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Research article

Unified integral inequalities comprising pathway operators

A. M. Mishra[1], **D. Kumar**[2] **and S. D. Purohit**[1,*]

[1] Department of HEAS (Mathematics), Rajasthan Technical University, Kota 324010, India
[2] Department of Mathematics, University of Rajasthan, Jaipur-302004, Rajasthan, India

* **Correspondence:** Email: sunil_a_purohit@yahoo.com; Tel: +917442473075.

Abstract: In this article, we established generalized version of unified integral inequalities, comprising pathway fractional operators related to bounded functions whose bounds are also bounded functions. We reduce these results in some useful particular forms and also some well-known inequalities of the literature.

Keywords: integral inequalities; pathway operator; fractional calculus; generalized fractional integral operators
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1. Introduction

Real world problems deal with mathematical modeling in different discipline of sciences and such mathematical model generally consists of partial differential equations, either in integer form or in fractional form (arbitrary), for instance see [17, 18, 28, 34, 35, 38, 39] and references therein. We often interested to find the bounds of solutions for such partial differential equations which can be achieved by fractional integral inequalities in an efficient manner. Because of this, fractional integral inequalities are always been a focusing area for researchers (see [1, 4, 5, 11, 12, 25, 29]). Many authors have studied a number of generalization of well known integral inequalities for suitable choices of functions [8, 10, 13, 19–24]. Tariboon *et al.* [36], Wang *et al.* [37] and Saxena *et al.* [33] studied different types of inequalities for the integrable functions. We try to find a general extensions of the results available in [36] and [37]. We have established new integral inequalities inherent in the integrable functions, concerning pathway fractional integral operators, given by Nair [26]. We have also discussed some ensuing results and special cases of the main results [20–22].

Pathway fractional integral operator is a generalized Riemann Liouville fractional integral operator in higher dimensions. Indeed, pathway fractional integral operator has been used to define certain probability density functions and having interesting applications in statistics also, see [20, 21].

Let λ be an arbitrary real number preferably less than 1. A class of conformal functions

$$C_\lambda^{(k)} = \{f(t) = t^p \tilde{f}(t); p > \lambda, \tilde{f} \in C^{(k)}([a,b])\}$$

be Banach space for our consideration for different non negative values of k. The space we have chosen for $k = 0$ [16] induced by the norm

$$\|f\|_r = \left[\int_a^b |f(x)|^r dx\right]^{\frac{1}{r}} < \infty.$$

Suppose $\phi(t) \in C_\lambda^{(k)}$ for $k = 0$ and $\nu \in \mathbb{C}$ which has positive real part and a is positive. Pathway fractional integral operator is defined as [20]

$$P_{0^+}^{(\nu,\lambda)}(\phi(t)) = \int_0^{\frac{t}{a(1-\lambda)}} t^\nu \left[1 - \frac{a(1-\lambda)u}{t}\right]^{\nu/(1-\lambda)} \phi(u)du. \tag{1.1}$$

We call λ the pathway parameter and ν, order of the integral operator. While λ converges to 1 from left side, the operator gets the form

$$P_{0^+}^{(\nu,1)}(\phi(t)) = t^\nu \mathfrak{L}_\phi \left\{\frac{a\nu}{t}\right\}, \tag{1.2}$$

where $\mathfrak{L}_\phi\{s\}$ is a Laplace transform of the function $\phi(.)$ with parameter s. Meaning thereby, the fractional integral operator of pathway type reduces to the Laplace transform with factor $(a\nu/t)$ for a particular value of λ.

Pathway integral operator reduces to left sided Riemann-Liouville fractional integral operator $I_{0+}^\nu(\phi(t))$ for particular value of $\lambda = 0$ and $a = 1$

$$\int_0^t (t-u)^{\nu-1}\phi(u)du = \Gamma(\nu) I_{0+}^\nu(\phi(t)), \tag{1.3}$$

by replacing ν to $\nu - 1$. The reader may refer the papers of Mathai and Haubold [21,22], Nair [26] and Nisar *et al.* [27] for more details of pathway operators. Pathway fractional integral operator yields the following expression on an application of polynomial functions:

