Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 249–258.
DOI:10.3934/math.2020016
Received: 11 May 2019
Accepted: 25 September 2019
Published: 06 November 2019

Research article

Development of analytical solution for a generalized Ambartsumian equation

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Abstract: Based on the conformable derivative, a generalized model of the Ambartsumian equation is analyzed in this paper. The solution is expressed as a power series of arbitrary powers. In addition, the convergence of the obtained series solution is theoretically proven. Furthermore, it is shown that the current series reduces to the corresponding one in the literature as the conformable derivative tends to one. It is also revealed that the obtained results are of acceptable accuracy. It is found that the residuals tend to zero in specific sub-domains. The diagonal Pade approximants are implemented to extend the domain of converge to include the whole domain.

Keywords: Ambartsumian equation; Milky way; series solution
Mathematics Subject Classification: 34A08, 65D15

1. Introduction

In this paper, we consider a generalized model of Ambartsumian delay equation (ADE) in the form:

$$D_t^{\alpha}\phi(t) = \phi(t) + \gamma\, \phi\,(\gamma\, t)\,, \tag{1.1}$$

subjected to

$$\phi(0) = \omega, \tag{1.2}$$

where $\gamma = \frac{1}{\mu}$, $\mu > 1$, and ω is a constant. The system (1.1–1.2) is a generalized form of the standard ADE which describes the surface brightness in the Milky Way [1]. As $\alpha \to 1$, the system (1.1–1.2) was solved by several authors [2–4] using various analytical approaches. It is interesting to point out to that Eq. (1.1) was derived more than 25 years earlier by Ambartsumian [1] to describe the absorption of light by the interstellar matter. Very recently, Patade and Bhalekar [2] derived a power series solution for the system (1.1–1.2), as $\alpha \to 1$, by applying the Daftardar-Gejji and Jafari Method. The convergence issue was also discuued in [2] for all $|\mu| > 1$. As $\alpha \to 1$, the existence and uniqueness for the system (1.1–1.2) were proved and discussed by Kato and McLeod [3]. After that, Bakodah and Ebaid [4] applied Laplace-Transform on the system (1.1–1.2) and they were successfully obtained an exact solution in the special case $\alpha \to 1$. However, the current generalized model (1.1–1.2) was solved in [5] using the homotopy transform analysis method by means of the Caputo's definition. Here, we consider the conformable derivative to deal with the present model. Such conformable derivative was implemented by several authors [6–9] for studying several applications.

Although the generalized model (1.1–1.2) can be solved by any of the familiar methods such as the Adomian decomposition method (ADM) [10–23], the Homotopy perturbation method (HPM) [24–27], and the Homotopy analysis method (HAM) [5], the authors believe that the power series method is a powerful one, especially, in proving the convergence of the obtained power series.

The motivation of this research is to investigate the generalized ADE (1.1–1.2) with the conformable derivative via a reliable approach. In order to achieve this goal, the power series method is suggested and developed in this paper to achieve this task. This is because the developed power series method allows to obtain the solution of Eqs. (1.1–1.2) in a closed-form. However, other approaches such as the ADM, DTM, and HPM give approximate solutions which could not be compacted in a closed-form in several cases. In addition, the main advantage of the closed-form power series solution over the other approximate solutions is that the closed-form one can be theoretically proven for convergence using any of the standard convergence tests. Therefore, the objective of this paper is to analyze the system (1.1–1.2) in view of the conformable derivative (2.1) using the power series approach. Morevoer, we will decalre that the current solution agrees with the corresponding one in the literature as $\alpha \to 1$ (the ordinary case).

2. Power series solution

The conformable derivative of arbitrary order α, $0 < \alpha \leq 1$, of a function $\phi(t) : [0, \infty) \to \mathrm{R}$ is defined as [6–9]

$$D_t^{\alpha}\phi(t) = t^{1-\alpha}\phi'(t). \tag{2.1}$$

Accordingly, Eq. (1.1) becomes

$$t^{1-\alpha}\phi'(t) = \phi(t) + \gamma\, \phi\,(\gamma\, t)\,, \tag{2.2}$$

In this section, we search for a solution of Eq. (2.2) in the following form

$$\phi(t) = \sum_{n=0}^{\infty} a_n(\alpha, \gamma)\, t^{\alpha n}, \tag{2.3}$$

which yields,

$$t^{1-\alpha}\phi'(t) = \sum_{n=0}^{\infty} \alpha(n+1)a_{n+1}(\alpha, \gamma)t^{\alpha n}. \tag{2.4}$$

