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Research article

Existence theory for coupled nonlinear third-order ordinary differential equations with nonlocal multi-point anti-periodic type boundary conditions on an arbitrary domain

Bashir Ahmad[1,*], **Ahmed Alsaedi**[1], **Mona Alsulami**[1,2] **and Sotiris K. Ntouyas**[1,3]

[1] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics, Faculty of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia

[3] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

* **Correspondence:** Email: bashirahmad_qau@yahoo.com.

Abstract: In this paper, we derive existence and uniqueness results for a coupled system of nonlinear third order ordinary differential equations equipped with nonlocal multi-point anti-periodic type coupled boundary conditions. Leray-Schauder alternative and Banach contraction mapping principle are the main tools of our study. Examples are constructed for illustrating the obtained results. Under appropriate conditions, our results correspond to the ones for an ant-periodic boundary value problem of nonlinear third order ordinary differential equations.

Keywords: system of ordinary differential equations; Leray-Schauder; Banach; existence; fixed point
Mathematics Subject Classification: 34B10, 34B15

1. Introduction

We introduce and study a coupled system of nonlinear third-order ordinary differential equations on an arbitrary domain:

$$\begin{aligned} u'''(t) &= f(t, u(t), v(t), w(t)),\ t \in [a, b],\\ v'''(t) &= g(t, u(t), v(t), w(t)),\ t \in [a, b],\\ w'''(t) &= h(t, u(t), v(t), w(t)),\ t \in [a, b], \end{aligned} \tag{1.1}$$

supplemented with nonlocal multi-point anti-periodic type coupled boundary conditions of the form:

$$\begin{aligned}
&u(a)+u(b)=\sum_{j=1}^{m}\alpha_j v(\eta_j),\ \ u'(a)+u'(b)=\sum_{l=1}^{m}\beta_l v'(\eta_l),\ \ u''(a)+u''(b)=\sum_{n=1}^{m}\gamma_n v''(\eta_n),\\
&v(a)+v(b)=\sum_{e=1}^{m}\delta_e w(\eta_e),\ \ v'(a)+v'(b)=\sum_{q=1}^{m}\rho_q w'(\eta_q),\ \ v''(a)+v''(b)=\sum_{r=1}^{m}\sigma_r w''(\eta_r),\\
&w(a)+w(b)=\sum_{k=1}^{m}\xi_k u(\eta_k),\ \ w'(a)+w'(b)=\sum_{p=1}^{m}\zeta_p u'(\eta_p),\ \ w''(a)+w''(b)=\sum_{d=1}^{m}\kappa_d u''(\eta_d),
\end{aligned} \tag{1.2}$$

where $f, g,$ and $h : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ are given continuous functions, $a < \eta_1 < \eta_2 < \cdots < \eta_m < b$, and $\alpha_j,\ \beta_l,\ \gamma_n,\ \delta_e,\ \rho_q,\ \sigma_r,\ \xi_k,\ \zeta_p$ and $\kappa_d \in \mathbb{R}^+$ $(j, l, n, e, q, r, k, p$ and $d = 1, 2, \ldots, m)$.

Boundary value problems arise in the mathematical modeling of several real world phenomena occurring in diverse disciplines such as fluid mechanics, mathematical physics, etc. [1]. The available literature on the topic deals with the existence and uniqueness of solutions, analytic and numerical methods, stability properties of solutions, etc., for instance, see [2–5]. Classical boundary conditions cannot cater the complexities of the physical and chemical processes occurring within the specified domain. In order to resolve this issue, the concept of nonlocal boundary conditions was introduced. The details on theoretical development of nonlocal boundary value problems can be found in the articles [6–10] and the references cited therein. For some recent works on the topic, we refer the reader to the articles [11–16] and the references cited therein.

Nonlinear third-order ordinary differential equations appear in the study of many applied and technical problems. In [2], third-order nonlinear boundary value problems associated with nano-boundary layer fluid flow over stretching-surfaces were investigated. Systems of third order nonlinear ordinary differential equations are involved in the study of magnetohydrodynamic flow of second-grade nanofluid over a nonlinear stretching-sheet [17] and in the analysis of magneto Maxwell nano-material by a surface of variable thickness [18]. In heat conduction problems, the boundary conditions of the form (1.2) help to accommodate the nonuniformities occurring at nonlocal positions on the heat sources (finite many segments separated by points of discontinuity). Moreover, the conditions (1.2) are also helpful in modeling finitely many edge-scattering problems. For engineering applications, see [19–21]. It is expected that the results presented in this work will help establish the theoretical aspects of nonlinear coupled systems occurring in the aforementioned applications.

The main objective of the present paper is to establish the existence theory for the problems (1.1) and (1.2). We arrange the rest of the paper as follows. In Section 2, we present an auxiliary lemma, while the main results for the given problem are presented in Section 3. The paper concludes with some interesting observations.

2. An auxiliary lemma

The following lemma plays a key role in the study of the problems (1.1) and (1.2).

