1} = [1/w_2^\alpha, 1/w_1^\alpha]$

provided that w is a proper fuzzy number, where the multiplications between intervals are performed in the standard interval arithmetic setting.

The fuzzy g-division $\div_g$ is well defined if the α-cuts $[w]^\alpha$ are such that $w \in \mathbb{R}_\mathcal{F}$ (w_1^α is nondecreasing, w_2^α nonincreasing, $w_1^1 \le w_2^1$).

Clearly, if $u \div_g v \in \mathbb{R}_\mathcal{F}$ exists, it has the properties already illustrated for the interval case.

Proposition 1. *[14] Let $u, v \in \mathbb{R}_\mathcal{F}$ (here 1 is the same as $\{1\}$). We have:*

1. *if $0 \notin [u]^\alpha\ \forall\alpha$, then $u \div_g u = 1$,*
2. *if $0 \notin [v]^\alpha\ \forall\alpha$, then $uv \div_g v = u$,*
3. *if $0 \notin [v]^\alpha\ \forall\alpha$, then $1 \div_g v = v^{-1}$ and $1 \div_g v^{-1} = v$*
4. *if $v \div_g u$ exists then either $u(v \div_g u) = v$ or $u(v \div_g u)^{-1} = u$ and both equalities hold if and only if $v \div_g u$ is crisp*

It is easy to see that if $w = u \div_g v$ exists according to case (i) then also $z = u \oslash v$ of [1] exists and $w = z$; but the existence of $v \div_g u$ according to case (ii) is not al lowed for $u \oslash v$.

In the fuzzy case, it is possible that the g-division of two fuzzy numbers does not exist. For example we can consider a triangular fuzzy number $u = (1, 1.5, 5)$ and $v = (-4, -2, -1)$ level-wise, the g-divisions exist but the resulting intervals are not the α-cuts of a fuzzy number.

To solve this shortcoming, in [14] a new division between fuzzy numbers was proposed, a division that always exists.

Definition 3. *The generalized division (G-division for short) of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ and $0 \notin [v]^\alpha$ $\forall\alpha \in [0,1]$, is given by its level sets as*

$$[u \div_G v]^\alpha = cl \bigcup_{\beta \geq \alpha} ([u]^\beta \div_g [v]^\beta), \qquad \forall \alpha \in [0,1] \tag{3.2}$$

where the g-division $\div_g$ is with interval operands $[u]^\beta$ and $[v]^\beta$.

Remark 1.
- *By [14] the g-division exist but the resulting intervals are not the α-cuts of fuzzy number, applying G-division* (3.2) *we obtain the fuzzy number.*
- *$w = u \div_G v$ can be considered as a generalized division of fuzzy numbers, existing for any u, v with $0 \notin [v]^\alpha$ for all $\alpha \in [0,1]$.*
- *If A is G-division, then is G_i-division (g-division satisfies (i)) or is G_{ii}-division (g-division satisfies (ii)).*

Proposition 2. *The G-division* (3.2) *is given by the expression*

$$[u \div_G v]^\alpha = \Big[\inf_{\beta \geq \alpha} \min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}, \sup_{\beta \geq \alpha} \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}\Big]. \tag{3.3}$$

Proof. Let $\alpha \in [0,1]$ be fixed . We observe that for any $\beta \geq \alpha$, $0 \notin [v_1^0, v_2^0]$ we have

$$\begin{aligned}
[u]^\beta \div_g [v]^\beta &= [\min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}, \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}] \\
&\subseteq \Big[\inf_{\lambda \geq \beta} \min\Big\{\frac{u_1^\lambda}{v_1^\lambda}, \frac{u_1^\lambda}{v_2^\lambda}, \frac{u_2^\lambda}{v_1^\lambda}, \frac{u_2^\lambda}{v_2^\lambda}\Big\}, \sup_{\lambda \geq \beta} \max\Big\{\frac{u_1^\lambda}{v_1^\lambda}, \frac{u_1^\lambda}{v_2^\lambda}, \frac{u_2^\lambda}{v_1^\lambda}, \frac{u_2^\lambda}{v_2^\lambda}\Big\}\Big]
\end{aligned}$$

and it follows that

$$cl \bigcup_{\beta \geq \alpha} ([u]^\beta \div_g [v]^\beta) \subseteq \Big[\inf_{\beta \geq \alpha} \min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}, \sup_{\beta \geq \alpha} \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}\Big].$$