On inserting Eqs. (2.3–2.4) into Eq. (2.2), we have

$$\sum_{n=0}^{\infty} \alpha(n+1)a_{n+1}(\alpha,\gamma)t^{\alpha n} = -\sum_{n=0}^{\infty} a_n(\alpha,\gamma)t^{\alpha n} + \gamma \sum_{n=0}^{\infty} a_n(\alpha,\gamma)(\gamma t)^{\alpha n}, \tag{2.5}$$

or

$$\sum_{n=0}^{\infty} \alpha(n+1)a_{n+1}(\alpha,\gamma)t^{\alpha n} = \sum_{n=0}^{\infty} \left(\gamma^{\alpha n+1} - 1\right) a_n(\alpha,\gamma)t^{\alpha n}, \tag{2.6}$$

Collecting the like terms of same powers, we obtain

$$\sum_{n=0}^{\infty} \left[\alpha(n+1)a_{n+1}(\alpha,\gamma) - \left(\gamma^{\alpha n+1} - 1\right) a_n(\alpha,\gamma)\right] t^{\alpha n} = 0. \tag{2.7}$$

Hence,

$$a_{n+1}(\alpha,\gamma) = \left(\frac{\gamma^{\alpha n+1} - 1}{\alpha(n+1)}\right) a_n(\alpha,\gamma), \quad n \geq 0. \tag{2.8}$$

Accordingly,

$$\begin{aligned}
a_1 &= \frac{1}{\alpha}(\gamma - 1)\, a_0, \\
a_2 &= \frac{1}{2\alpha^2}(\gamma - 1)\left(\gamma^{\alpha+1} - 1\right) a_0, \\
a_3 &= \frac{1}{6\alpha^3}(\gamma - 1)\left(\gamma^{\alpha+1} - 1\right)\left(\gamma^{2\alpha+1} - 1\right) a_0, \\
a_4 &= \frac{1}{24\alpha^3}(\gamma - 1)\left(\gamma^{\alpha+1} - 1\right)\left(\gamma^{2\alpha+1} - 1\right)\left(\gamma^{3\alpha+1} - 1\right) a_0, \\
&\quad . \\
&\quad . \\
&\quad . \\
a_{n+1} &= \frac{1}{(n+1)!\,\alpha^{n+1}}(\gamma - 1)\left(\gamma^{\alpha+1} - 1\right)\left(\gamma^{2\alpha+1} - 1\right)\ldots\left(\gamma^{\alpha n+1} - 1\right) a_0,
\end{aligned} \tag{2.9}$$

and therefore,

$$a_{n+1}(\alpha,\gamma) = \frac{a_0}{(n+1)!\,\alpha^{n+1}} \prod_{k=0}^{n} \left(\gamma^{k\alpha+1} - 1\right), \quad n \geq 0. \tag{2.10}$$

From (2.3), we have

$$\begin{aligned}
\phi(t) &= a_0 + \sum_{n=0}^{\infty} a_{n+1}(\alpha,\gamma)\, t^{\alpha(n+1)}, \\
&= a_0 + a_0 \sum_{n=0}^{\infty} \frac{t^{\alpha(n+1)}}{(n+1)!\,\alpha^{n+1}} \prod_{k=0}^{n} \left(\gamma^{k\alpha+1} - 1\right), \\
&= a_0\left[1 + \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n} \left(\gamma^{k\alpha+1} - 1\right)\right) \frac{t^{\alpha(n+1)}}{(n+1)!\,\alpha^{n+1}}\right].
\end{aligned} \tag{2.11}$$

On using the initial condition in Eq. (1.2), we finally obtain

$$\phi(t) = \omega\left[1 + \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n}\left(\gamma^{k\alpha+1} - 1\right)\right)\frac{t^{\alpha(n+1)}}{(n+1)!\ \alpha^{n+1}}\right], \tag{2.12}$$

which can be rewritten as

$$\phi(t) = \omega\left[1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\left(\gamma^{(k-1)\alpha+1} - 1\right)\right)\frac{t^{\alpha n}}{n!\ \alpha^{n}}\right]. \tag{2.13}$$