Lemma 2.1. *Let $f_1, g_1, h_1 \in C[a, b]$. Then the solution of the following linear system of differential equations:*

$$u'''(t) = f_1(t),\ \ v'''(t) = g_1(t),\ \ w'''(t) = h_1(t),\ t \in [a, b], \tag{2.1}$$

subject to the boundary conditions (1.2) *is equivalent to the system of integral equations:*

$$
\begin{aligned}
u(t) \;=\; & \int_a^t \frac{(t-s)^2}{2} f_1(s)ds + \frac{1}{\Lambda}\Big\{ -\int_a^b \Big[2\Lambda_1(b-s)^2 + G_1(t)(b-s) + P_1(t)\Big] f_1(s)ds \\
& - \int_a^b \Big[\Lambda_1 \sum_{j=1}^m \alpha_j (b-s)^2 + G_2(t)(b-s) + P_2(t)\Big] g_1(s)ds \\
& - \int_a^b \Big[\Lambda_1 S_{11} \frac{(b-s)^2}{2} + G_3(t)(b-s) + P_3(t)\Big] h_1(s)ds \\
& + P_3(t)\Big(\sum_{d=1}^m \kappa_d \int_a^{\eta_d} f_1(s)ds\Big) + P_1(t)\Big(\sum_{n=1}^m \gamma_n \int_a^{\eta_n} g_1(s)ds\Big) \\
& + P_2(t)\Big(\sum_{r=1}^m \sigma_r \int_a^{\eta_r} h_1(s)ds\Big) + G_3(t)\Big(\sum_{p=1}^m \zeta_p \int_a^{\eta_p} (\eta_p - s) f_1(s)ds\Big) \qquad (2.2) \\
& + G_1(t)\Big(\sum_{l=1}^m \beta_l \int_a^{\eta_l} (\eta_l - s) g_1(s)ds\Big) + G_2(t)\Big(\sum_{q=1}^m \rho_q \int_a^{\eta_q} (\eta_q - s) h_1(s)ds\Big) \\
& + \Lambda_1 S_{11}\Big(\sum_{k=1}^m \xi_k \int_a^{\eta_k} \frac{(\eta_k - s)^2}{2} f_1(s)ds\Big) + 2\Lambda_1\Big(\sum_{j=1}^m \alpha_j \int_a^{\eta_j} (\eta_j - s)^2 g_1(s)ds\Big) \\
& + \Lambda_1 \sum_{j=1}^m \alpha_j \Big(\sum_{e=1}^m \delta_e \int_a^{\eta_e} (\eta_e - s)^2 h_1(s)ds\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
v(t) \;=\; & \int_a^t \frac{(t-s)^2}{2} g_1(s)ds + \frac{1}{\Lambda}\Big\{ -\int_a^b \Big[\Lambda_1 S_{12} \frac{(b-s)^2}{2} + G_4(t)(b-s) + P_4(t)\Big] f_1(s)ds \\
& - \int_a^b \Big[2\Lambda_1 (b-s)^2 + G_5(t)(b-s) + P_5(t)\Big] g_1(s)ds \\
& - \int_a^b \Big[\Lambda_1 \sum_{e=1}^m \delta_e (b-s)^2 + G_6(t)(b-s) + P_6(t)\Big] h_1(s)ds \\
& + P_6(t)\Big(\sum_{d=1}^m \kappa_d \int_a^{\eta_d} f_1(s)ds\Big) + P_4(t)\Big(\sum_{n=1}^m \gamma_n \int_a^{\eta_n} g_1(s)ds\Big) \\
& + P_5(t)\Big(\sum_{r=1}^m \sigma_r \int_a^{\eta_r} h_1(s)ds\Big) + G_6(t)\Big(\sum_{p=1}^m \zeta_p \int_a^{\eta_p} (\eta_p - s) f_1(s)ds\Big) \qquad (2.3) \\
& + G_4(t)\Big(\sum_{l=1}^m \beta_l \int_a^{\eta_l} (\eta_l - s) g_1(s)ds\Big) + G_5(t)\Big(\sum_{q=1}^m \rho_q \int_a^{\eta_q} (\eta_q - s) h_1(s)ds\Big) \\
& + \Lambda_1 \sum_{e=1}^m \delta_e \Big(\sum_{k=1}^m \xi_k \int_a^{\eta_k} (\eta_k - s)^2 f_1(s)ds\Big) + \Lambda_1 S_{12}\Big(\sum_{j=1}^m \alpha_j \int_a^{\eta_j} \frac{(\eta_j - s)^2}{2} g_1(s)ds\Big) \\
& + 2\Lambda_1\Big(\sum_{e=1}^m \delta_e \int_a^{\eta_e} (\eta_e - s)^2 h_1(s)ds\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
w(t) \;=\; & \int_a^t \frac{(t-s)^2}{2} h_1(s)ds + \frac{1}{\Lambda}\Big\{ -\int_a^b \Big[\Lambda_1 \sum_{k=1}^m \xi_k \frac{(b-s)^2}{2} + G_7(t)(b-s) + P_7(t)\Big] f_1(s)ds \\
& - \int_a^b \Big[\Lambda_1 S_{13} \frac{(b-s)^2}{2} + G_8(t)(b-s) + P_8(t)\Big] g_1(s)ds \\
& - \int_a^b \Big[2\Lambda_1 (b-s)^2 + G_9(t)(b-s) + P_9(t)\Big] h_1(s)ds \\
& + P_9(t)\Big(\sum_{d=1}^m \kappa_d \int_a^{\eta_d} f_1(s)ds\Big) + P_7(t)\Big(\sum_{n=1}^m \gamma_n \int_a^{\eta_n} g_1(s)ds\Big) \\
& + P_8(t)\Big(\sum_{r=1}^m \sigma_r \int_a^{\eta_r} h_1(s)ds\Big) + G_9(t)\Big(\sum_{p=1}^m \zeta_p \int_a^{\eta_p} (\eta_p - s) f_1(s)ds\Big) \\
& + G_7(t)\Big(\sum_{l=1}^m \beta_l \int_a^{\eta_l} (\eta_l - s) g_1(s)ds\Big) + G_8(t)\Big(\sum_{q=1}^m \rho_q \int_a^{\eta_q} (\eta_q - s) h_1(s)ds\Big) \\
& + 2\Lambda_1\Big(\sum_{k=1}^m \xi_k \int_a^{\eta_k} (\eta_k - s)^2 f_1(s)ds\Big) + \Lambda_1 \sum_{k=1}^m \xi_k \Big(\sum_{j=1}^m \alpha_j \int_a^{\eta_j} (\eta_j - s)^2 g_1(s)ds\Big) \\
& + \Lambda_1 S_{13}\Big(\sum_{e=1}^m \delta_e \int_a^{\eta_e} \frac{(\eta_e - s)^2}{2} h_1(s)ds\Big)\Big\},
\end{aligned} \tag{2.4}
$$

