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Research article

Ostrowski type inequalities via the Katugampola fractional integrals

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Abstract: The main aim of this study is to reveal new generalized-Ostrowski-type inequalities using Katugampola fractional integral operator which generalizes Riemann-Liouville and Hadamard fractional integral operators into a single form. For this purpose, at first, a new fractional integral identity is generated by the researchers. Then, by using this identity, some inequalities for the class of functions whose certain powers of absolute values of derivatives are $p-$convex are derived. Some applications to special means for positive real numbers are also given. It is observed that the obtained inequalities are generalizations of some well known results.

Keywords: Katugampola fractional integral; Ostrowski type inequalities; p-convex functions
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1. Introduction

Fractional calculus was first suggested for consideration by Leibnitz in his letter to L'Hospital which dealt with derivatives of order $\alpha = \frac{1}{2}$ (see [10]). Hereupon, this theory has been used in many fields of science such as economics, biology, engineering, physics and mathematics for sure. Many types of fractional derivatives and integrals were studied by Hadamard, Caputo, Riemann-Liouville, Grünwald-Letnikov, etc. Various properties of these operators have been summarized in [9]. For the last decades, this theory has been used in inequality theory frequently because it enables scientists to obtain integral inequalities for also non-integer orders. One of the most famous inequality is Ostrowski's which has lead to gain many practical inequalities with fractional calculus as well.

Ostrowski proved an important integral inequality in 1938 which gives an upper bound for difference between the value $f(x)$ and mean value of f for functions whose derivatives' absolute values are bounded, which can be seen in [11] as the following.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$ then,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

holds for all $x \in [a, b]$. Here $\frac{1}{4}$ is the best possible constant.

Zhang and Wan introduced $p-$convex functions in [14], and İşcan gave a different version of this definition in [5] as follows.

Definition 1. Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \backslash \{0\}$. A function $f : I \to \mathbb{R}$ is said to be a $p-$convex function , if

$$f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) \leq tf(x) + (1-t) f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

It is easy to see that $p-$convexity reduces to ordinary convexity for $p = 1$ and harmonically convexity for $p = -1$.

$p-$convex functions are frequently considered in the inequalities especially when using fractional integral calculations. Some fractional integral operators are used to do these calculations. Therefore, some new definitions about fractional calculus are given. First of them is Riemann-Liouville fractional integration operator (see [9]) which ables to integrate functions on fractional orders.

Definition 2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J_{a+}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$. Here $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

Definition 3. [9] The left and right-side Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ are defined as

$$\begin{aligned} \mathfrak{I}_{a+}^{\alpha} \varphi &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{\left(\ln \frac{x}{t}\right)^{1-\alpha}} \frac{dt}{t}, \quad x > a > 0, \\ \mathfrak{I}_{b-}^{\alpha} \varphi &= \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{\left(\ln \frac{t}{x}\right)^{1-\alpha}} \frac{dt}{t}, \quad 0 < x < b. \end{aligned}$$

where Γ is the gamma function.

Definition 4. [8] Let the space $X_c^p(a, b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) of those complex-valued Lebesque measurable functions f on $[a, b]$ for which $\|f\|\, x_c^p < \infty$, where the norm is defined by

$$\|f\|\, x_c^p = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \tag{1.2}$$

for $1 \leq p \leq \infty$, $c \in \mathbb{R}$ and for the case $p = \infty$,

$$\|f\| x_c^p = ess \sup_{a \leq t \leq b} \left[t^c \left| f(t) \right| \right] \qquad (c \in \mathbb{R}). \tag{1.3}$$

Katugampola revealed a new fractional integration operator which generalizes both Riemann-Liouville and Hadamard fractional integration operators. This integration operator also holds semigroup property (see [6, 7]) and is defined as the following statement.

Definition 5. Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left and right-side Katugampola fractional integrals of order ($\alpha > 0$) of $f \in X_c^p(a, b)$ are defined by

$${}^{\rho}I_{a+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)\, dt$$

and

$${}^{\rho}I_{b-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)\, dt$$

with $a < x < b$ and $\rho > 0$ if the integral exists.