As $\alpha \to 1$, we have from (2.13) that

$$\begin{aligned}\lim_{\alpha\to 1}\phi(t) &= \omega\left[1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\left(\gamma^{k} - 1\right)\right)\frac{t^{n}}{n!}\right],\\ &= \omega\left[1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\left(\frac{1}{\mu^{k}} - 1\right)\right)\frac{t^{n}}{n!}\right]. \qquad (2.14)\end{aligned}$$

Equation (2.14) is the same closed-form solution obtained by Patade and Bhalekar [2] as a special case when $\alpha \to 1$.

3. Convergence analysis

In order to prove the convergence of (2.13), we suppose that

$$c_n = \frac{1}{n!\ \alpha^n}\left(\prod_{k=1}^{n}\left(\gamma^{(k-1)\alpha+1} - 1\right)\right), \tag{3.1}$$

and hence we have the following theorem.

Theorem 1. *The power series* (2.13) *has an infinite radius of convergence for* $|\mu| > 1\ \forall\ \alpha \in (0, 1]$.

Proof. Suppose that ρ is the radius of convergence of the series (2.13). On using the ratio test, we have

$$\begin{aligned}\frac{1}{\rho} &= \lim_{n\to\infty}|\frac{c_{n+1}}{c_n}|,\\ &= \lim_{n\to\infty}|\frac{n!\ \alpha^n}{(n+1)!\ \alpha^{n+1}}\left(\prod_{k=1}^{n+1}\left(\gamma^{(k-1)\alpha+1} - 1\right)\right)\times\left(\prod_{k=1}^{n}\left(\gamma^{(k-1)\alpha+1} - 1\right)\right)^{-1}|,\\ &= \lim_{n\to\infty}|\frac{1}{\alpha(n+1)}\left(\gamma^{\alpha n+1} - 1\right)|,\\ &= \frac{1}{\alpha}\times\lim_{n\to\infty}|\frac{1}{n+1}|\times\lim_{n\to\infty}|\frac{1}{\mu^{\alpha n+1}} - 1|,\\ &= \frac{1}{\alpha}\times 0\times 1, \quad \text{where} \quad \mu > 1\\ &= 0\ \forall\ \alpha \in (0, 1]. \qquad (3.2)\end{aligned}$$

□

4. Results

In order to declare the effectiveness of the present approach and derive some results, the infinite series (2.13) is approximated by taking m term. Hence, the m-term approximate solution is given as

$$\Omega_m(t) = \omega\left[1 + \sum_{n=1}^{m}\left(\prod_{k=1}^{n}\left(\gamma^{(k-1)\alpha+1} - 1\right)\right)\frac{t^{\alpha n}}{n!\,\alpha^n}\right], \qquad m \geq 1. \tag{4.1}$$

In this section we discuss some issues related to the convergence, behavior, and accuracy of the present solution. First of all, it should be noted that the power series (4.1), although converges in the limit as proved by theorem 1, has a certain domain of convergence at selected values for the parameters α, ω, and μ. Regarding, it is indicated here that such domain of convergence depends on the number of terms m taken to approximate the infinite series. For declaration, it is shown in Figure 1 that the approximate solutions $\Omega_{60}(t)$, $\Omega_{70}(t)$, $\Omega_{80}(t)$, and $\Omega_{90}(t)$ have different domains of convergence at $\omega = 1$ and $\mu = 1.4$. Moreover, these approximations are not valid in the whole domain $t \in [0, \infty)$.

However, this difficulty will be solved later in this section by applying the Pade approximants (PA) on the truncated power series (4.1), where each PA converges for $t \in [0, \infty)$. It is also noticed from Figure 1 that the domain of convergence increases as the number of terms increases.

In Figure 2, the 100-term approximate solution $\Omega_{100}(t)$ is depicted versus the order of the derivative α at $\omega = 1$ and $\mu = 1.4$. This figure shows that the curves in the cases of non-integer values α, i.e., $\alpha = 0.7$, $\alpha = 0.8$, and $\alpha = 0.9$, lie above the curve of the standard case, i.e., at $\alpha \to 1$. This means that the surface brightness in the Milky Way is influenced by the arbitrary order α. In addition, the domain of convergence is extended for the non-integer values α when compared with the standard case $\alpha \to 1$.