Let us consider now

$$cl \bigcup_{\beta \geq \alpha} ([u]^\beta \div_g [v]^\beta) = cl \bigcup_{\beta \geq \alpha} \Big[\min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}, \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}\Big].$$

For any $n \geq 1$, there exist $a_n \in \{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta} | \beta \geq \alpha\}$ such that

$$\inf_{\beta \geq \alpha} \min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\} > a_n - \frac{1}{n}$$

Also there exist $b_n \in \Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta} | \beta \geq \alpha\Big\}$ such that

$$\sup_{\beta \geq \alpha} \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\} < b_n + \frac{1}{n}$$

We have $[a_n, b_n] \subseteq cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta), \forall n \geq 1$ and we obtain

$$(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n) \subseteq \bigcup_{n\geq1}[a_n, b_n] \subseteq cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta)$$

and finally

$$\Big[\inf_{\beta\geq\alpha} \min\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\}, \sup_{\beta\geq\alpha} \max\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\}\Big] \subseteq cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta)$$

The conclusion

$$\Big[\inf_{\beta\geq\alpha} \min\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\}, \sup_{\beta\geq\alpha} \max\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\}\Big] = cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta)$$

of the proposition follows. □

The following proposition gives a simplified notation for $u \div_G v$ and $v \div_G u$.

Proposition 3. *For any two fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$ the two G-divisions $u \div_G v$ and $v \div_G u$ exist and, for any $\alpha \in [0, 1]$, we have $u \div_G v = (v \div_G u)^{-1}$ with $0 \notin [v]^\beta$,$0 \notin [u]^\beta$ and*

$$[u \div_G v]^\alpha = [d_1^\alpha, d_2^\alpha] \quad \textit{and} \quad [v \div_g u]^\alpha = \Big[\frac{1}{d_2^\alpha}, \frac{1}{d_1^\alpha}\Big]$$

where

$$d_\alpha^1 = \inf(D_\alpha) \qquad d_\alpha^2 = \sup(D_\alpha)$$

and the sets D_α are

$$D_\alpha = \Big\{\frac{u_1^\beta}{v_1^\beta} | \beta \geq \alpha\Big\} \cup \Big\{\frac{u_1^\beta}{v_2^\beta} | \beta \geq \alpha\Big\} \cup \Big\{\frac{u_2^\beta}{v_1^\beta} | \beta \geq \alpha\Big\} \cup \Big\{\frac{u_2^\beta}{v_2^\beta} | \beta \geq \alpha\Big\}. \tag{3.4}$$

Proof. Consider a fixed $\alpha \in [0, 1]$. Clearly, using Proposition (3.2)

$$\begin{aligned} [u \div_G v]^\alpha &= \Big[\inf_{\beta\geq\alpha} \min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}, \sup_{\beta\geq\alpha} \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\}\Big] \\ &\subseteq [\inf(D_\alpha), \sup(D_\alpha)] = [d_1^\alpha, d_2^\alpha] \end{aligned}$$

Vice versa, for all $n \geq 1$ and from the definition of d_1^α and d_2^α there exist $a_n, b_n \in D_\alpha$ such that

$$d_1^\alpha \leq a_n < d_1^\alpha + \frac{1}{n}, \qquad d_2^\alpha - \frac{1}{n} < b_n \leq d_2^\alpha$$

and the following limits exist:

$$\lim a_n = d_1^\alpha, \qquad \lim b_n = d_2^\alpha,$$

on the other hand,[$[a_n, b_n] \subseteq cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta), \forall n \geq 1$ and then

$$\bigcup_{n\geq1}[a_n, b_n] \subseteq cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta)$$