Theorem 2. *[7] Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,*

$$\begin{aligned} 1.\ \lim_{\rho \to 1} {}^{\rho}I_{a+}^{\alpha} f(x) &= J_{a+}^{\alpha} f(x) \\ 2.\ \lim_{\rho \to 0^+} {}^{\rho}I_{a+}^{\alpha} f(x) &= \mathfrak{J}_{a+}^{\alpha} f(x). \end{aligned}$$

Similar results also hold for right-sided operators.

Erdélyi et al. deeply involved in hypergeometric functions which Whittaker discovered in 1904 and gave the definition of it in [4] as:

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, b-c)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt,\ c > b > 0,\ |z| < 1 \tag{1.4}$$

Throughout the paper the notation $Y_f(\alpha, \rho; a, x, b)$ will be used in the meaning of following statement.

$$\begin{aligned} Y_f(\alpha, \rho; a, x, b) &= \frac{\rho f(x)}{b-a} \left[(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha \right] \\ &\quad - \frac{\rho^{\alpha+1} \Gamma(\alpha+1)}{b-a} \left[{}^{\rho}I_{x-}^{\alpha} f(a) + {}^{\rho}I_{x+}^{\alpha} f(b) \right]. \end{aligned} \tag{1.5}$$

where Γ is Euler Gamma function, i.e., $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$.

Alomari et al. proved the following lemma in 2010 in [2] to obtain new Ostrowski-type results.

Lemma 1. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds*

$$\begin{aligned} & f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \\ =\ & \frac{(x-a)^2}{b-a} \int_0^1 t f'(tx + (1-t)a)\, dt - \frac{(b-x)^2}{b-a} \int_0^1 t f'(tx + (1-t)b)\, dt \end{aligned} \tag{1.6}$$

for each $x \in [a, b]$.

Set proved the next lemma in 2012 which helps to obtain Ostrowski-type inequalities for Riemann-Liouville fractional integrals in [13].

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then for all $x \in [a, b]$ and $\alpha > 0$ the following identity holds*

$$\begin{aligned} & \frac{f(x)}{b-a}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x-}^{\alpha} f(a)+J_{x+}^{\alpha} f(b)\right] \\ = & \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} f'(tx+(1-t)a)\,dt-\frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha} f'(tx+(1-t)b)\,dt \end{aligned} \tag{1.7}$$

where Γ is Euler Gamma function.

To see more studies involving Ostrowski-type inequalities, one can see references [1, 3, 12]. Also in [2] and [5], Ostrowski-type inequalities using integer order integrals and in [13], Ostrowski-type inequalities using Riemann-Liouville integral operator were obtained. On the other hand, the findings in this study were obtained using Katugampola fractional integration operator, which gives more general results than inequalities using integer order integral or Riemann-Liouville fractional integral operator.

In this paper, a new lemma including Katugampola fractional integral has been proved inspired by Lemma 2. Then with the help of some properties and inequalities like Hölder and power mean, new Ostrowski-type inequalities are proved. It is seen that results are supported by the literature.

2. Main results

Lemma 3. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then for all $x \in [a, b]$ the following identity holds*

$$\begin{aligned} Y_f(\alpha, \rho; a, x, b) = & \frac{(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a} \int_0^1 \frac{t^{\alpha} f'\left(\left[tx^{\rho}+(1-t)a^{\rho}\right]^{\frac{1}{\rho}}\right)}{\left(tx^{\rho}+(1-t)a^{\rho}\right)^{1-\frac{1}{\rho}}}\,dt \\ & -\frac{(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a} \int_0^1 \frac{t^{\alpha} f'\left(\left[tx^{\rho}+(1-t)b^{\rho}\right]^{\frac{1}{\rho}}\right)}{\left(tx^{\rho}+(1-t)b^{\rho}\right)^{1-\frac{1}{\rho}}}\,dt \end{aligned} \tag{2.1}$$

where $\alpha > 0$, $\rho > 0$.