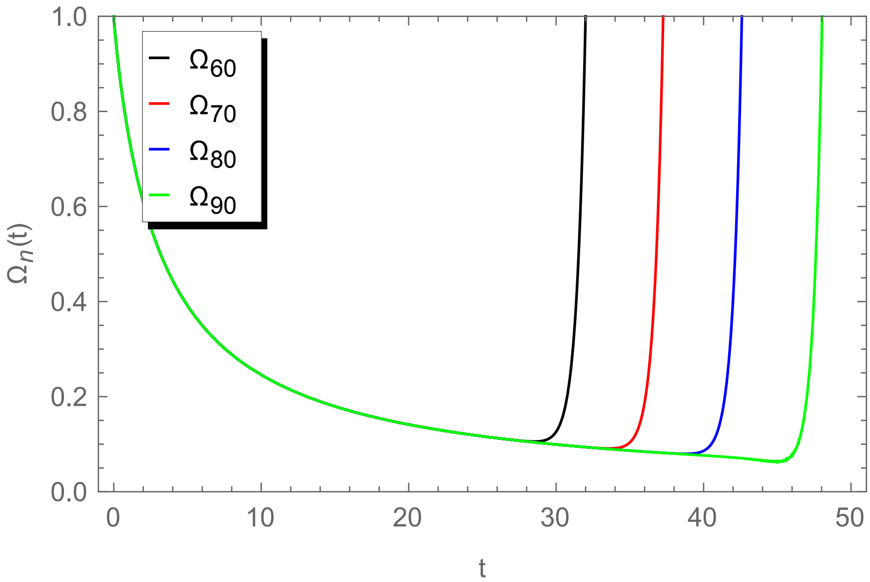

Figure 1. Domain of convergence of the approximate solutions (4.1) at ω =1 and μ =1.4.

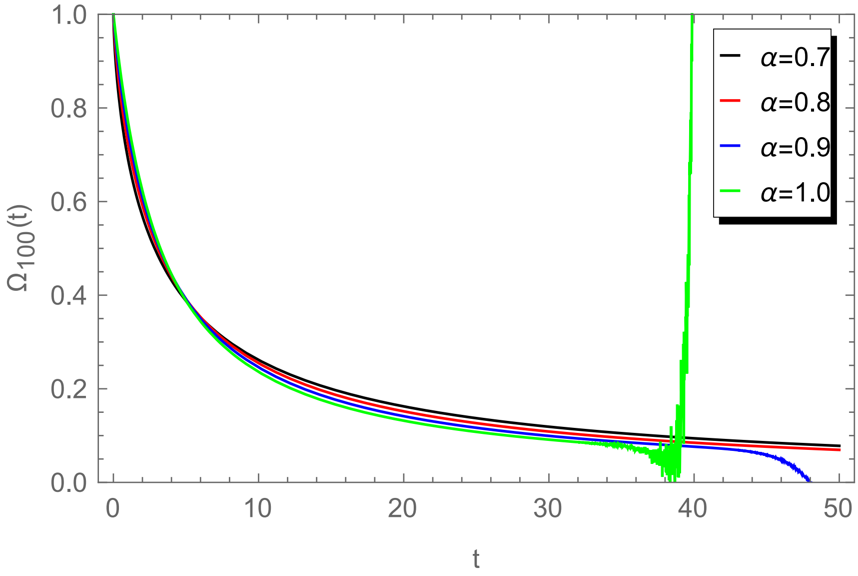

Figure 2. The 100-term approximate solution at different values of α when ω =1 and μ = 1.4.

