Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 259–272.
DOI:10.3934/math.2020017
Received: 11 August 2019
Accepted: 23 October 2019
Published: 07 November 2019

Research article

Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces

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Abstract: We introduce a more general class of fractional-order boundary value problems involving the Caputo-Hadamard fractional derivative. Existence results for the given problem are established by applying the Mönch's fixed point theorem and the technique of measures of noncompactness. an example is given to illustrate our results. The boundary conditions introduced in this work are of quite general nature and reduce to many special cases by fixing the parameters involved in the conditions.

Keywords: fractional differential equation; fractional integral conditions; Caputo-Hadamard fractional derivative; Kuratowski measures of noncompactness; Mönch fixed point theorems; Banach space
Mathematics Subject Classification: 26A33, 34A60

1. Introduction

In the last few decades, the investigation of fractional differential equations has been picking up much attention of researchers. This is due to the fact that fractional differential equations have various applications in engineering and scientific disciplines, for example, fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panel in supersonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nonlinear oscillation of earthquake, models of population growth, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, nuclear reactors and theory of population dynamics. The fractional differential equation is an important tool to describe the memory and hereditary properties of various materials and phenomena. The details on the theory and its applications may be found in books [35, 38, 40, 42] and references therein.

It has also been many subjects in fractional calculus that have been developed in various fields, from pure mathematical theory to applied sciences such as modeling of heat transfer in heterogeneous media [43], modeling of ultracapacitor and beams heating [25], etc. These applications are mainly due to the fact that many physical systems are related to fractional-order dynamics and their behaviors are governed by fractional differential equations (FDEs) [39]. The significant importance of using FDEs describes the non-local property [31], which means the current state and all its previous states affect the next state of a dynamical system. We remind that an essential issue about fractional calculus problems is difficult in obtaining analytical solutions. Therefore, numerical and approximation methods are commonly proposed to obtain approximate solutions for this kind of problems,e.g., [8–10, 28, 32, 33, 41].

Recently, fractional-order differential equations equipped with a variety of boundary conditions have been studied. The literature on the topic includes the existence and uniqueness results related to classical, initial value problem, periodic/anti-periodic, nonlocal, multi-point, integral boundary conditions, and Integral Fractional Boundary Condition, for instance, the monographs of Ahmed et al. [4], Benchohra et al. [13], W, Benhamida et al. [16], D. Chergui et al. [23], Chen et al. [24], Goodrich et al. [29] and Zhang et al. [47].

On the other hand, the nonlocal problem has been studied by many authors. The existence of a solution for abstract Cauchy differential equations with nonlocal conditions in a Banach space has been considered first by Byszewski [19]. In physical science, the nonlocal condition may be connected with better effect in applications than the classical initial condition since nonlocal conditions are normally more exact for physical estimations than the classical initial condition. For the study of nonlocal problems, we refer to [20–22, 26, 27, 29, 47] and references given therein.

This paper deals with the existence of solutions to the boundary value problem for fractional-order differential equations:

$$^{C}D^{r}x(t) = f(t, x(t)),\ t \in J := [1, T],\ 0 < r \leq 1, \tag{1.1}$$

with fractional boundary condition:

$$\alpha x(1) + \beta x(T) = \lambda I^{q} x(\eta) + \delta, \qquad q \in (0, 1]. \tag{1.2}$$

where D^r is the Caputo-Hadamard fractional derivative, $0 < r < 1$, $0 < q \leq 1$, and let E be a Banach space space with norm $\|.\|$, $f : J \times E \to E$ is given continuous function and satisfying some assumptions that will be specified later. α, β, λ are real constants, and $\eta \in (1, T)$, $\delta \in E$.

In this paper, we present existence results for the problem (1.1)-(1.2) using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. that technique turns out to be a very useful tool in existence for several types of integral equations; details are found in A. Aghajani et al. [3], Akhmerov et al. [5], Alvàrez [6], Banaŝ et al. [11, 12], Benchohra et al. [14, 15], Guo et al. [30], Mönch [37], Szufla [44]. We can use a numerical method to solve the problem in Equation (1.1-1.2), for instance, see [8–10, 28, 32, 33, 41].