$$P_{0+}^{(\nu,\lambda)}(z^n) = \frac{\Gamma(n+1)}{[a(1-\lambda)]^{n+1}} \frac{\Gamma\left(1 + \frac{\nu}{1-\lambda}\right)}{\Gamma\left(2 + n + \frac{\nu}{1-\lambda}\right)} z^{\nu+n+1}, \tag{1.4}$$

generally known as image formula. Pathway fractional integral operator acts as vector space over the field of real numbers so value of pathway operator can be evaluated by

$$P_{0+}^{(\nu,\lambda)}(1) = \frac{1}{a(1-\lambda)} \frac{\Gamma\left(1 + \frac{\nu}{1-\lambda}\right)}{\Gamma\left(2 + \frac{\nu}{1-\lambda}\right)} z^{\nu+1}. \tag{1.5}$$

Recently, several extensions of the classical inequalities have been studied by many authors, see [2,3, 6,7,30,31] and references therein. Our article has organized as follows: First we state and prove the main inequality of the article and take a specific case when pathway parameters coincide. Moreover, we discuss certain generalized versions of this theorem, associated with the two bounded integrable functions. Special cases of these results have also been discussed in a separate section with some concluding remarks.

2. Main result

The results are stated as the following theorems:

Theorem 2.1. *Let F be an integrable bounded function such that*

$$\gamma_1(z) \leq F(z) \leq \gamma_2(z)$$

for all $z \in [a,b]$. Then,

$$\begin{aligned} P_{0+}^{(\nu',\lambda')}(\gamma_1(z))P_{0+}^{(\nu,\lambda)}(F(z)) + P_{0+}^{(\nu,\lambda)}(\gamma_2(z))P_{0+}^{(\nu',\lambda')}(F(z)) \geq P_{0+}^{(\nu,\lambda)}(\gamma_2(z))P_{0+}^{(\nu',\lambda')}(\gamma_1(z)) \\ + P_{0+}^{(\nu,\lambda)}(F(z))P_{0+}^{(\nu',\lambda')}((F(z)) \end{aligned} \tag{2.1}$$

where $\gamma_1(z), \gamma_2(z)$ and $F(z)$ are integrable functions over the interval $[a,b]$.

Proof. Taking into account of the inequality $\gamma_1(z) \leq F(z) \leq \gamma_2(z)$, for any $u, v > 0$, we have

$$(\gamma_2(u) - F(u))(F(v) - \gamma_1(v)) \geq 0, \tag{2.2}$$

and it follows that

$$\gamma_2(u)F(v) + \gamma_1(v)F(u) \geq \gamma_1(v)\gamma_2(u) + F(u)F(v). \tag{2.3}$$

Let us consider

$$\mathfrak{F}(z,u) = z^{\nu}\left[1 - \frac{a(1-\lambda)u}{z}\right]^{\nu/(1-\lambda)}, \tag{2.4}$$

$u \in [0,z], z > 0.$

Multiply (2.3) by $\mathfrak{F}(z,u)p(u)$ (where $p(u)$ is a weight function) and integrate u over 0 to $z/a(1-\lambda)$, we get

$$\begin{aligned} \int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)\gamma_2(u)F(v)du + \int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)\gamma_1(v)F(u)du \\ \geq \int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)\gamma_1(v)\gamma_2(u)du + \int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)F(u)F(v)du \end{aligned} \tag{2.5}$$

i.e.,

$$\begin{aligned} F(v)\int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)\gamma_2(u)du + \gamma_1(v)\int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)F(u)du \\ \geq \gamma_1(v)\int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)\gamma_2(u)du + F(v)\int_0^{z/a(1-\lambda)} \mathfrak{F}(z,u)p(u)F(u)du. \end{aligned} \tag{2.6}$$

This implies,

$$F(v)P_{0+}^{(\nu,\lambda)}(\gamma_2(z)) + \gamma_1(v)P_{0+}^{(\nu,\lambda)}(F(z)) \geq \gamma_1(v)P_{0+}^{(\nu,\lambda)}(\gamma_2(z)) + F(v)P_{0+}^{(\nu,\lambda)}(F(z)). \tag{2.7}$$

In the same way, we consider the function

$$\mathfrak{H}(z,v) = z^{\nu'}\left[1-\frac{a(1-\lambda')v}{z}\right]^{\nu'/(1-\lambda')}. \tag{2.8}$$