In order to improving and extending the domain of convergence, the PA can be applied, as usual, on the truncated series (4.1). The diagonal PA is the best to achieve this task. However, such PA can not be applied directly on (4.1) because the independent variable t in (4.1) is not of integer powers. To overcome this problem, the series (4.1) can be expressed in terms of a new variable τ, where $\tau = t^{\alpha}$. Hence, (4.1) becomes

$$\Omega_m(t) = \omega \left[1 + \sum_{n=1}^{m-1} \left(\prod_{k=1}^{n} \left(\gamma^{(k-1)\alpha+1} - 1 \right) \right) \frac{\tau^n}{n!\ \alpha^n} \right]. \tag{4.2}$$

Four different PA, [3/3], [4/4], [5/5], and [6/6] are displayed in Figure 3 in the same domain of Figure 1 and at the same selected values of ω and μ. Really, the the domain of convergence is enhanced even by using a low-order PA. This point can be also confirmed through the results introduced in Figure 4 which represents the PA [6/6] versus α. Here, it should be noted that the system (1.1–1.2) describes a linear delay dynamical system which has been solved in the closed-form (2.13). The system (1.1–1.2) can be also solved via a recent approach introduced by Turkyilmazoglu [28] using the ADM. Such approach was based on adding a predetermined parameter into the ADM formulation which helps in both preventing its divergence and speeding up its convergence. Consequently, via the improved ADM, the convergence region of the series approximation is found to be enlarged to a bigger physical domain. In addition, other approaches such as HAM [29] and HPM or the traditional Taylor series expansion [30] can be effectively applied on the current model. The achievement of the HAM is mainly due to a so-called convergence control parameter plugged into the studied system as shown by Turkyilmazoglu [29]. A simple algorithm to determine such parameter was also discussed by in [29]. Moreover, the techniques introduced by Turkyilmazoglu in [28–30] may be extended to the current model as a future work.

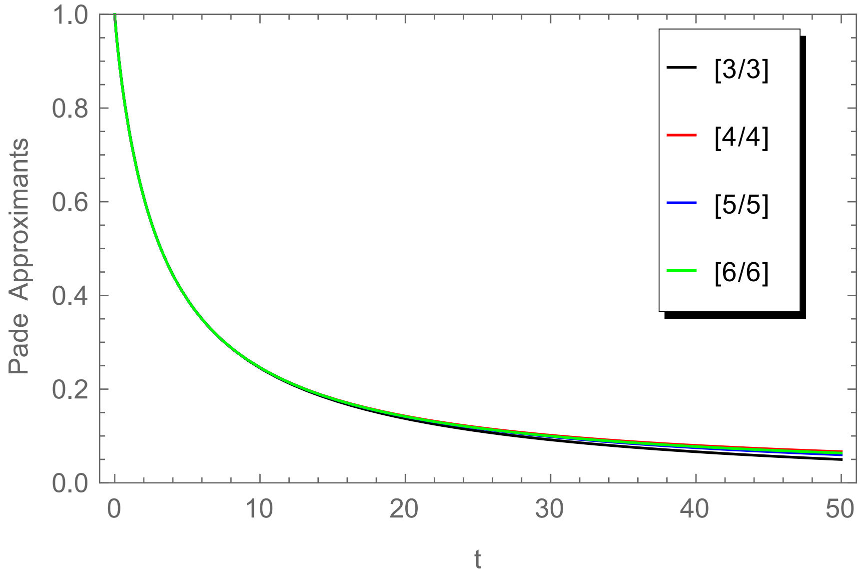

Figure 3. Pade Approximants at ω =1 and μ =1.4.