Proof. By integrating by parts, the following statement is obtained

$$\begin{aligned} I_1 & = \int_0^1 \frac{t^{\alpha} f'\left(\left[tx^{\rho}+(1-t)a^{\rho}\right]^{\frac{1}{\rho}}\right)}{\left(tx^{\rho}+(1-t)a^{\rho}\right)^{1-\frac{1}{\rho}}}\,dt \\ & = \frac{\rho f(x)}{x^{\rho}-a^{\rho}}-\frac{\alpha\rho}{x^{\rho}-a^{\rho}} \int_0^1 t^{\alpha-1} f\left(\left[tx^{\rho}+(1-t)a^{\rho}\right]^{\frac{1}{\rho}}\right) dt. \end{aligned}$$

With changing the variable $u = \left[tx^{\rho}+(1-t)a^{\rho}\right]^{\frac{1}{\rho}}$, it is easy to get

$$I_1 = \frac{\rho f(x)}{x^{\rho}-a^{\rho}}-\frac{\alpha\rho}{x^{\rho}-a^{\rho}} \int_a^x \left(\frac{u^{\rho}-a^{\rho}}{x^{\rho}-a^{\rho}}\right)^{\alpha-1} \frac{\rho u^{\rho-1}}{x^{\rho}-a^{\rho}} f(u)\,du$$

$$
\begin{aligned}
&= \frac{\rho f(x)}{x^\rho - a^\rho} - \frac{\alpha\rho^2}{(x^\rho - a^\rho)^{\alpha+1}} \int_a^x \frac{u^{\rho-1}}{(u^\rho - a^\rho)^{1-\alpha}} f(u)\, du \\
&= \frac{\rho f(x)}{x^\rho - a^\rho} - \frac{\alpha\rho^2 \Gamma(\alpha)}{(x^\rho - a^\rho)^{\alpha+1} \rho^{1-\alpha}} \, {}^\rho I_{x-}^\alpha f(a) \\
&= \frac{\rho f(x)}{x^\rho - a^\rho} - \frac{\rho^{\alpha+1}\Gamma(\alpha+1)}{(x^\rho - a^\rho)^{\alpha+1}} \, {}^\rho I_{x-}^\alpha f(a). \qquad (2.2)
\end{aligned}
$$

In the same way, integrating by parts I_2 can be revealed as

$$
\begin{aligned}
I_2 &= \int_0^1 \frac{t^\alpha f'\left([tx^\rho + (1-t) b^\rho]^{\frac{1}{\rho}}\right)}{(tx^\rho + (1-t) b^\rho)^{1-\frac{1}{\rho}}} dt \\
&= \frac{\rho f(x)}{x^\rho - b^\rho} - \frac{\alpha\rho}{x^\rho - b^\rho} \int_0^1 t^{\alpha-1} f\left([tx^\rho + (1-t) b^\rho]^{\frac{1}{\rho}}\right) dt\,.
\end{aligned}
$$

With same change of variable, it can be seen that

$$
\begin{aligned}
I_2 &= \frac{\rho f(x)}{x^\rho - b^\rho} - \frac{\alpha\rho}{x^\rho - b^\rho} \int_b^x \left(\frac{u^\rho - b^\rho}{x^\rho - b^\rho}\right)^{\alpha-1} \frac{\rho u^{\rho-1}}{x^\rho - b^\rho} f(u)\, du \\
&= -\frac{\rho f(x)}{b^\rho - x^\rho} + \frac{\alpha\rho^2}{(b^\rho - x^\rho)^{\alpha+1}} \int_x^b \frac{u^{\rho-1}}{(b^\rho - u^\rho)^{1-\alpha}} f(u)\, du \\
&= -\frac{\rho f(x)}{b^\rho - x^\rho} + \frac{\alpha\rho^2\Gamma(\alpha)}{(b^\rho - x^\rho)^{\alpha+1}\rho^{1-\alpha}} \, {}^\rho I_{x+}^\alpha f(b) \\
&= -\frac{\rho f(x)}{b^\rho - x^\rho} + \frac{\rho^{\alpha+1}\Gamma(\alpha+1)}{(b^\rho - x^\rho)^{\alpha+1}} \, {}^\rho I_{x+}^\alpha f(b). \qquad (2.3)
\end{aligned}
$$