It follows that

$$[d_1^\alpha, d_2^\alpha] = [\lim a_n, \lim b_n] \subseteq \bigcup_{n\geq1}[a_n, b_n] \subseteq cl \bigcup_{\beta\geq\alpha}([u]^\beta \div_g [v]^\beta)$$

and the proof is complete. □

Remark 2. *We observe that there are other possible different expressions for the G-division as e.g.,*

$$[u \div_G v]^\alpha = \Big[\min\Big\{\inf_{\beta\geq\alpha}(\frac{u_1^\beta}{v_1^\beta}), \inf_{\beta\geq\alpha}(\frac{u_1^\beta}{v_2^\beta}), \inf_{\beta\geq\alpha}(\frac{u_2^\beta}{v_1^\beta}), \inf_{\beta\geq\alpha}(\frac{u_2^\beta}{v_2^\beta})\Big\},$$

$$\max\Big\{\sup_{\beta\geq\alpha}(\frac{u_1^\beta}{v_1^\beta}), \sup_{\beta\geq\alpha}(\frac{u_1^\beta}{v_2^\beta}), \sup_{\beta\geq\alpha}(\frac{u_2^\beta}{v_1^\beta}), \sup_{\beta\geq\alpha}(\frac{u_2^\beta}{v_2^\beta})\Big\}\Big].$$

The next proposition shows that the G-division is well defined for any two fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$.

Proposition 4. *[14] For any fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$, $0 \notin [v_1^0, v_2^0]$ the G-division $u \div_G v$ exists and it is a fuzzy number.*

Proof. We regard the fuzzy quantity $u \div_G v$ Then according to the previous result, if we denote $w_1 = (u \div_G v)_1$ and $w_2 = (u \div_G v)_2$ with $0 \notin [v]^\alpha$, $\forall\alpha \in [0, 1]$ we have

$$w_1^\alpha = \inf_{\beta\geq\alpha} \min\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\} \leq w_2^\alpha = \sup_{\beta\geq\alpha} \max\Big\{\frac{u_1^\beta}{v_1^\beta}, \frac{u_1^\beta}{v_2^\beta}, \frac{u_2^\beta}{v_1^\beta}, \frac{u_2^\beta}{v_2^\beta}\Big\}$$

Obviously w_1 is bounded and decreasing inverse while w_2 is bounded. Also w_1, w_2 are left continuous on $(0, 1]$, since $\frac{u_1}{v_1}, \frac{u_1}{v_2}, \frac{u_2}{v_1}, \frac{u_2}{v_2}$ are left continuous on $(0, 1]$ and they are right continuous at 0 since so are the functions $\frac{u_1}{v_1}, \frac{u_1}{v_2}, \frac{u_2}{v_1}, \frac{u_2}{v_2}$ □

Proposition 5. *Let $u, v \in \mathbb{R}_\mathcal{F}$ (here 1 is the same as $\{1\}$). We have:*

1. *$u \div_G v = u \div_g v$, if $0 \notin [v]^\alpha$ $\forall\alpha \in [0, 1]$ whenever the expression on the right exists; in particular $u \div_G u = 1$ if $0 \notin [u]^\alpha$ $\forall\alpha$*
2. *$(uv) \div_G v = u$,*
3. *$1 \div_G v = v^{-1}$ and $1 \div_G v^{-1} = v$*
4. *if $0 \notin [u]^\alpha$ and $0 \notin [v]^\alpha$ $\forall\alpha \in [0, 1]$ then $1 \div_g (v \div_g u) = u \div_g v$*
5. *$v \div_G u = u \div_G v = w$ if and only if $w = w^{-1}$, furthermore, $w = 1$ if and only if $u = v$.*

Proof. The proof of (1) is immediate.