The organization of this work is as follows. In Section 2, we introduce some notations, definitions, and lemmas that will be used later. Section 3 treats the existence of solutions in Banach spaces. In Section 4, we illustrate the obtained results by an example. Finally, the paper concludes with some interesting observations in Section 5.

2. Preliminaires

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel. For more details, we refer to [1, 2, 5, 11, 35, 36, 42, 44].
Denote by $C(J, E)$ the Banach space of continuous functions $x : J \to E$, with the usual supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\|, t \in J\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $x : J \to E$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_J |x(t)|\, dt.$$

Let the space

$$AC^n_\delta([a, b], E) = \left\{h : [a, b] \to \mathbb{R} : \delta^{n-1}h(t) \in AC([a, b], E)\right\}.$$

where $\delta = t\frac{d}{dt}$ is the Hadamard derivative and $AC([a, b], E)$ is the space of absolutely continuous functions on $[a, b]$.

Now, we give some results and properties of fractional calculus.

Definition 2.1. (Hadamard fractional integral) (see [35])
The left-sided fractional integral of order $\alpha > 0$ of a function $y : (a, b) \to \mathbb{R}$ is given by

$$I^\alpha_{a^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s} \tag{2.1}$$

provided the right integral converges.

Definition 2.2. (Hadamard fractional derivative) (see [35])
The left-sided Hadamard fractional derivative of order $\alpha \geq 0$ of a continuous function $y : (a, b) \to \mathbb{R}$ is given by

$$\begin{aligned} D^\alpha_{a^+} f(t) &= \delta^n I^{n-\alpha}_{a^+} y(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s)\frac{ds}{s} \end{aligned} \tag{2.2}$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α and $\delta = t\frac{d}{dt}$.
provided the right integral converges.

There is a recent generalization introduced by Jarad and al in [34], where the authors define the generalization of the Hadamard fractional derivatives and present properties of such derivatives. This new generalization is now known as the Caputo-Hadamard fractional derivatives and is given by the following definition:

Definition 2.3. (Caputo-Hadamard fractional derivative) (see [34, 46])
Let $\alpha = 0$, and $n = [\alpha] + 1$. If $y(x) \in AC^n_\delta[a, b]$, where $0 < a < b < \infty$ and

$$AC_{\delta}^{n}([a,b],E)=\left\{h:[a,b]\to\mathbb{R}:\delta^{n-1}h(t)\in AC([a,b],E)\right\}.$$

The left-sided Caputo-type modification of left-Hadamard fractional derivatives of order α is given by

$$^{C}D_{a^+}^{\alpha}y(t)=D_{a^+}^{\alpha}\left(y(t)-\sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}(\log\frac{t}{s})^k\right) \tag{2.3}$$

Theorem 2.4. *(See [34])*
Let $\alpha\geq 0$, and $n=[\alpha]+1$. If $y(t)\in AC_{\delta}^{n}[a,b]$, where $0<a<b<\infty$. Then $^{C}D_{a^+}^{\alpha}f(t)$ exist everywhere on $[a,b]$ and
(i) if $\alpha\notin\mathbb{N}-\{0\}$, $^{C}D_{a^+}^{\alpha}f(t)$ can be represented by

$$\begin{aligned}^{C}D_{a^+}^{\alpha}y(t)&=I_{a^+}^{n-\alpha}\delta^n y(t)\\&=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n y(s)\frac{ds}{s}\end{aligned} \tag{2.4}$$

(ii) if $\alpha\in\mathbb{N}-\{0\}$, then

$$^{C}D_{a^+}^{\alpha}y(t)=\delta^n y(t) \tag{2.5}$$

In particular

$$^{C}D_{a^+}^{0}y(t)=y(t) \tag{2.6}$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing a by 0 in formula (2.4) provided that $y(t)\in AC_{\delta}^{n}(\mathbb{R}^+)$. Thus one has

$$^{C}D_{a^+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n y(s)\frac{ds}{s} \tag{2.7}$$

Proposition 2.5. *(see [34, 35])*
Let $\alpha>0,\beta>0,n=[\alpha]+1$, and $a>0$, then

$$\begin{aligned}I_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}(x)&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}\\^{C}D_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}(x)&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1},\beta>n,\\^{C}D_{a^+}^{\alpha}\left(\log\frac{t}{a}\right)^{k}&=0,k=0,1,...,n-1.\end{aligned} \tag{2.8}$$