Multiplying (2.2) with $\frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a}$ and (2.3) with $\left(-\frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a}\right)$ and summing them side by side, the following calculations can be performed.

$$
\begin{aligned}
&\frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha f'\left([tx^\rho + (1-t) a^\rho]^{\frac{1}{\rho}}\right)}{(tx^\rho + (1-t) a^\rho)^{1-\frac{1}{\rho}}} dt \\
&- \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha f'\left([tx^\rho + (1-t) b^\rho]^{\frac{1}{\rho}}\right)}{(tx^\rho + (1-t) b^\rho)^{1-\frac{1}{\rho}}} dt \\
= \; &\frac{\rho f(x)(x^\rho - a^\rho)^\alpha}{b-a} - \frac{\rho^{\alpha+1}\Gamma(\alpha+1)\, {}^\rho I_{x-}^\alpha f(a)}{b-a} \\
&+ \frac{\rho f(x)(b^\rho - x^\rho)^\alpha}{b-a} - \frac{\rho^{\alpha+1}\Gamma(\alpha+1)\, {}^\rho I_{x+}^\alpha f(b)}{b-a}.
\end{aligned}
$$

With rearranging the last statement

$$
\frac{\rho f(x)}{b-a} \left[(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha\right] - \frac{\rho^{\alpha+1}\Gamma(\alpha+1)}{b-a} \left[\, {}^\rho I_{x-}^\alpha f(a) + {}^\rho I_{x+}^\alpha f(b)\right]
$$

$$
\begin{aligned}
&= \frac{(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}\int_0^1 \frac{t^{\alpha} f'\left(\left[tx^{\rho}+(1-t)\,a^{\rho}\right]^{\frac{1}{\rho}}\right)}{(tx^{\rho}+(1-t)\,a^{\rho})^{1-\frac{1}{\rho}}}dt \\
&\quad -\frac{(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\int_0^1 \frac{t^{\alpha} f'\left(\left[tx^{\rho}+(1-t)\,b^{\rho}\right]^{\frac{1}{\rho}}\right)}{(tx^{\rho}+(1-t)\,b^{\rho})^{1-\frac{1}{\rho}}}dt
\end{aligned}
$$

is obtained, which completes the proof. □

Remark 1. Under necessary conditions of Lemma 3 with choosing $\rho = 1$, we get Lemma 2 which is proven in [13].

Remark 2. By choosing $\alpha = 1$ in Remark 1, it is easy to obtain Lemma 1 which is proven in [2].

Theorem 3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $a < b$ such that $f' \in L[a,b]$. If $|f'|$ is $p-$convex on I and $|f'(x)| \leq M$ for all $x \in \left[a, 2^{\frac{1}{\rho}}a\right)$ (if $2^{\frac{1}{\rho}}a < b$, otherwise $x \in [a,b]$), then the following inequality holds*

$$
\left|Y_f(\alpha,\rho;a,x,b)\right| \leq M\frac{(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}\;\{R(a)+S(a)\} + M\frac{(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\{R(b)+S(b)\} \tag{2.4}
$$

where

$$
\begin{aligned}
R(\lambda) &= \frac{\lambda^{1-\rho}}{\alpha+2}\;{}_2F_1\left(\alpha+2,\;\frac{\rho-1}{\rho};\;\alpha+3;\;1-\frac{x^{\rho}}{\lambda^{\rho}}\right) \\
S(\lambda) &= \frac{\lambda^{1-\rho}}{(\alpha+1)(\alpha+2)}\left[\begin{array}{l} (\alpha+2)\;{}_2F_1\left(\alpha+1,\;\frac{\rho-1}{\rho};\;\alpha+2;\;1-\frac{x^{\rho}}{\lambda^{\rho}}\right) \\ -(\alpha+1)\;{}_2F_1\left(\alpha+2,\;\frac{\rho-1}{\rho};\;\alpha+3;\;1-\frac{x^{\rho}}{\lambda^{\rho}}\right) \end{array}\right]
\end{aligned}
$$

and $\rho > 1$, $\alpha > 0$, $\lambda \in \{a,b\}$, ${}_2F_1(.,.;.;.)$ is hypergeometric function and $Y_f(\alpha,\rho;a,x,b)$ is as defined in (1.4).