For (2) we can use (1), Indeed in this case $uv \div_g u$ existe and we have

$$uv \div_G v = uv \div_g v = u$$

The proof of (3) is immediate.

The proof of (4) follows from(3.4) for all $\alpha \in [0, 1]$

$$\begin{aligned}
[1 \div_g (v \div_g u)]^\alpha &= [1, 1] \div_g [d_1^\alpha, d_2^\alpha] \\
&= [\min\{\frac{1}{d_1^\alpha}, \frac{1}{d_2^\alpha}\}, \max\{\frac{1}{d_1^\alpha}, \frac{1}{d_2^\alpha}\}] \\
&= [\frac{1}{d_2^\alpha}, \frac{1}{d_1^\alpha}] \\
&= \Big[1/\max\Big\{\frac{v_1^\alpha}{u_1^\alpha}, \frac{v_1^\alpha}{u_2^\alpha}, \frac{v_2^\beta}{u_1^\alpha}, \frac{v_2^\alpha}{u_2^\alpha}\Big\}, 1/\min\Big\{\frac{v_1^\alpha}{u_1^\alpha}, \frac{v_1^\alpha}{u_2^\alpha}, \frac{v_2^\alpha}{u_1^\alpha}, \frac{v_2^\beta}{u_2^\alpha}\Big\}\Big]
\end{aligned}$$

$$= \Big[\min\Big\{\frac{u_1^\alpha}{v_1^\alpha}, \frac{u_1^\alpha}{v_2^\alpha}, \frac{u_2^\alpha}{v_1^\alpha}, \frac{u_2^\alpha}{v_2^\alpha}\Big\}, \max\Big\{\frac{u_1^\alpha}{v_1^\alpha}, \frac{u_1^\alpha}{v_2^\alpha}, \frac{u_2^\alpha}{v_1^\alpha}, \frac{u_2^\alpha}{v_2^\alpha}\Big\}\Big]$$
$$= [u \div_g v]^\alpha$$

To prove(5), consider again (3.4); for all $\alpha \in [0,1]$ we have $[w]^\alpha = [v \div_G u]^\alpha = [d_1^\alpha, d_2^\alpha]$ and $[w]^\alpha = [u \div_G v]^\alpha = [\frac{1}{d_2^\alpha}, \frac{1}{d_1^\alpha}]$ so that $w = w^{-1}$ and vice versa, the last part of (5) follows from the last part of (1) and the fact that $w = w^{-1} = 1$ if and only if $d_1^\alpha = d_2^\alpha$ for all $\alpha \in [0,1]$ this is true if and only if $\frac{u_1^\alpha}{v_1^\alpha} = 1$ and $\frac{u_2^\alpha}{v_2^\alpha} = 1$ i. e., $u_1^\alpha = v_1^\alpha$ and $u_2^\alpha = v_2^\alpha$ for all $\alpha \in [0,1]$. □

4. Some results for hybrid differential equations

Were call the result which establishes the existence of solution for first order hybrid differential equation (in short HDE) with initial condition. This result will be useful in the study of the corresponding fuzzy problem.

We consider the initial value problem

$$\begin{cases} \frac{d}{dt}\Big[\frac{u(t)}{f(t,u(t))}\Big] = g(t,u(t)) & t \in J \\ u(t_0) = u_0 \in \mathbb{R} \end{cases} \tag{4.1}$$

where, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.
By a solution of the HDE (4.1) we mean a function $u \in AC(J, \mathbb{R})$ such that

(i) the function $t \mapsto \frac{u}{f(t,u)}$ is absolutely continuous for each $u \in \mathbb{R}$, and
(ii) u satisfies the equations in (4.1),

where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions defined on $J = [0, T]$.

Theorem 2. *[5] Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) A is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function ψ,
(b) B is completely continuous,
(c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$, and
(d) $M\psi(r) < r$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Then the operator equation $AxBx = x$ has a solution in S.