where

$$
\begin{aligned}
& G_1(t) = (8 - B_1)(\mu_1 + 4\Omega(t)), \;\; G_2(t) = (8 - B_1)(\mu_2 + 2\Omega(t) \sum_{l=1}^m \beta_l), \\
& G_3(t) = (8 - B_1)(\mu_3 + S_6 \Omega(t)), \; G_4(t) = (8 - B_1)(\mu_4 + S_8 \Omega(t)), \\
& G_5(t) = (8 - B_1)(\mu_5 + 4\Omega(t)), \; G_6(t) = (8 - B_1)(\mu_6 + 2\Omega(t) \sum_{q=1}^m \rho_q), \\
& G_7(t) = (8 - B_1)(\mu_7 + 2\Omega(t) \sum_{p=1}^m \zeta_p), \; G_8(t) = (8 - B_1)(\mu_8 + S_7 \Omega(t)), \\
& G_9(t) = (8 - B_1)(\mu_9 + 4\Omega(t)), \; P_1(t) = L_1 + A_1 \Omega(t) + 2\Lambda_2 (t-a)^2, \\
& P_2(t) = L_2 + A_2 \Omega(t) + \Lambda_2 (t-a)^2 \sum_{n=1}^m \gamma_n, \; P_3(t) = L_3 + A_3 \Omega(t) + S_1 \Lambda_2 \frac{(t-a)^2}{2}, \\
& P_4(t) = L_4 + A_7 \Omega(t) + S_3 \Lambda_2 \frac{(t-a)^2}{2}, \; P_5(t) = L_5 + A_8 \Omega(t) + 2\Lambda_2 (t-a)^2, \\
& P_6(t) = L_6 + A_9 \Omega(t) + \Lambda_2 (t-a)^2 \sum_{r=1}^m \sigma_r, \; P_7(t) = L_7 + A_4 \Omega(t) + \Lambda_2 (t-a)^2 \sum_{d=1}^m \kappa_d, \\
& P_8(t) = L_8 + A_5 \Omega(t) + S_2 \Lambda_2 \frac{(t-a)^2}{2}, \; P_9(t) = L_9 + A_6 \Omega(t) + 2\Lambda_2 (t-a)^2, \\
& \Omega(t) = (8 - B_3)(t-a),
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&S_1 = \Big(\sum_{r=1}^{m}\sigma_r\Big)\Big(\sum_{n=1}^{m}\gamma_n\Big),\ \ S_2 = \Big(\sum_{n=1}^{m}\gamma_n\Big)\Big(\sum_{d=1}^{m}\kappa_d\Big),\ \ S_3 = \Big(\sum_{r=1}^{m}\sigma_r\Big)\Big(\sum_{d=1}^{m}\kappa_d\Big),\\
&S_4 = \Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{d=1}^{m}\kappa_d\Big),\ \ S_5 = \Big(\sum_{r=1}^{m}\sigma_r\Big)\Big(\sum_{l=1}^{m}\beta_l\Big),\ \ S_6 = \Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{q=1}^{m}\rho_q\Big),\\
&S_7 = \Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{p=1}^{m}\zeta_p\Big),\ \ S_8 = \Big(\sum_{p=1}^{m}\zeta_p\Big)\Big(\sum_{q=1}^{m}\rho_q\Big),\ \ S_9 = \Big(\sum_{d=1}^{m}\kappa_d\Big)\Big(\sum_{q=1}^{m}\rho_q\Big),\\
&S_{10} = \Big(\sum_{n=1}^{m}\gamma_n\Big)\Big(\sum_{q=1}^{m}\rho_q\Big),\ \ S_{11} = \Big(\sum_{e=1}^{m}\delta_e\Big)\Big(\sum_{j=1}^{m}\alpha_j\Big),\ \ S_{12} = \Big(\sum_{k=1}^{m}\xi_k\Big)\Big(\sum_{e=1}^{m}\delta_e\Big),\\
&S_{13} = \Big(\sum_{k=1}^{m}\xi_k\Big)\Big(\sum_{j=1}^{m}\alpha_j\Big),\ E_1 = \sum_{j=1}^{m}\alpha_j(\eta_j-a),\ E_2 = \sum_{j=1}^{m}\alpha_j\frac{(\eta_j-a)^2}{2},\\