Theorem 2.6. *(see [45])*
Let $y(t)\in AC_{\delta}^{n}[a,b], 0<a<b<\infty$ and $\alpha\geq 0,\beta\geq 0$, Then

$$\begin{aligned}^{C}D_{a^+}^{\alpha}\left(I_{a^+}^{\alpha}y\right)(t)&=\left(I_{a^+}^{\beta-\alpha}y\right)(t),\\^{C}D_{a^+}^{\alpha}\left(^{C}D_{a^+}^{\beta}y\right)(t)&=\left(^{C}D_{a^+}^{\alpha+\beta}y\right)(t).\end{aligned} \tag{2.9}$$

Lemma 2.7. *(see [34])*
Let $\alpha \geq 0$*, and* $n = [\alpha] + 1$*. If* $y(t) \in AC_{\delta}^{n}[a, b]$*, then the Caputo-Hadamard fractional differential equation*

$$^{C}D_{a^+}^{\alpha} y(t) = 0 \tag{2.10}$$

has a solution:

$$y(t) = \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k \tag{2.11}$$

and the following formula holds:

$$I_{a^+}^{\alpha} \left({}^{C}D_{a^+}^{\alpha} y\right)(t) = y(t) + \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a}\right)^k \tag{2.12}$$

where $c_k \in \mathbb{R}, k = 1, 2, ..., n - 1$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.8. ([5, 11]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0, \infty]$ defined by

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

This measure of noncompactness satisfies some important properties [5, 11]:
(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\mu(B) = \mu(\overline{B})$.
(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
(d) $\mu(A + B) \leq \mu(A) + \mu(B)$
(e) $\mu(cB) = |c|\mu(B); c \in \mathbb{R}$.
(f) $\mu(convB) = \mu(B)$.
Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively. The details of μ and its properties can be found in ([5, 11]).

Definition 2.9. A map $f : J \times E \to E$ is said to be Caratheodory if
(i) $t \mapsto f(t, u)$ is measurable for each $u \in E$;
(ii) $u \mapsto F(t, u)$ is continuous for almost all $t \in J$.

Notation 2.10. for a given set V of functions $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, t \in J,$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.11. *([2, 37, 44]) Let D be a bounded, closed and convex subset of a Banach space such that* $0 \in D$*, and let N be a continuous mapping of D into itself. If the implication*
$V = \overline{conv}N(V)$ *or* $V = N(V) \cup 0 \Rightarrow \mu(V) = 0$
holds for every subset V of D, then N has a fixed point.

Lemma 2.12. *([44]) Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$, G a continuous function on $J \times J$ and f a function from $J \times E \longrightarrow E$ which satisfies the Caratheodory conditions, and suppose there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h\to 0^+} \mu(f(J_{t,h} \times B)) \leq p(t)\mu(B); \text{ here } J_{t,h} = [t-h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t)f(s,y(s))ds : y \in V\right\}\right) \leq \int_J \|G(t,s)\|p(s)\mu(V(s))ds.$$

3. Main results

This section is devoted to the existence results for problem (1.1)-(1.2).

Definition 3.1. A function $x \in AC^1_\delta(J, E)$ is said to be a solution of the problem (1.1)-(1.2) if x satisfies the equation ${}^C D^r x(t) = f(t, x(t))$ on J, and the conditions (1.2).

For the existence of solutions for the problem (1.1)-(1.2), we need the following auxiliary lemma.

Lemma 3.2. *Let $h : [1, T) \to E$ be a continuous function. A function x is a solution of the fractional integral equation*

$$x(t) = I^r h(t) + \frac{1}{\Lambda}\{\lambda I^{r+q}h(\eta) - \beta I^r h(T) + \delta\} \tag{3.1}$$

if and only if x is a solution of the fractional BVP

$${}^C D^r x(t) = h(t), \quad t \in J, \quad r \in (0,1] \tag{3.2}$$

$$\alpha x(1) + \beta x(T) = \lambda I^q x(\eta) + \delta, q \in (0,1] \tag{3.3}$$