Proof. By using Lemma 3 and properties of modulus, it can be written

$$
\begin{aligned}
\left|Y_f(\alpha,\rho;a,x,b)\right| &\leq \frac{(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}\int_0^1 \frac{t^{\alpha}\left|f'\left(\left[tx^{\rho}+(1-t)\,a^{\rho}\right]^{\frac{1}{\rho}}\right)\right|}{(tx^{\rho}+(1-t)\,a^{\rho})^{1-\frac{1}{\rho}}}dt \\
&\quad +\frac{(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\int_0^1 \frac{t^{\alpha}\left|f'\left(\left[tx^{\rho}+(1-t)\,b^{\rho}\right]^{\frac{1}{\rho}}\right)\right|}{(tx^{\rho}+(1-t)\,b^{\rho})^{1-\frac{1}{\rho}}}dt.
\end{aligned}
$$

By means of $p-$convexity of $|f'|$, following computations can be performed

$$
\begin{aligned}
\left|Y_f(\alpha,\rho;a,x,b)\right| &\leq \frac{(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}\int_0^1 \frac{t^{\alpha}\left[t\,|f'(x)|+(1-t)\,|f'(a)|\right]}{(tx^{\rho}+(1-t)\,a^{\rho})^{1-\frac{1}{\rho}}}dt \\
&\quad +\frac{(b^{\rho}-x^{\rho})^{\alpha+1}}{b-a}\int_0^1 \frac{t^{\alpha}\left[t\,|f'(x)|+(1-t)\,|f'(b)|\right]}{(tx^{\rho}+(1-t)\,b^{\rho})^{1-\frac{1}{\rho}}}dt \\
&= \frac{(x^{\rho}-a^{\rho})^{\alpha+1}}{b-a}
\end{aligned}
$$

$$\times\left\{\begin{array}{c}|f'(x)|\int_0^1 t^{\alpha+1}(tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}dt\\ +|f'(a)|\int_0^1\left(t^\alpha-t^{\alpha+1}\right)(tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}dt\end{array}\right\}$$
$$+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}$$
$$\times\left\{\begin{array}{c}|f'(x)|\int_0^1 t^{\alpha+1}(tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}dt\\ +|f'(b)|\int_0^1\left(t^\alpha-t^{\alpha+1}\right)(tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}dt\end{array}\right\}.$$

With necessary computations, it is easy to see that

$$\begin{aligned}\left|Y_f(\alpha,\rho;a,x,b)\right| &\leq \frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}\{|f'(x)|R(a)+|f'(a)|S(a)\}\\ &\quad+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}\{|f'(x)|R(b)+|f'(b)|S(b)\}.\end{aligned}$$

By using boundedness of $f'(x)$, that is, $|f'(x)|\leq M$, it is easy to see

$$\left|Y_f(\alpha,\rho;a,x,b)\right|\leq M\frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}\{R(a)+S(a)\}+M\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}\{R(b)+S(b)\}$$

which completes the proof. □

Remark 3. By choosing $\rho=1$ in Theorem 3, it reduces to Theorem 7 with $s=1$ in [13] where we used the fact that ${}_2F_1(x,\ 0;\ y;\ z)=1$.