&E_3 = \sum_{l=1}^{m}\beta_l(\eta_l-a),\ E_4 = \sum_{e=1}^{m}\delta_e(\eta_e-a),\ E_5 = \sum_{e=1}^{m}\delta_e\frac{(\eta_e-a)^2}{2},\ E_6 = \sum_{q=1}^{m}\rho_q(\eta_q-a),\\
&E_7 = \sum_{k=1}^{m}\xi_k(\eta_k-a),\ E_8 = \sum_{k=1}^{m}\xi_k\frac{(\eta_k-a)^2}{2},\ E_9 = \sum_{p=1}^{m}\zeta_p(\eta_p-a),
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
A_1 &= -2(b-a)\Big[8 + S_6\Big(\sum_{d=1}^{m}\kappa_d\Big) + S_3\Big(\sum_{l=1}^{m}\beta_l\Big)\Big] + 4S_6E_9 + 4S_3E_3 + 4S_4E_6,\\
A_2 &= -(b-a)\Big[S_2S_6 + 8\Big(\sum_{n=1}^{m}\gamma_n\Big) + 8\Big(\sum_{l=1}^{m}\beta_l\Big)\Big] + 2S_6E_9\Big(\sum_{n=1}^{m}\gamma_n\Big) + 16E_3 + 2S_2E_6\Big(\sum_{l=1}^{m}\beta_l\Big),\\
A_3 &= -4(b-a)\Big[S_6 + S_1 + S_5\Big] + S_1S_6E_9 + 8E_3\Big(\sum_{r=1}^{m}\sigma_r\Big) + 8E_6\Big(\sum_{l=1}^{m}\beta_l\Big),\\
A_4 &= -(b-a)\Big[8\Big(\sum_{d=1}^{m}\kappa_d\Big) + 8\Big(\sum_{p=1}^{m}\zeta_p\Big) + S_3S_7\Big] + 16E_9 + 2S_3E_3\Big(\sum_{p=1}^{m}\zeta_p\Big) + 2S_4E_6\Big(\sum_{p=1}^{m}\zeta_p\Big),\\
A_5 &= -4(b-a)\Big[S_2 + \Big(\sum_{n=1}^{m}\gamma_n\Big)\Big(\sum_{p=1}^{m}\zeta_p\Big) + S_7\Big] + 8E_9\Big(\sum_{n=1}^{m}\gamma_n\Big) + 8E_3\Big(\sum_{p=1}^{m}\zeta_p\Big) + S_2S_7E_6,\\
A_6 &= -2(b-a)\Big[8 + S_1\Big(\sum_{p=1}^{m}\zeta_p\Big) + S_5\Big(\sum_{p=1}^{m}\zeta_p\Big)\Big] + 4S_1E_9 + 4E_3\Big(\sum_{r=1}^{m}\sigma_r\Big)\Big(\sum_{p=1}^{m}\zeta_p\Big) + 4S_7E_6,\\
A_7 &= -4(b-a)\Big[S_8 + S_3 + S_9\Big] + S_3S_8E_3 + 8E_6\Big(\sum_{d=1}^{m}\kappa_d\Big) + 8E_9\Big(\sum_{q=1}^{m}\rho_q\Big),\\
A_8 &= -2(b-a)\Big[8 + S_8\Big(\sum_{n=1}^{m}\gamma_n\Big) + S_2\Big(\sum_{q=1}^{m}\rho_q\Big)\Big] + 4S_8E_3 + 4S_2E_6 + 4S_{10}E_9,\\
A_9 &= -(b-a)\Big[S_1S_8 + 8\Big(\sum_{r=1}^{m}\sigma_r\Big) + 8\Big(\sum_{q=1}^{m}\rho_q\Big)\Big] + 2S_8E_3\Big(\sum_{r=1}^{m}\sigma_r\Big) + 16E_6 + 2S_1E_9\Big(\sum_{q=1}^{m}\rho_q\Big),
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
J_1 &= E_1A_7 - A_1(b-a) + (8-B_2)(S_3E_2 - 2(b-a)^2),\\
J_2 &= E_1A_8 - A_2(b-a) + (8-B_2)(4E_2 - \sum_{n=1}^{m}\gamma_n(b-a)^2),\\
J_3 &= E_1A_9 - A_3(b-a) + (8-B_2)\Big(2E_2\sum_{r=1}^{m}\sigma_r - S_1\frac{(b-a)^2}{2}\Big),\\