We consider the following hypotheses in what follows.
(A_0) The function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.
(A_1) There exists a constant $L > 0$ such that

$$| f(t,x) - f(t,y) | \le L \,| x - y | \tag{4.2}$$

for all $t \in J$ and $x, y \in \mathbb{R}$.
(A_2) There exists a function $h \in L^1(J, \mathbb{R})$ such that

$$| g(t,x) | \le h(t) \qquad t \in J$$

In the following section, we consider a fuzzy differential equation which is a fuzzy analogue to (4.1).

5. Hybrid fuzzy differential equations

We shall consider the initial value problem,

$$\frac{d}{dt}\Big[\frac{u(t)}{f(t,u(t))}\Big] = g(t,u(t)) \quad t \in J \tag{5.1}$$
$$u(0) = u_0 \in \mathbb{R}_{\mathcal{F}}$$

The extension principle of Zadeh leads to the following definition of $f(t,u)$ and $g(t,u)$ when are a fuzzy numbers

$$f(t,u)(y) = \sup\{u(x) : y = f(t,x), \quad x \in \mathbb{R},$$
$$g(t,u)(y) = \sup\{u(x) : y = g(t,x), \quad x \in \mathbb{R}.$$

It follows that

$$[f(t,u)]^\alpha = \Big[\min\{f(t,x) : x \in [u_1^\alpha, u_2^\alpha]\},\ \max\{f(t,x) : x \in [u_1^\alpha, u_2^\alpha]\}\Big],$$
$$[g(t,u)]^\alpha = \Big[\min\{g(t,x) : x \in [u_1^\alpha, u_2^\alpha]\},\ \max\{g(t,x) : x \in [u_1^\alpha, u_2^\alpha]\}\Big]$$

for $u \in \mathbb{R}_{\mathcal{F}}$ with α-level sets $[u]^\alpha = [u_1^\alpha, u_2^\alpha],\ 0 < \alpha \leq 1$. We call $u : J \to \mathbb{R}_{\mathcal{F}}$ a fuzzy solution of (5.1), if

$$\Big[\frac{d}{dt}\Big[u(t) \div_G f(t,u(t))\Big]\Big]^\alpha = \Big[g(t,u(t))\Big]^\alpha \text{ and } [u(0)]^\alpha = [u_0]^\alpha \tag{5.2}$$

for all $t \in J$ and $\alpha \in [0,1]$. Denote $\widetilde{f} = (f_1, f_2)$ and $\widetilde{g} = (g_1, g_2)$,

$$f_1(t,u) = \min\{f(t,x) : x \in [u_1,u_2]\},\ f_2(t,u) = \max\{f(t,x) : x \in [u_1,u_2]\}\ \textit{and}$$
$$g_1(t,u) = \min\{g(t,x) : x \in [u_1,u_2]\},\ g_2(t,u) = \max\{g(t,x) : x \in [u_1,u_2]\}\ (\textit{resp}),$$

where $u = (u_1, u_2) \in \mathbb{R}^2$. Thus for fixed α we have an initial value problem in $\mathbb{R}^2$

$$\begin{aligned} \frac{d}{dt}\Big[\frac{u_1^\alpha(t)}{\widetilde{f}(t,u_1^\alpha(t),u_2^\alpha(t))}\Big] &= \widetilde{g}(t,u_1^\alpha(t),u_2^\alpha(t)) \\ u_1^\alpha(0) &= u_{01}^\alpha \\ \text{and}& \\ \frac{d}{dt}\Big[\frac{u_2^\alpha(t)}{\widetilde{f}(t,u_1^\alpha(t),u_2^\alpha(t))}\Big] &= \widetilde{g}(t,u_1^\alpha(t),u_2^\alpha(t)) \\ u_2^\alpha(0) &= u_{02}^\alpha \end{aligned} \tag{5.3}$$