Proof. Assume x satisfies (3.2). Then Lemma 2.8 implies that

$$x(t) = I^r h(t) + c_1. \tag{3.4}$$

The condition (3.3) implies that

$$x(1) = c_1$$
$$x(T) = I^r h(T) + c_1$$
$$I^q x(1) = I^{r+q}h(\eta)) + c_1 \frac{(\log \eta)^q}{\Gamma(q+1)}$$

So

$$\alpha c_1 + \beta I^r h(T) + \beta c_1 = \lambda I^{r+q}h(\eta)) + c_1 \frac{\lambda(\log \eta)^q}{\Gamma(q+1)} + \delta$$

Thus,

$$c_1\left(\alpha + \beta - \frac{\lambda(\log \eta)^q}{\Gamma(q+1)}\right) = \lambda I^{r+q}h(\eta)) - \beta I^r h(T) + \delta.$$

Consequently,

$$c_1 = \frac{1}{\Lambda}\left\{\lambda I^{r+q}h(\eta)) - \beta I^r h(T) + \delta\right\}.$$

Where,

$$\Lambda = \left(\alpha + \beta - \frac{\lambda(\log\eta)^q}{\Gamma(q+1)}\right)$$

Finally, we obtain the solution (3.1)

$$x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q}h(\eta) - \beta I^r h(T) + \delta\right\}$$

□

In the following,we prove existence results, for the boundary value problem (1.1)-(1.2) by using a Mönch fixed point theorem.
(H1) $f : J \times E \to E$ satisfies the Caratheodory conditions;
(H2) There exists $p \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$, such that,

$$\|f(t, x)\| \leq p(t)\|x\|, \text{ for } t \in J \text{ and each } x \in E;$$

(H3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h\to 0^+} \mu(f(J_{t,h} \times B)) \leq p(t)\mu(B); \quad here \quad J_{t,h} = [t-h, t] \cap J.$$

Theorem 3.3. *Assume that conditions (H1)-(H3) hold. Let* $p^* = \sup_{t\in J} p(t)$*. If*

$$p^* M < 1 \tag{3.5}$$

With

$$M := \left\{\frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}$$

then the BVP (1.1)-(1.2) has at least one solution.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $\mathfrak{F} : C(J, E) \to C(J, E)$ defined by

$$\mathfrak{F}x(t) = I^r h(t) + \frac{1}{\Lambda}\left\{\lambda I^{r+q}h(\eta) - \beta I^r h(T) + \delta\right\} \tag{3.6}$$

Clearly, the fixed points of the operator $\mathfrak{F}$ are solutions of the problem (1.1)-(1.2). Let

$$R \geq \frac{|\delta|}{|\Lambda|(1 - p^* M)}. \tag{3.7}$$

and consider

$$D = \{x \in C(J, E) : \|x\| \leq R\}.$$

Clearly, the subset D is closed, bounded and convex. We shall show that $\mathfrak{F}$ satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps. □

Step 1: First we show that $\mathfrak{F}$ is continuous:
Let x_n be a sequence such that $x_n \to x$ in $C(J, E)$. Then for each $t \in J$,

$$\begin{aligned}
\|(\mathfrak{F}x_n)(t) - (\mathfrak{F}x)(t)\| &\leq \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \|f(s, x_n(s)) - f(s, x(s))\| \frac{ds}{s} \\
&+ \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} \|f(s, x_n(s)) - f(s, x(s))\| \frac{ds}{s} \\
&+ \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} \|f(s, x_n(s)) - f(s, x(s))\| \frac{ds}{s} \\
&\leq \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} \|f(s, x_n(s)) - f(s, x(s))\|
\end{aligned}$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|\mathfrak{F}(x_n) - \mathfrak{F}(x)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: Second we show that $\mathfrak{F}$ maps D into itself :
Take $x \in D$, by (H2), we have, for each $t \in J$ and assume that $\mathfrak{F}x(t) \neq 0$.