J_4 &= E_4A_4 - A_7(b-a) + (8-B_2)\Big(2E_5\sum_{d=1}^{m}\kappa_d - S_3\frac{(b-a)^2}{2}\Big),\\
J_5 &= E_4A_5 - A_8(b-a) + (8-B_2)(E_5S_2 - 2(b-a)^2),\\
J_6 &= E_4A_6 - A_9(b-a) + (8-B_2)(4E_5 - \sum_{r=1}^{m}\sigma_r(b-a)^2),\\
J_7 &= E_7A_1 - A_4(b-a) + (8-B_2)(4E_8 - \sum_{d=1}^{m}\kappa_d(b-a)^2),\\
J_8 &= E_7A_2 - A_5(b-a) + (8-B_2)\Big(2E_8\sum_{n=1}^{m}\gamma_n - S_2\frac{(b-a)^2}{2}\Big),\\
J_9 &= E_7A_3 - A_6(b-a) + (8-B_2)(S_1E_8 - 2(b-a)^2),
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&\mu_1 = 4S_8E_1 - (b-a)\Big[16 + 2\Big(\sum_{j=1}^{m}\alpha_j\Big)S_8 + 2S_{11}\Big(\sum_{p=1}^{m}\zeta_p\Big)\Big] + 4S_{11}E_7\\
&+4E_4\Big(\sum_{p=1}^{m}\zeta_p\Big)\Big(\sum_{j=1}^{m}\alpha_j\Big),\ \ \mu_2 = 16E_1 - (b-a)\Big[8\Big(\sum_{l=1}^{m}\beta_l\Big) + 8\Big(\sum_{j=1}^{m}\alpha_j\Big) + S_{11}S_7\Big]\\
&+2E_4S_7\Big(\sum_{j=1}^{m}\alpha_j\Big) + 2S_{11}E_7\Big(\sum_{l=1}^{m}\beta_l\Big),\ \ \mu_3 = 8E_1\Big(\sum_{q=1}^{m}\rho_q\Big) + 8E_4\Big(\sum_{j=1}^{m}\alpha_j\Big) + S_6S_{11}E_7,\\
&-4(b-a)\Big[S_6 + S_{11} + \Big(\sum_{q=1}^{m}\rho_q\Big)\Big(\sum_{j=1}^{m}\alpha_j\Big)\Big],\\
&\mu_4 = S_8S_{12}E_1 - 4(b-a)\Big[S_{12} + S_8 + \Big(\sum_{p=1}^{m}\zeta_p\Big)\Big(\sum_{e=1}^{m}\delta_e\Big)\Big] + 8E_7\Big(\sum_{e=1}^{m}\delta_e\Big) + 8E_4\Big(\sum_{p=1}^{m}\zeta_p\Big),\\
&\mu_5 = 4S_{12}E_1 - 2(b-a)\Big[8 + S_{12}\Big(\sum_{l=1}^{m}\beta_l\Big) + S_7\Big(\sum_{e=1}^{m}\delta_e\Big)\Big] + 4S_7E_4 + 4E_7\Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{e=1}^{m}\delta_e\Big),\\
&\mu_6 = 2\Big(\sum_{q=1}^{m}\rho_q\Big)S_{12}E_1 - (b-a)\Big[S_6S_{12} + 8\Big(\sum_{q=1}^{m}\rho_q\Big) + 8\Big(\sum_{e=1}^{m}\delta_e\Big)\Big] + 16E_4 + 2\Big(\sum_{e=1}^{m}\delta_e\Big)S_6E_7,\\
&\mu_7 = 2\Big(\sum_{p=1}^{m}\zeta_p\Big)S_{13}E_4 - (b-a)\Big[S_8S_{13} + 8\Big(\sum_{k=1}^{m}\xi_k\Big) + 8\Big(\sum_{p=1}^{m}\zeta_p\Big)\Big] + 2S_8E_1\Big(\sum_{k=1}^{m}\xi_k\Big) + 16E_7,\\
&\mu_8 = S_7S_{13}E_4 - 4(b-a)\Big[S_{13} + \Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{k=1}^{m}\xi_k\Big) + S_7\Big] + 8E_1\Big(\sum_{k=1}^{m}\xi_k\Big) + 8E_7\Big(\sum_{l=1}^{m}\beta_l\Big),
\end{aligned}
\tag{2.9}
$$