Theorem 4. *Let $f: I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is $p-$convex on I and $|f'(x)|\leq M$ for all $x\in I-\{a,b\}$, then the following inequality holds*

$$\left|Y_f(\alpha,\rho;a,x,b)\right|\leq\frac{M}{b-a}\left(\frac{1}{\alpha q+1}\right)^{\frac{1}{q}}\left[(x^\rho-a^\rho)^{\alpha+1}K^{\frac{1}{r}}(a)+(b^\rho-x^\rho)^{\alpha+1}K^{\frac{1}{r}}(b)\right]$$

where

$$K(\lambda)=\frac{\rho\left(x^{r(1-\rho)+\rho}-\lambda^{r(1-\rho)+\rho}\right)}{(x^\rho-\lambda^\rho)(r(1-\rho)+\rho)}$$

and $\rho>0$, $\alpha>0$, $\lambda\in\{a,b\}$, $r>1$, $\frac{1}{r}+\frac{1}{q}=1$, $r\neq\frac{\rho}{\rho-1}$ and $Y_f(\alpha,\rho;a,x,b)$ is as defined in (1.4).

Proof. With the help of Lemma 3 and properties of modulus, one can write

$$\begin{aligned}&\left|Y_f(\alpha,\rho;a,x,b)\right|\\ &\leq\frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1\frac{t^\alpha\left|f'\left([tx^\rho+(1-t)a^\rho]^{\frac{1}{\rho}}\right)\right|}{(tx^\rho+(1-t)a^\rho)^{1-\frac{1}{\rho}}}dt\\ &\quad+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1\frac{t^\alpha\left|f'\left([tx^\rho+(1-t)b^\rho]^{\frac{1}{\rho}}\right)\right|}{(tx^\rho+(1-t)b^\rho)^{1-\frac{1}{\rho}}}dt.\end{aligned}$$

By using Hölder inequality, it can be written as

$$\begin{aligned}
&\left|Y_f(\alpha,\rho;a,x,b)\right| \\
\leq\ & \frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}\left(\int_0^1\left((tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}\right)^r dt\right)^{\frac{1}{r}} \\
&\times\left(\int_0^1 t^{\alpha q}\left|f'\left([tx^\rho+(1-t)a^\rho]^{\frac{1}{\rho}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}\left(\int_0^1\left((tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}\right)^r dt\right)^{\frac{1}{r}} \\
&\times\left(\int_0^1 t^{\alpha q}\left|f'\left([tx^\rho+(1-t)b^\rho]^{\frac{1}{\rho}}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

From the $p-$convexity of $|f'|^q$ and $|f'(x)|\leq M$, it follows that

$$\begin{aligned}
\left|Y_f(\alpha,\rho;a,x,b)\right| \leq\ & \frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(a) \\
&\times\left(\int_0^1 t^{\alpha q+1}|f'(x)|^q dt+\int_0^1 t^{\alpha q}(1-t)|f'(a)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(b) \\
&\times\left(\int_0^1 t^{\alpha q+1}|f'(x)|^q dt+\int_0^1 t^{\alpha q}(1-t)|f'(b)|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(a)\left(M^q\frac{1}{\alpha q+2}+M^q\frac{1}{(\alpha q+1)(\alpha q+2)}\right)^{\frac{1}{q}} \\
&+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}K^{\frac{1}{r}}(b)\left(M^q\frac{1}{\alpha q+2}+M^q\frac{1}{(\alpha q+1)(\alpha q+2)}\right)^{\frac{1}{q}} \\
=\ & \frac{M}{b-a}\left(\frac{1}{\alpha q+1}\right)^{\frac{1}{q}}\left[(x^\rho-a^\rho)^{\alpha+1}K^{\frac{1}{r}}(a)+(b^\rho-x^\rho)^{\alpha+1}K^{\frac{1}{r}}(b)\right]
\end{aligned}$$

which completes the proof. □

Theorem 5. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I$ with $a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is $p-$convex on I and $|f'(x)|\leq M$ for all $x\in\left[a,2^{\frac{1}{\rho}}a\right)$ (if $2^{\frac{1}{\rho}}a<b$, otherwise $x\in[a,b]$), then the following inequality holds*