If we can solve it (uniquely) we have only to verify that the intervals $[u_1^\alpha(t), u_2^\alpha(t)]$, $\alpha \in [0,1]$, define a fuzzy number $u(t)$ in $\mathbb{R}_{\mathcal{F}}$. see [2–4]. Since f and g are assumed continue and Caratheodory (resp), the initial value problem (5.3) is equivalent to the following nonlinear hybrid integral equation (HIE)

$$u(t) = \widetilde{f}(t,u(t))\Big(\frac{u_0}{\widetilde{f}(0,u(0)} + \int_0^t \widetilde{g}(s,u(s))ds\Big) \tag{5.4}$$

Theorem 3. *Assume sign($u(0)$) = sign($u(t)$), for all $t \in J$, let $z(t) = u \div_G f(t, u(t))$ and $0 \notin [f(t, u)]^\alpha$ $\alpha \in [0, 1]$ and*

$$r(t) = f(t, u(t)) \div_G \Big(z(0) + \int_0^t g(s, u(s))ds\Big), \quad 0 \notin \Big[z(0) + \int_0^t g(s, u(s))\Big]^\alpha$$

1. *If $z(0) * g(t, u) > 0$ then the function $u(t) \in AC((0, T], \mathbb{R}_{\mathcal{F}})$ is a fuzzy solution of* (5.1)
   - *If $z(t)$ is G_i-division*
   - *or $z(t)$ is G_i-division and $r(t)$ is G_i-division*
2. *Or if $z(t)$ is G_i-division and $z_1^\alpha(0) \le 0 \le z_2^\alpha(0)$ then $u(t)$ is a fuzzy solution.*

Proof. We solve initial value problem in $\mathbb{R}^2$

$$\begin{aligned} \frac{d}{dt} z_1^\alpha &= \min\{g(t, x) : x \in [u_1^\alpha(t), u_2^\alpha(t)]\},\ u_1^\alpha(0) = u_{01}^\alpha \\ \frac{d}{dt} z_2^\alpha &= \max\{g(t, x) : x \in [u_1^\alpha(t), u_2^\alpha(t)]\},\ u_2^\alpha(0) = u_{02}^\alpha \end{aligned} \tag{5.5}$$

Step 1 :

It can be assume that (4.2) implies

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|, \quad \text{for all } t \in J, \quad x, y \in \mathbb{R} \tag{5.6}$$

where the $\|.\|$ is defined by $\|u\| = \max\{|u_1|, |u_2|$. It is well known that (5.6) and the assumptions on g Theorem 2 guarantee the existence and continuous dependence on initial of a solution to

$$\begin{cases} \frac{d}{dt}\Big[\frac{u(t)}{\widetilde{f}(t,u(t))}\Big] = \widetilde{g}(t, u(t)), \\ u(0) = u_0 \end{cases} \tag{5.7}$$

and that for any continuous function $u_0 \in \mathbb{R}^2$ we have (5.4).

By choosing $u_0 = (u_{01}^\alpha, u_{02}^\alpha)$ in (5.7) we get a solution $u^\alpha(t) = (u_1^\alpha(t), u_2^\alpha(t))$ to (5) for all $\alpha \in (0, 1]$.

Step 2 :

We will show that the intervals $[u_1^\alpha(t), u_2^\alpha(t)]$, $\alpha \in [0, 1]$, define a fuzzy number $u(t) \in \mathbb{R}_{\mathcal{F}}$. For simplicity assume $[u(0)]^\alpha \le 0$, $[f(t, u(t))]^\alpha > 0$ and $[g(t, u(t))]^\alpha < 0$ for all $\alpha \in [0, 1]$ (The proof for other cases is similar and omitted), then wa have tow cases.