$$\mu_9 = 4S_{13}E_4 - 2(b-a)\Big[8 + S_{13}\Big(\sum_{q=1}^{m}\rho_q\Big) + S_6\Big(\sum_{k=1}^{m}\xi_k\Big)\Big] + 4E_1\Big(\sum_{q=1}^{m}\rho_q\Big)\Big(\sum_{k=1}^{m}\xi_k\Big) + 4S_6E_7,$$

$$L_1 = 4J_1 + J_7S_{11} + 2J_4\sum_{j=1}^{m}\alpha_j,\ L_2 = 4J_2 + J_8S_{11} + 2J_5\sum_{j=1}^{m}\alpha_j,$$

$$L_3 = 4J_3 + J_9S_{11} + 2J_6\sum_{j=1}^{m}\alpha_j,\ L_4 = 4J_4 + J_1S_{12} + 2J_7\sum_{e=1}^{m}\delta_e,$$

$$L_5 = 4J_5 + J_2S_{12} + 2J_8\sum_{e=1}^{m}\delta_e,\ L_6 = 4J_6 + J_3S_{12} + 2J_9\sum_{e=1}^{m}\delta_e, \tag{2.10}$$

$$L_7 = 4J_7 + J_4S_{13} + 2J_1\sum_{k=1}^{m}\xi_k,\ L_8 = 4J_8 + J_5S_{13} + 2J_2\sum_{k=1}^{m}\xi_k,$$

$$L_9 = 4J_9 + J_6S_{13} + 2J_3\sum_{k=1}^{m}\xi_k,$$

and it is assumed that

$$\Lambda = (8-B_1)(8-B_2)(8-B_3) \neq 0, \tag{2.11}$$

$$\Lambda_1 = \Lambda/(8-B_3),\ \ \Lambda_2 = \Lambda/(8-B_1), B_1 = \Big(\sum_{r=1}^{m}\sigma_r\Big)\Big(\sum_{d=1}^{m}\kappa_d\Big)\Big(\sum_{n=1}^{m}\gamma_n\Big),$$

$$B_2 = \Big(\sum_{p=1}^{m}\zeta_p\Big)\Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{q=1}^{m}\rho_q\Big), B_3 = \Big(\sum_{k=1}^{m}\xi_k\Big)\Big(\sum_{j=1}^{m}\alpha_j\Big)\Big(\sum_{e=1}^{m}\delta_e\Big). \tag{2.12}$$

Proof. We know that the general solution of the linear differential equations (2.1) can be written as

$$u(t) = c_0 + c_1(t-a) + c_2\frac{(t-a)^2}{2} + \int_a^t \frac{(t-s)^2}{2} f_1(s)ds, \tag{2.13}$$

$$v(t) = c_3 + c_4(t-a) + c_5\frac{(t-a)^2}{2} + \int_a^t \frac{(t-s)^2}{2} g_1(s)ds, \tag{2.14}$$

$$w(t) = c_6 + c_7(t-a) + c_8\frac{(t-a)^2}{2} + \int_a^t \frac{(t-s)^2}{2} h_1(s)ds, \tag{2.15}$$

where $c_i \in \mathbb{R},\ i = 1, \ldots, 8$ are arbitrary real constants. Using the boundary conditions (1.2) in (2.13), (2.14) and (2.15), we obtain

$$\begin{aligned} &2c_0 + (b-a)c_1 + \frac{(b-a)^2}{2}c_2 - \Big(\sum_{j=1}^{m}\alpha_j\Big)c_3 - \Big(\sum_{j=1}^{m}\alpha_j(\eta_j - a)\Big)c_4 - \Big(\sum_{j=1}^{m}\alpha_j\frac{(\eta_j-a)^2}{2}\Big)c_5 \\ = &-\int_a^b \frac{(b-s)^2}{2} f_1(s)ds + \sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}\frac{(\eta_j-s)^2}{2}g_1(s)ds, \end{aligned} \tag{2.16}$$

$$2c_1 + (b-a)c_2 - \Big(\sum_{l=1}^{m}\beta_l\Big)c_4 - \Big(\sum_{l=1}^{m}\beta_l(\eta_l - a)\Big)c_5$$

$$= -\int_a^b (b-s)f_1(s)ds + \sum_{l=1}^{m}\beta_l \int_a^{\eta_l} (\eta_l - s)g_1(s)ds, \tag{2.17}$$

$$2c_2 - \Big(\sum_{n=1}^{m}\gamma_n\Big)c_5 = -\int_a^b f_1(s)ds + \sum_{n=1}^{m}\gamma_n \int_a^{\eta_n} g_1(s)ds, \tag{2.18}$$