$$\begin{aligned}
\|(\mathfrak{F}x)(t)\| &\leq \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \|f(s, x(s))\| \frac{ds}{s} + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} \|f(s, x(s))\| \frac{ds}{s} \\
&+ \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} \|f(s, x(s))\| \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
&\leq \frac{1}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} p(s)\|x(s)\| \frac{ds}{s} + \frac{|\lambda|}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} p(s)\|x(s)\| \frac{ds}{s} \\
&+ \frac{|\beta|}{|\Lambda|\Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} p(s)\|x(s)\| \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
&\leq \frac{P^*R}{\Gamma(r)} \int_1^t (\log \frac{t}{s})^{r-1} \frac{ds}{s} + \frac{|\lambda|P^*R}{|\Lambda|\Gamma(r+q)} \int_1^\eta (\log \frac{\eta}{s})^{r+q-1} \frac{ds}{s} \\
&+ \frac{|\beta|P^*R}{|\Lambda|\Gamma(r)} \int_1^T (\log \frac{T}{s})^{r-1} \frac{ds}{s} + \frac{|\delta|}{|\Lambda|} \\
&\leq P^*R \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} + \frac{|\delta|}{|\Lambda|} \\
&\leq P^*RM + \frac{|\delta|}{|\Lambda|} \\
&\leq R.
\end{aligned}$$

Step 3: we show that $\mathfrak{F}(D)$ is equicontinuous :
By Step 2, it is obvious that $\mathfrak{F}(D) \subset C(J, E)$ is bounded. For the equicontinuity of $\mathfrak{F}(D)$, let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in D$, so $\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1) \neq 0$. Then

$$\|\mathfrak{F}x(t_2) - \mathfrak{F}x(t_1)\| \leq \frac{1}{\Gamma(r)} \int_1^{t_1} \left[(\log \frac{t_2}{s})^{r-1} - (\log \frac{t_1}{s})^{r-1} \right] \|f(s, x(s))\| \frac{ds}{s}$$

$$+\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}\|f(s,x(s))\|\frac{ds}{s}$$
$$\le\frac{R}{\Gamma(r)}\int_1^{t_1}\left[(\log\frac{t_2}{s})^{r-1}-(\log\frac{t_1}{s})^{r-1}\right]p(s)\frac{ds}{s}+\frac{R}{\Gamma(r)}\int_{t_1}^{t_2}(\log\frac{t_2}{s})^{r-1}p(s)\frac{ds}{s}$$
$$\le\frac{Rp^*}{\Gamma(r+1)}\left[(\log t_2)^r-(\log t_1)^r\right].$$

As $t_1\to t_2$, the right hand side of the above inequality tends to zero.
Hence $N(D)\subset D$.

Finally we show that the implication holds:
Let $V\subset D$ such that $V=\overline{conv}(\mathfrak{F}(V)\cup\{0\})$. Since V is bounded and equicontinuous, and therefore the function $v\to v(t)=\mu(V(t))$ is continuous on J. By assumption (H2), and the properties of the measure μ we have for each $t\in J$.

$$\begin{aligned}
v(t) &\le \mu(\mathfrak{F}(V)(t)\cup\{0\}))\le\mu((\mathfrak{F}V)(t))\\
&\le\frac{1}{\Gamma(r)}\int_1^t(\log\frac{t}{s})^{r-1}p(s)\mu(V(s))\frac{ds}{s}+\frac{|\lambda|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}p(s)\mu(V(s))\frac{ds}{s}\\
&+\frac{|\beta|}{|\Lambda|\Gamma(r)}\int_1^T(\log\frac{T}{s})^{r-1}p(s)\mu(V(s))\frac{ds}{s}\\
&\le\frac{\|v\|}{\Gamma(r)}\int_1^t(\log\frac{t}{s})^{r-1}p(s)\frac{ds}{s}+\frac{|\lambda|\|v\|}{|\Lambda|\Gamma(r+q)}\int_1^\eta(\log\frac{\eta}{s})^{r+q-1}p(s)\frac{ds}{s}\\
&+\frac{|\beta|\|v\|}{|\Lambda|\Gamma(r)}\int_1^T(\log\frac{T}{s})^{r-1}p(s)\frac{ds}{s}\\
&\le p^*\|v\|\left\{\frac{(\log T)^r}{\Gamma(r+1)}+\frac{|\lambda|(\log\eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)}+\frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)}\right\}\\
&:=p^*\|v\|M.
\end{aligned}$$

This means that

$$\|v\|(1-p^*M)\le 0$$

By (3.5) it follows that $\|v\|=0$, that is $v(t)=0$ for each $t\in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem 2.11, we conclude that $\mathfrak{F}$ has a fixed point which is a solution of the problem (1.1)-(1.2).