Now, Multiply (2.7) by $\mathfrak{H}(z,v)p(v)$ and integrate with respect to v from 0 to $\nu'/a(1-\lambda')$, we get:

$$\begin{aligned}&\int_0^{z/a(1-\lambda')} \mathfrak{H}(z,v)p(v)F(v)dv\left(P_{0+}^{(\nu,\lambda)}(\gamma_2(z)\right) + \int_0^{z/a(1-\lambda')} \mathfrak{H}(z,v)p(v)\gamma_1(v)dv\left(P_{0+}^{(\nu,\lambda)}(F(z))\right)\\ \geq &\int_0^{z/a(1-\lambda')} \mathfrak{H}(z,v)p(v)\gamma_1(v)dv\left(P_{0+}^{(\nu,\lambda)}(\gamma_2(z))\right) + \int_0^{z/a(1-\lambda')} \mathfrak{H}(z,v)p(v)F(v)dv\left(P_{0+}^{(\nu,\lambda)}(F(z))\right)\end{aligned} \tag{2.9}$$

or,

$$\begin{aligned}P_{0+}^{(\nu',\lambda')}(\gamma_1(z))P_{0+}^{(\nu,\lambda)}(F(z)) + P_{0+}^{(\nu,\lambda)}(\gamma_2(z))P_{0+}^{(\nu',\lambda')}(F(z)) \geq P_{0+}^{(\nu,\lambda)}(\gamma_2(z))P_{0+}^{(\nu',\lambda')}(\gamma_1(z))\\ +P_{0+}^{(\nu,\lambda)}(F(z))P_{0+}^{(\nu',\lambda')}((F(z)).\end{aligned} \tag{2.10}$$

This proves the theorem. □

The following corollary is an immediate result of the above theorem, when both pathway parameter and order of pathway operator coincides.

Corollary 1. *Under the hypothesis of Theorem 1, the following inequality holds:*

$$P_{0+}^{(\nu,\lambda)}(\gamma_1(z))P_{0+}^{(\nu,\lambda)}(F(z)) + P_{0+}^{(\nu,\lambda)}(\gamma_2(z))P_{0+}^{(\nu,\lambda)}(F(z)) \geq P_{0+}^{(\nu,\lambda)}(\gamma_2(z))P_{0+}^{(\nu,\lambda)}(\gamma_1(z)) + \left(P_{0+}^{(\nu,\lambda)}(F(z))\right)^2. \tag{2.11}$$

Now, we can also extend this result for two bounded functions.

Theorem 2.2. *Suppose F and G are two integrable functions over the interval $[a,b]$ and $\gamma_1,\gamma_2,\delta_1$ and δ_2 are four integrable functions on $[a,b]$, such that*

$$\gamma_1(z) \leq F(z) \leq \gamma_2(z), \quad \delta_1(z) \leq G(z) \leq \delta_2(z), \quad for\ all\ z \in [a,b]. \tag{2.12}$$

Then the following inequalities holds true:

$$P_{0+}^{(\nu,\lambda)}\delta_1(z)P_{0+}^{(\nu',\lambda')}F(z) + P_{0+}^{(\nu',\lambda')}\gamma_2(z)P_{0+}^{(\nu,\lambda)}G(z) \geq P_{0+}^{(\nu',\lambda')}\gamma_2(z)P_{0+}^{(\nu,\lambda)}\delta_1(z) + P_{0+}^{(\nu',\lambda')}F(z)P_{0+}^{(\nu,\lambda)}G(z) \tag{2.13}$$

$$P_{0+}^{(\nu,\lambda)}\gamma_1(z)P_{0+}^{(\nu',\lambda')}G(z) + P_{0+}^{(\nu',\lambda')}\delta_2(z)P_{0+}^{(\nu,\lambda)}F(z) \geq P_{0+}^{(\nu',\lambda')}\delta_2(z)P_{0+}^{(\nu,\lambda)}\gamma_1(z) + P_{0+}^{(\nu',\lambda')}G(z)P_{0+}^{(\nu,\lambda)}F(z) \tag{2.14}$$