$$\begin{aligned} & 2c_3 + (b-a)c_4 + \frac{(b-a)^2}{2}c_5 - \Big(\sum_{e=1}^{m}\delta_e\Big)c_6 - \Big(\sum_{e=1}^{m}\delta_e(\eta_e - a)\Big)c_7 - \Big(\sum_{e=1}^{m}\delta_e\frac{(\eta_e-a)^2}{2}\Big)c_8 \\ = & -\int_a^b \frac{(b-s)^2}{2}g_1(s)ds + \sum_{e=1}^{m}\delta_e \int_a^{\eta_e} \frac{(\eta_e-s)^2}{2}h_1(s)ds, \end{aligned} \tag{2.19}$$

$$\begin{aligned} & 2c_4 + (b-a)c_5 - \Big(\sum_{q=1}^{m}\rho_q\Big)c_7 - \Big(\sum_{q=1}^{m}\rho_q(\eta_q - a)\Big)c_8 \\ & = -\int_a^b (b-s)g_1(s)ds + \sum_{q=1}^{m}\rho_q \int_a^{\eta_q} (\eta_q - s)h_1(s)ds, \end{aligned} \tag{2.20}$$

$$2c_5 - \Big(\sum_{r=1}^{m}\sigma_r\Big)c_8 = -\int_a^b g_1(s)ds + \sum_{r=1}^{m}\sigma_r \int_a^{\eta_r} h_1(s)ds, \tag{2.21}$$

$$\begin{aligned} & -\Big(\sum_{k=1}^{m}\xi_k\Big)c_0 - \Big(\sum_{k=1}^{m}\xi_k(\eta_k - a)\Big)c_1 - \Big(\sum_{k=1}^{m}\xi_k\frac{(\eta_k-a)^2}{2}\Big)c_2 + 2c_6 + (b-a)c_7 + \frac{(b-a)^2}{2}c_8 \\ = & -\int_a^b \frac{(b-s)^2}{2}h_1(s)ds + \sum_{k=1}^{m}\xi_k \int_a^{\eta_k} \frac{(\eta_k-s)^2}{2}f_1(s)ds, \end{aligned} \tag{2.22}$$

$$\begin{aligned} & -\Big(\sum_{p=1}^{m}\zeta_p\Big)c_1 - \Big(\sum_{p=1}^{m}\zeta_p(\eta_p - a)\Big)c_2 + 2c_7 + (b-a)c_8 \\ & = -\int_a^b (b-s)h_1(s)ds + \sum_{p=1}^{m}\zeta_p \int_a^{\eta_p} (\eta_p - s)f_1(s)ds, \end{aligned} \tag{2.23}$$

$$-\Big(\sum_{d=1}^{m}\kappa_d\Big)c_2 + 2c_8 = -\int_a^b h_1(s)ds + \sum_{d=1}^{m}\kappa_d \int_a^{\eta_d} f_1(s)ds. \tag{2.24}$$

Solving (2.18), (2.21) and (2.24) for c_2, c_5 and c_8, together with the notations S_1, S_2 and S_3 given by (2.6), we get

$$\begin{aligned} c_2 = & \frac{1}{8-B_1}\Big\{-4\int_a^b f_1(s)ds - 2\Big(\sum_{n=1}^{m}\gamma_n\Big)\int_a^b g_1(s)ds - S_1\int_a^b h_1(s)ds \\ & + S_1\Big(\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d} f_1(s)ds\Big) + 4\Big(\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n} g_1(s)ds\Big) + 2\Big(\sum_{n=1}^{m}\gamma_n\Big)\Big(\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r} h_1(s)ds\Big)\Big\}, \end{aligned}$$

$$
\begin{aligned}
c_5 &= \frac{1}{8-B_1}\Big\{-S_3\int_a^b f_1(s)ds-4\int_a^b g_1(s)ds-2\Big(\sum_{r=1}^m\sigma_r\Big)\int_a^b h_1(s)ds\\
&\quad+2\Big(\sum_{r=1}^m\sigma_r\Big)\Big(\sum_{d=1}^m\kappa_d\int_a^{\eta_d} f_1(s)ds\Big)+S_3\Big(\sum_{n=1}^m\gamma_n\int_a^{\eta_n} g_1(s)ds\Big)+4\Big(\sum_{r=1}^m\sigma_r\int_a^{\eta_r} h_1(s)ds\Big)\Big\},\\
c_8 &= \frac{1}{8-B_1}\Big\{-2\Big(\sum_{d=1}^m\kappa_d\Big)\int_a^b f_1(s)ds-S_2\int_a^b g_1(s)ds-4\int_a^b h_1(s)ds\\
&\quad+4\Big(\sum_{d=1}^m\kappa_d\int_a^{\eta_d} f_1(s)ds\Big)+2\Big(\sum_{d=1}^m\kappa_d\Big)\Big(\sum_{n=1}^m\gamma_n\int_a^{\eta_n} g_1(s)ds\Big)+S_2\Big(\sum_{r=1}^m\sigma_r\int_a^{\eta_r} h_1(s)ds\Big)\Big\}.
\end{aligned}
$$