4. Example

Let

$$E=l^1=\{x=(x_1,x_2,...,x_n,...):\sum_{n=1}^{\infty}|x_n|<\infty\}$$

with the norm

$$\|x\|_E=\sum_{n=1}^{\infty}|x_n|$$

We consider the problem for Caputo-Hadamard fractional differential equations of the form:

$$\begin{cases} D^{\frac{2}{3}}x(t) = f(t, x(t)), (t, x) \in ([1, e], E), \\ x(1) + x(e) = \frac{1}{2}\left(I^{\frac{1}{2}}x(2)\right) + \frac{3}{4}. \end{cases} \tag{4.1}$$

Here $r = \frac{2}{3}$, $q = \frac{1}{2}$, $\delta = \frac{3}{4}$, $\lambda = \frac{1}{2}$, $\eta = 2$, $T = e$.
With

$$f(t, y(t)) = \frac{t\sqrt{\pi} - 1}{16} y(t), t \in [1, e]$$

Clearly, the function f is continuous. For each $x \in \mathbb{R}^+$ and $t \in [1, e]$, we have

$$|f(t, x(t))| \leq \frac{t\sqrt{\pi}}{16}|x|$$

Hence, the hypothesis (H2) is satisfied with $p^* = \frac{t\sqrt{\pi}}{16}$. We shall show that condition (3.5) holds with $T = e$. Indeed,

$$p^* \left\{ \frac{(\log T)^r}{\Gamma(r+1)} + \frac{|\lambda|(\log \eta)^{r+q}}{|\Lambda|\Gamma(r+q+1)} + \frac{|\beta|(\log T)^r}{|\Lambda|\Gamma(r+1)} \right\} \simeq 0.6109 < 1$$

Simple computations show that all conditions of Theorem 3.3 are satisfied. It follows that the problem (4.1) has at least solution defined on $[1, e]$.

5. Conclusion

In this paper, we obtained some existence results of nonlinear Caputo-Hadamard fractional differential equations with three-point boundary conditions by using a method involving a measure of noncompactness and a fixed point theorem of Mönch type. Though the technique applied to establish the existence results for the problem at hand is a standard one, yet its exposition in the present framework is new. An illustration to the present work is also given by presenting some examples. Our results are quite general give rise to many new cases by assigning different values to the parameters involved in the problem. For an explanation, we enlist some special cases.

- We remark that when $\lambda = 0$, problem (1.1)-(1.2), the boundary conditions take the form: $\alpha x(1) + \beta x(T) = \delta$ and the resulting problem corresponds to the one considered in [17, 18].
- If we take $\alpha = q = 1$, $\beta = 0$, in (1.2), then our results correspond to the case integral boundary conditions take the form: $x(1) = \lambda \int_1^e x(s)ds + \delta$ considered in [7].
- By fixing $\alpha = 1, \beta = \lambda = 0$, in (1.2), our results correspond to the ones for initial value problem take the form:$x(1) = \delta$.
- In case we choose $\alpha = \beta = 1$, $\lambda = \delta = 0$, in (1.2), our results correspond to periodic/anti-periodic type boundary conditions take the form: $x(1) = -(\beta/\alpha)x(T)$. In particular, we have the results for anti-periodic type boundary conditions when $(\beta/\alpha) = 1$. For more details on anti-periodic fractional order boundary value problems, see [4].
- Letting $\alpha = 1, \beta = \delta = 0$, in (1.2), then our results correspond to the case fractional integral boundary conditions take the form:$x(1) = \lambda I^q x(\eta)$.

- When, $\alpha = \beta = 1$, $\delta = 0$, in (1.2), our results correspond to fractional integral and anti-periodic type boundary conditions.

In the nutshell, the boundary value problem studied in this paper is of fairly general nature and covers a variety of special cases and we can use a numerical method to solve the problem in equation (1.1-1.2). The possible generalization is to consider the problem (1.1-1.2) on Banach space with another technique, other fixed point theorem and determine the conditions that befit closer to obtain the best results. As another proposal, considering some type of fractional derivative (Hilfer-Hadamard, Hilfer-Katugampola) with respect to another function. we will use the numerical method to solve this problem. These suggestions will be treated in the future.

Acknowledgments

The authors would like to express their thanks and are grateful to the referees for their helpful comments and suggestions.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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