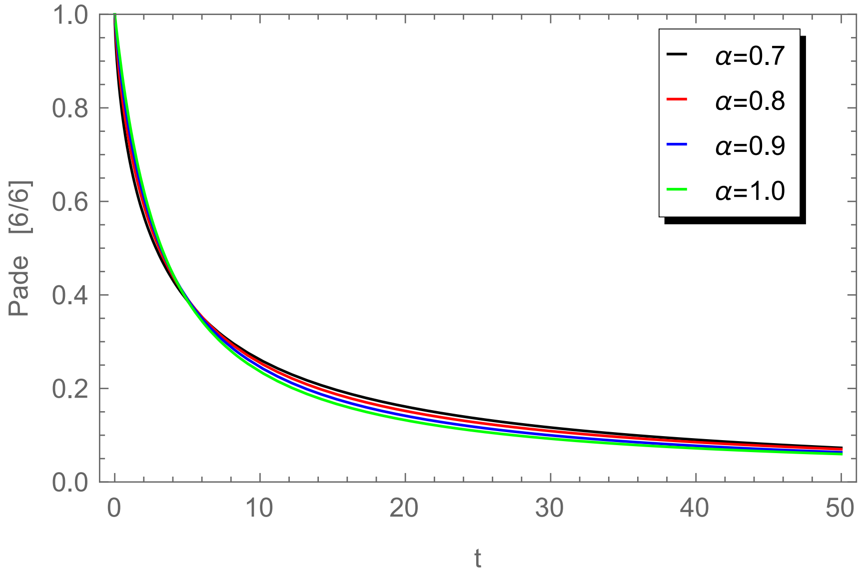

Figure 4. The diagonal Pade [6/6] at ω =1 and μ =1.4.

Regarding the accuracy of the current results, the residual $RE_m(t)$ defined by

$$RE_m(t) = t^{1-\alpha}\, \Omega'_m(t) + \Omega_m(t) - \frac{1}{\mu}\, \Omega_m\left(\frac{t}{\mu}\right), \quad m \geq 2, \tag{4.3}$$

is calculated when $m = 60$ and introduced in Figure 5. It is revealed from this figures that the present results are of acceptable accuracy. In addition, such accuracy can be easily improved by increasing the number of terms m. This proves the efficiency of the current analysis to analyzing the conformable form of the Ambartsumian equation.

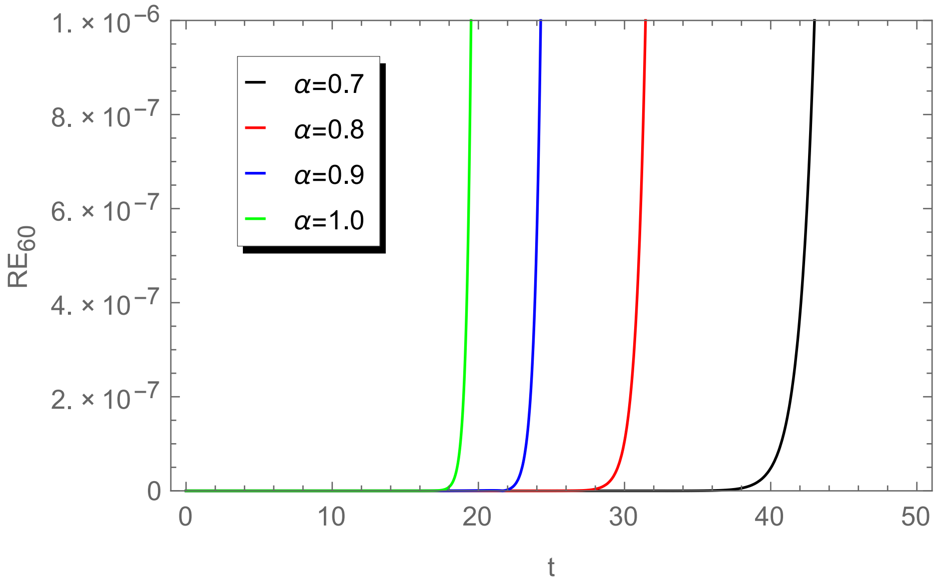

Figure 5. The residual at ω =1 and μ =1.4.

5. Conclusion

In this paper, a generalized model of the Ambartsumian equation was introduced. The power series solution was obtained in a closed-form. The obtained closed-form solution was theoretically proved for convergence using the standard ratio tests. Moreover, it was shown that the present series solution agrees with the corresponding series in the literature as a special case, when the conformable derivative tends to one. The present calculations and results revealed that they were of good accuracy, where the residuals tends to zero in several sub-domains and less than 10^{-6} in the rest of such domains. In order to apply Pade approximants, a simple transformation was implemented to express the obtained power series of arbitrary orders in terms of integer powers. Accordingly, several diagonal Pade approximants were calculated and plotted. Therefore, the domain of converge of the obtained power series of arbitrary orders was extended to include the whole domain by means of the resulting diagonal Pade approximants.

Acknowledgments

This project was supported by the Deanship of Scientific Research at Prince Sattam bin Abdulaziz University under the research project No. 2019/1/10067.

The authors would like to express his gratitude to the anonymous referees for their helpful comments and suggestions, which have greatly improved the paper.

Conflict of interest

The authors declare no conflict of interest.

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