$$\begin{aligned}
\left|Y_f(\alpha,\rho;a,x,b)\right| \leq\ & \frac{M}{b-a}(x^\rho-a^\rho)^{\alpha+1}L^{1-\frac{1}{q}}(a)(R(a)+S(a))^{\frac{1}{q}} \\
&+\frac{M}{b-a}(b^\rho-x^\rho)^{\alpha+1}L^{1-\frac{1}{q}}(b)(R(b)+S(b))^{\frac{1}{q}}
\end{aligned}$$

where

$$R(\lambda) = \frac{\lambda^{1-\rho}}{\alpha+2}\,{}_2F_1\left(\alpha+2,\ \frac{\rho-1}{\rho};\ \alpha+3;\ 1-\frac{x^\rho}{\lambda^\rho}\right)$$

$$
\begin{aligned}
S(\lambda) &= \frac{\lambda^{1-\rho}}{(\alpha+1)(\alpha+2)}\left[\begin{array}{c}(\alpha+2)\ {}_2F_1\left(\alpha+1,\ \frac{\rho-1}{\rho};\ \alpha+2;\ 1-\frac{x^\rho}{\lambda^\rho}\right)\\ -(\alpha+1)\ {}_2F_1\left(\alpha+2,\ \frac{\rho-1}{\rho};\ \alpha+3;\ 1-\frac{x^\rho}{\lambda^\rho}\right)\end{array}\right]\\
L(\lambda) &= \frac{\lambda^{1-\rho}}{\alpha+1}\ {}_2F_1\left(\alpha+1,\frac{\rho-1}{\rho};\alpha+2;1-\frac{x^\rho}{\lambda^\rho}\right)
\end{aligned}
$$

and $\rho > 1$, $\alpha > 0$, $q > 1$, $\lambda \in \{a, b\}$, ${}_2F_1(.,.;.;.)$ is hypergeometric function and $Y_f(\alpha,\rho;a,x,b)$ is as defined in (1.4).

Proof. Making use of Lemma 3 and properties of absolute value, it can be seen that

$$
\begin{aligned}
&\left|Y_f(\alpha,\rho;a,x,b)\right|\\
&\le \frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1 \frac{t^\alpha\left|f'\left([tx^\rho+(1-t)a^\rho]^{\frac{1}{\rho}}\right)\right|}{(tx^\rho+(1-t)a^\rho)^{1-\frac{1}{\rho}}}dt\\
&\quad+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 \frac{t^\alpha\left|f'\left([tx^\rho+(1-t)b^\rho]^{\frac{1}{\rho}}\right)\right|}{(tx^\rho+(1-t)b^\rho)^{1-\frac{1}{\rho}}}dt.
\end{aligned}
$$

Then, making use of Power-Mean inequality, the following computations can be performed

$$
\begin{aligned}
\left|Y_f(\alpha,\rho;a,x,b)\right| &\le \frac{(x^\rho-a^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha(tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1 t^\alpha(tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}\left|f'\left([tx^\rho+(1-t)a^\rho]^{\frac{1}{\rho}}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b^\rho-x^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha(tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1 t^\alpha(tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}\left|f'\left([tx^\rho+(1-t)b^\rho]^{\frac{1}{\rho}}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Hence $|f'|^q$ is chosen as $p-$convex on I

$$
\begin{aligned}
\left|Y_f(\alpha,\rho;a,x,b)\right| &\le \frac{(x^\rho-a^\rho)^{\alpha+1}L^{1-\frac{1}{q}}(a)}{b-a}\\
&\quad\times\left(\begin{array}{c}\int_0^1 t^{\alpha+1}(tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}|f'(x)|^q dt\\ +\int_0^1\left(t^\alpha-t^{\alpha+1}\right)(tx^\rho+(1-t)a^\rho)^{\frac{1}{\rho}-1}|f'(a)|^q dt\end{array}\right)^{\frac{1}{q}}\\
&\quad+\frac{(b^\rho-x^\rho)^{\alpha+1}L^{1-\frac{1}{q}}(b)}{b-a}\\
&\quad\times\left(\begin{array}{c}\int_0^1 t^{\alpha+1}(tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}|f'(x)|^q dt\\ +\int_0^1\left(t^\alpha-t^{\alpha+1}\right)(tx^\rho+(1-t)b^\rho)^{\frac{1}{\rho}-1}|f'(b)|^q dt\end{array}\right)^{\frac{1}{q}}.
\end{aligned}
$$