Case I:

By Eq. (5.3), we have the two following HDE with initial conditions

$$\begin{cases} \frac{d}{dt}\Big[\frac{u_1^\alpha(t)}{f_2^\alpha(t,u(t))}\Big] = g_1^\alpha(t, u_(t)) \\ u_1^\alpha(0) = u_{01}^\alpha \end{cases} \tag{5.8}$$

and

$$\begin{cases} \frac{d}{dt}\Big[\frac{u_2^\alpha(t)}{f_1^\alpha(t,u(t))}\Big] = g_2^\alpha(t, u(t)) \\ u_2^\alpha(0) = u_{02}^\alpha \end{cases} \tag{5.9}$$

In consequence by Step 1, we deduce that, for every $\alpha \in [0,1]$, the solution to problems (5.8)–(5.9) are respectively

$$\begin{aligned} u_1^\alpha(t) &= f_2^\alpha(t,u(t))\Big[\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \int_0^t g_2^\alpha(s,u(s))ds\Big] \\ u_2^\alpha(t) &= f_1^\alpha(t,u(t))\Big[\frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \int_0^t g_1^\alpha(s,u(s))ds\Big] \end{aligned}$$

We check that $\{[u_1^\alpha(t), u_2^\alpha(t)],\ \alpha \in [0,1]\}$ represent the level set of a fuzzy set $u(t)$ in $\mathbb{R}_{\mathcal{F}}$, for each $t \in J$ fixed, by applying the stacking Theorem 1. Indeed, we fix $t \in J$ and check the validity of the three conditions.
(1) : First, we check that $u_1^\alpha(t) \leq u_2^\alpha(t)$, for every $\alpha \in [0,1]$ and $t \in J$, Indeed, for each $\alpha \in [0,1]$ and $t \in J$ we have that $f_1^\alpha(t,u(t)) \leq f_2^\alpha(t,u(t))$ and

$$\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \int_0^t g_1^\alpha(s,u(s))ds \leq \frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \int_0^t g_2^\alpha(s,u(s))ds$$

and by classical arithmetic we have

$$\begin{aligned} u_1^\alpha(t) &= f_2^\alpha(t,u(t))\Big[\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \int_0^t g_1^\alpha(s,u(s))ds\Big] \\ &\leq f_1^\alpha(t,u(t))\Big[\frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \int_0^t g_2^\alpha(s,u(s))ds\Big] = u_2^\alpha(t) \end{aligned}$$

(2) : Let $0 \leq \alpha \leq \beta \leq 1$. Since $u_0 \in \mathbb{R}_{\mathcal{F}}$, we have that $f_2^\beta(t,u(t)) \leq f_2^\alpha(t,u(t))$ and

$$\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \int_0^t g_1^\alpha(s,u(s))ds \leq \frac{u_1^\beta(0)}{f_2^\beta(0,u(0))} + \int_0^t g_1^\beta(s,u(s))ds$$

we deduce that

$$\begin{aligned} u_1^\alpha(t) &= f_2^\alpha(t,u(t))\Big[\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))} + \int_0^t g_1^\alpha(s,u(s))ds\Big] \\ &\leq f_2^\beta(t,u(t))\Big[\frac{u_1^\beta(0)}{f_2^\beta(0,u(0))} + \int_0^t g_1^\beta(s,u(s))ds\Big] = u_1^\beta(t) \end{aligned}$$

and, similarly, $f_1^\alpha(t,u(t)) \leq f_1^\beta(t,u(t))$ and

$$\frac{u_2^\beta(0)}{f_1^\beta(0,u(0))} + \int_0^t g_2^\beta(s,u(s))ds \leq \frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))} + \int_0^t g_2^\alpha(s,u(s))ds$$

so

$$u_2^\beta(t) = f_1^\beta(t,u(t))\Big[\frac{u_2^\beta(0)}{f_1^\beta(0,u(0))} + \int_0^t g_2^\beta(s,u(s))ds\Big]$$

$$\leq \quad f_1^\alpha(t,u(t))\Big[\frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))}+\int_0^t g_2^\alpha(s,u(s))ds\Big]=u_2^\alpha(t)$$

which proves that $[u_1^\beta(t),u_2^\beta(t)]\subseteq[u_1^\alpha(t),u_2^\alpha(t)]$.
(3) : Given a nondecreasing sequence $\{\alpha_i\}$ in $(0,1]$ such that $\alpha_i\uparrow\alpha\in(0,1]$, we prove that $[u_1^\alpha(t),u_2^\alpha(t)]=\bigcap_{i=1}^\infty[u_1^{\alpha_i}(t),u_2^{\alpha_i}(t)]$.