$$P_{0+}^{(\nu,\lambda)}\gamma_2(z)P_{0+}^{(\nu',\lambda')}\delta_2(z) + P_{0+}^{(\nu',\lambda')}F(z)P_{0+}^{(\nu,\lambda)}G(z) \geq P_{0+}^{(\nu',\lambda')}\gamma_2(z)P_{0+}^{(\nu,\lambda)}G(z) + P_{0+}^{(\nu',\lambda')}F(z)P_{0+}^{(\nu,\lambda)}\delta_2(z) \tag{2.15}$$

$$P_{0+}^{(\nu,\lambda)}\gamma_1(z)P_{0+}^{(\nu',\lambda')}\delta_1(z) + P_{0+}^{(\nu',\lambda')}F(z)P_{0+}^{(\nu,\lambda)}G(z) \geq P_{0+}^{(\nu',\lambda')}\gamma_1(z)P_{0+}^{(\nu,\lambda)}G(z) + P_{0+}^{(\nu',\lambda')}F(z)P_{0+}^{(\nu,\lambda)}\delta_1(z) \tag{2.16}$$

Proof. Assuming F and G are two integrable functions which are satisfying inequality (2.12), then in order to prove (2.13), we can write

$$(\gamma_2(u) - F(u))(G(v) - \delta_1(v)) \geq 0, \tag{2.17}$$

or we can deduce

$$\gamma_2(u)G(v) + \delta_1(v)F(u) \geq \delta_1(v)\gamma_2(u) + F(u)G(v). \tag{2.18}$$

Multiplying both sides of (2.18) by $\mathfrak{F}(z, u)$ and integrating with respect to u from 0 to z, we get

$$G(v)P_{0+}^{(v',\lambda')}\gamma_2(z) + \delta_1(v)P_{0+}^{(v',\lambda')}F(z) \geq \delta_1(v)P_{0+}^{(v',\lambda')}\gamma_2(z) + G(v)P_{0+}^{(v',\lambda')}F(z). \tag{2.19}$$

Again we multiply both sides of (2.19) by $\mathfrak{H}(z, v)$, given in (2.8), and integrate with respect to v from 0 to z, we can easily find the result (2.13).
The proofs of remaining inequalities (2.14), (2.15) and (2.16) follows on similar manners, as we did in the case of (2.13), we can prove other inequalities by taking into account the following identities, respectively:

$$(\delta_2(u) - G(u))(F(v) - \gamma_1(v)) \geq 0,$$

$$(\gamma_2(u) - F(u))(G(v) - \delta_2(v)) \geq 0$$

and

$$(\gamma_1(u) - F(u))(G(v) - \delta_1(v)) \geq 0.$$

We omit details of the proof. □

3. Special cases and concluding remarks

On the account of general nature of the pathway fractional integral operator, a number of new and known results involving Riemann-Liouville and Laplace transforms follow as special cases of theorems given above. To this end, let us set $\lambda = 0$, and use the relation (1.3), we recognize that Theorem 1 yields known integral inequality involving Riemann Liouville integral operators, due to [36]

Corollary 2. *Suppose F is an integrable bounded function such that*

$$\gamma_1(z) \leq F(z) \leq \gamma_2(z)$$

for all $z \in [a, b]$. Then,

$$I_{0+}^{(v')}(\gamma_1(z))I_{0+}^{(v)}(F(z)) + I_{0+}^{(v)}(\gamma_2(z))I_{0+}^{(v')}(F(z)) \geq I_{0+}^{(v)}(\gamma_2(z))I_{0+}^{(v')}(\gamma_1(z)) + I_{0+}^{(v)}(F(z))I_{0+}^{(v')}((F(z)) \tag{3.1}$$

where $\gamma_1(z), \gamma_2(z)$ and $F(z)$ are integrable functions over the interval $[a, b]$.

Similarly, if λ tends to 1 from left side then we get the following inequality:

Corollary 3. *Suppose F is an integrable bounded function such that*

$$\gamma_1(z) \leq F(z) \leq \gamma_2(z)$$

for all $z \in [a, b]$. Then,

$$\mathfrak{L}_{\gamma_1}\{u\}\mathfrak{L}_F\{v\} + \mathfrak{L}_{\gamma_2}\{v\}\mathfrak{L}_F\{u\} \geqslant \mathfrak{L}_{\gamma_1}\{u\}\mathfrak{L}_{\gamma_2}\{v\} + \mathfrak{L}_F\{v\}\mathfrak{L}_F\{u\} \tag{3.2}$$

where $u = \frac{av'}{z}$ and $v = \frac{av}{z}$.