With necessary computations, it can be easily seen that

$$
\left|Y_f(\alpha,\rho;a,x,b)\right| = \frac{(x^\rho-a^\rho)^{\alpha+1}L^{1-\frac{1}{q}}(a)}{b-a}\left(|f'(x)|^q R(a)+|f'(a)|^q S(a)\right)^{\frac{1}{q}}
$$

$$+\frac{\left(b^{\rho}-x^{\rho}\right)^{\alpha+1}L^{1-\frac{1}{q}}(b)}{b-a}\left(\left|f'(x)\right|^{q}R(b)+\left|f'(b)\right|^{q}S(b)\right)^{\frac{1}{q}}$$

With using boundedness of $|f'(x)|$, it can be written that

$$\begin{aligned}\left|Y_f(\alpha,\rho;a,x,b)\right| &\leq M\frac{\left(x^{\rho}-a^{\rho}\right)^{\alpha+1}L^{1-\frac{1}{q}}(a)}{b-a}\left(R(a)+S(a)\right)^{\frac{1}{q}}\\ &\quad+M\frac{\left(b^{\rho}-x^{\rho}\right)^{\alpha+1}L^{1-\frac{1}{q}}(b)}{b-a}\left(R(b)+S(b)\right)^{\frac{1}{q}}.\end{aligned}$$

So the proof is completed. □

Remark 4. By choosing $\rho=1$ in Theorem 5, it reduces to Theorem 9 with $s=1$ in [13] where we used the fact that ${}_2F_1(x,\ 0;\ y;\ z)=1$.

3. Applications to special means

Let us recall the following means for two positive real numbers.
(1) The arithmetic mean:

$$A=A(a,b)=\frac{a+b}{2};\ a,b>0;\ a,b\in\mathbb{R},$$

(2) The logarithmic mean:

$$L=L(a,b)=\frac{b-a}{\ln b-\ln a};\ a,b>0;\ a,b\in\mathbb{R}.$$

Proposition 1. *Let* $0<a<b$ *and* $\frac{a+b}{2}<2^{\frac{1}{\rho}}a$. *Then the following inequality holds*

$$\begin{aligned}&\left|4A(a,b)\ln(A(a,b))-2\frac{b^{\left(b^2\right)}-a^{\left(a^2\right)}}{b-a}L^{-1}\left(a^{\left(a^2\right)},b^{\left(b^2\right)}\right)+2A(a,b)\right|\\ \leq\ &|M|\frac{\left(b^2-a^2\right)}{2}\end{aligned}$$

where $M=\max\{|\ln a|,|\ln b|\}$.

Proof. The proof follows from Theorem 3 on applying $\alpha=1,\rho=2,x=\frac{a+b}{2}$ and $f(x)=-\ln x$ which is $p-$convex on $(0,\infty)$ for $p\geq 1$. □

Proposition 2. *Let* $0<a<b$ *and* $\frac{a+b}{2}<2^{\frac{1}{\rho}}a$. *Then the following inequality holds*

$$\left|4A^{-1}(a,b)-4L^{-1}(a,b)\right|\leq\frac{b^2-a^2}{2a^2}.$$

Proof. The proof is immediate from Theorem 3 on applying $\alpha=1,\rho=2,x=\frac{a+b}{2}$ and $f(x)=x^{-2}$ which is $p-$convex on $(0,\infty)$. □

4. Conclusion

In this study, new lemma and theorems are put forward to obtain new upper bounds for Ostrowski-type inequalities including Katugampola fractional operator. Researchers can obtain new lemmas and theorems by using similar method used in this study or use the obtained results in many fields of science.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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