Indeed, by the Dominated Convergence Theorem,

$$\lim_{\alpha_i\uparrow\alpha}\int_0^t g_1^{\alpha_i}(s,u(s))=\int_0^t\lim_{\alpha_i\uparrow\alpha}g_1^{\alpha_i}(s,u(s))ds=\int_0^t g_1^\alpha(s,u(s))ds$$

and, hence,

$$\begin{aligned}\lim_{\alpha_i\uparrow\alpha}u_1^{\alpha_i}(t) &= \lim_{\alpha_i\uparrow\alpha}\Big(f_2^{\alpha_i}(t,u(t))\Big[\frac{u_1^{\alpha_i}(0)}{f_2^{\alpha_i}(0,u(0))}+\int_0^t g_1^{\alpha_i}(s,u(s))ds\Big]\Big)\\ &= \lim_{\alpha_i\uparrow\alpha}\Big(f_2^{\alpha}(t,u(t))\Big[\frac{u_1^{\alpha}(0)}{f_2^{\alpha}(0,u(0))}+\int_0^t g_1^{\alpha}(s,u(s))ds\Big]\Big)=u_1^\alpha(t)\end{aligned}$$

Hence, $u(t)\in\mathbb{R}_{\mathcal{F}}$.

Case II: By Eq. (5.3), we have the two following HDE with initial conditions

$$\begin{cases}\frac{d}{dt}\Big[\frac{u_2^\alpha(t)}{f_1^\alpha(t,u(t))}\Big]=g_1^\alpha(t,u_(t))\\ u_2^\alpha(0)=u_{02}^\alpha\end{cases}\tag{5.10}$$

and

$$\begin{cases}\frac{d}{dt}\Big[\frac{u_1^\alpha(t)}{f_2^\alpha(t,u(t))}\Big]=g_2^\alpha(t,u(t))\\ u_1^\alpha(0)=u_{01}^\alpha\end{cases}\tag{5.11}$$

The solution to problems (5.10)–(5.11) are respectively

$$\begin{aligned}u_2^\alpha(t) &= f_1^\alpha(t,u(t))\Big[\frac{u_2^\alpha(0)}{f_1^\alpha(0,u(0))}+\int_0^t g_1^\alpha(s,u(s))ds\Big]\\ u_1^\alpha(t) &= f_2^\alpha(t,u(t))\Big[\frac{u_1^\alpha(0)}{f_2^\alpha(0,u(0))}+\int_0^t g_2^\alpha(s,u(s))ds\Big]\end{aligned}$$

By applying step 1 and we consider the situation where $0\notin[z(0)+\int_0^t\widetilde{g}(s,u(s))ds]^\alpha$,

$$\frac{f_2^\alpha(t,u(t))}{z_1(0)+\int_0^t g_1^\alpha(s,u(s))ds}\leq\frac{f_1^\alpha(t,u(t))}{z_2(0)+\int_0^t g_2^\alpha(s,u(s))ds}\tag{5.12}$$

i.e., $(u_1^\alpha(t)\leq u_2^\alpha(t))$ similarly by applying theorem 1 the details for the case I are analogous, and if the situation (5.12) does not hold i.e., $(u_2^\alpha(t)\leq u_1^\alpha(t))$, then by theorem 1, $u(t)$ is not a fuzzy solution of (5.7).

□

6. Conclusions and further work

We have investigated generalized division concepts for fuzzy number. The G-division introduced here is a very general division concept, being also practically applicable. Developed the theory of hybrid differential equation with fuzzy condition involving their compact and convex level-cuts. The next step in the research direction proposed here is to investigate hybrid fuzzy fractional differential equations with G-division and applications.

Conflict of interest

The authors declare no conflict of interest.

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