Further, if we set $\gamma_1(t) = m, \gamma_2(t) = M, \delta_1(t) = p$ and $\delta_2(t) = P$, where $m, M, p, P \in \mathbb{R}, \forall\ t \in [0,\infty)$ and use of equation (1.6), then Theorems 1 & 2 lead to the following inequalities:

Corollary 4. *Suppose F is an integrable function defined on $[a,b]$, such that*

$$A \leq F(z) \leq B, \quad A, B \in \mathbb{R} \ \ for\ all\ z \in [a,b],$$

Thereupon, for $z > 0$, we have

$$\begin{aligned}
&\frac{A}{a(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+1} P_{0+}^{(\nu,\lambda)}(F(z)) + \frac{A}{a(1-\lambda)}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)} z^{\nu+1} P_{0+}^{(\nu',\lambda')}(F(z)) \\
&\geq \frac{AB}{a^2(1-\lambda)(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+\nu+2} + P_{0+}^{(\nu,\lambda)}(F(z)) P_{0+}^{(\nu',\lambda')}(F(z)).
\end{aligned} \tag{3.3}$$

Corollary 5. *Suppose F and G are two integrable functions over $[a,b]$, such that*

$$A \leq F(z) \leq B, \ \ C \leq G(z) \leq D, \ \ A, B, C, D \in \mathbb{R} \ \ for\ all\ z \in [a,b]$$

Then, for $z > 0$, these results follow:

$$\begin{aligned}
&\frac{C}{a(1-\lambda)}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)} z^{\nu+1} P_{0+}^{(\nu',\lambda')} F(z) + \frac{B}{a(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+1} P_{0+}^{(\nu,\lambda)} G(z) \\
&\geq \frac{CB}{a^2(1-\lambda)(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+\nu+2} + P_{0+}^{(\nu',\lambda')} F(z) P_{0+}^{(\nu,\lambda)} G(z),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\frac{A}{a(1-\lambda)}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)} z^{\nu+1} P_{0+}^{(\nu',\lambda')} G(z) + \frac{D}{a(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+1} P_{0+}^{(\nu,\lambda)} F(z) \\
&\geq \frac{AD}{a^2(1-\lambda)(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+\nu+2} + P_{0+}^{(\nu',\lambda')} G(z) P_{0+}^{(\nu,\lambda)} F(z),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\frac{BD}{a^2(1-\lambda)(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+\nu+2} + P_{0+}^{(\nu',\lambda')} F(z) P_{0+}^{(\nu,\lambda)} G(z) \\
&\geq \frac{B}{a(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+1} P_{0+}^{(\nu,\lambda)} G(z) + P_{0+}^{(\nu',\lambda')} F(z) \frac{D}{a(1-\lambda)}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)} z^{\nu+1},
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\frac{AC}{a^2(1-\lambda)(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+\nu+2} + P_{0+}^{(\nu',\lambda')} F(z) P_{0+}^{(\nu,\lambda)} G(z) \\
&\geq \frac{A}{a(1-\lambda')}\frac{\Gamma\left(1+\frac{\nu'}{1-\lambda'}\right)}{\Gamma\left(2+\frac{\nu'}{1-\lambda'}\right)} z^{\nu'+1} P_{0+}^{(\nu,\lambda)} G(z) + \frac{C}{a(1-\lambda)}\frac{\Gamma\left(1+\frac{\nu}{1-\lambda}\right)}{\Gamma\left(2+\frac{\nu}{1-\lambda}\right)} z^{\nu+1} P_{0+}^{(\nu',\lambda')} F(z).
\end{aligned} \tag{3.7}$$

By setting specific functions in place of $\gamma_1(z)$ and $\gamma_2(z)$ we can get a number of inequalities. For example, we may take $\gamma_1(z) = z$ and $\gamma_2(z) = z+1$. Same will work with corollaries, i.e., we can suitably choose $\delta_1(z)$ and $\delta_2(z)$. Pathway fractional integral operator converges to Riemann-Liouville fractional integral operator and Laplace transform under suitable values of parameters. Thus we can deduce standard inequalities involving Riemann-Liouville fractional integral operator and Laplace transform from the theorems stated in this paper.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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