Inserting the values of c_2, c_5 and c_8 in (2.17), (2.20) and (2.23), and using (2.6), we obtain

$$
\begin{aligned}
2c_1-\Big(\sum_{l=1}^m\beta_l\Big)c_4 &= \frac{1}{8-B_1}\Big\{-\int_a^b\Big[(b-s)(8-B_1)+S_3E_3-4(b-a)\Big]f_1(s)ds\\
&\quad-\int_a^b\Big[4E_3-2(b-a)\Big(\sum_{n=1}^m\gamma_n\Big)\Big]g_1(s)ds\\
&\quad-\int_a^b\Big[2E_3\Big(\sum_{r=1}^m\sigma_r\Big)-S_1(b-a)\Big]h_1(s)ds\\
&\quad+\sum_{d=1}^m\kappa_d\int_a^{\eta_d}\Big[2E_3\sum_{r=1}^m\sigma_r-S_1(b-a)\Big]f_1(s)ds\\
&\quad+\sum_{n=1}^m\gamma_n\int_a^{\eta_n}\Big[S_3E_3-4(b-a)\Big]g_1(s)ds\\
&\quad+\sum_{r=1}^m\sigma_r\int_a^{\eta_r}\Big[4E_3-2(b-a)\Big(\sum_{n=1}^m\gamma_n\Big)\Big]h_1(s)ds\Big\}\\
&\quad+\sum_{l=1}^m\beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds,
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
2c_4-\Big(\sum_{q=1}^m\rho_q\Big)c_7 &= \frac{1}{8-B_1}\Big\{-\int_a^b\Big[2E_6\Big(\sum_{d=1}^m\kappa_d\Big)-S_3(b-a)\Big]f_1(s)ds\\
&\quad-\int_a^b\Big[(b-s)(8-B_1)+S_2E_6-4(b-a)\Big]g_1(s)ds\\
&\quad-\int_a^b\Big[4E_6-2(b-a)\Big(\sum_{r=1}^m\sigma_r\Big)\Big]h_1(s)ds\\
&\quad+\sum_{d=1}^m\kappa_d\int_a^{\eta_d}\Big[4E_6-2(b-a)\Big(\sum_{r=1}^m\sigma_r\Big)\Big]f_1(s)ds\\
&\quad+\sum_{n=1}^m\gamma_n\int_a^{\eta_n}\Big[2E_6\Big(\sum_{d=1}^m\kappa_d\Big)-S_3(b-a)\Big]g_1(s)ds
\end{aligned}
\tag{2.26}
$$

$$+\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}\Big[S_2E_6-4(b-a)\Big]h_1(s)ds\Big\}+\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds,$$

$$\begin{aligned}
-\Big(\sum_{p=1}^{m}\zeta_p\Big)c_1+2c_7 \;=\; & \frac{1}{8-B_1}\Big\{-\int_a^b\Big[4E_9-2\Big(\sum_{d=1}^{m}\kappa_d\Big)(b-a)\Big]f_1(s)ds \\
& -\int_a^b\Big[2E_9\Big(\sum_{n=1}^{m}\gamma_n\Big)-S_2(b-a)\Big]g_1(s)ds \\
& -\int_a^b\Big[(b-s)(8-B_1)+S_1E_9-4(b-a)\Big]h_1(s)ds \\
& +\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}\Big[S_1E_9-4(b-a)\Big]f_1(s)ds \\
& +\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}\Big[4E_9-2\Big(\sum_{d=1}^{m}\kappa_d\Big)(b-a)\Big]g_1(s)ds \\
& +\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}\Big[2E_9\Big(\sum_{n=1}^{m}\gamma_n\Big)-S_2(b-a)\Big]h_1(s)ds\Big\} \\
& +\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds.
\end{aligned} \tag{2.27}$$

Solving the systems (2.25) – (2.27) for c_1, c_4 and c_7 together with the notations (2.7) we find that

$$\begin{aligned}
c_1 \;=\; & \frac{1}{\Lambda_1}\Big\{-\int_a^b\Big[4(8-B_1)(b-s)+A_1\Big]f_1(s)ds-\int_a^b\Big[2(8-B_1)(b-s)\Big(\sum_{l=1}^{m}\beta_l\Big) \\
& +A_2\Big]g_1(s)ds-\int_a^b\Big[S_6(8-B_1)(b-s)+A_3\Big]h_1(s)ds+A_3\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}f_1(s)ds \\
& +A_1\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}g_1(s)ds+A_2\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}h_1(s)ds \\
& +S_6(8-B_1)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds\Big)+4(8-B_1)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds\Big) \\
& +2(8-B_1)(\sum_{l=1}^{m}\beta_l)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds\Big)\Big\}, \\
c_4 \;=\; & \frac{1}{\Lambda_1}\Big\{-\int_a^b\Big[S_8(8-B_1)(b-s)+A_7\Big]f_1(s)ds-\int_a^b\Big[4(8-B_1)(b-s)+A_8\Big]g_1(s)ds \\
& -\int_a^b\Big[2(\sum_{q=1}^{m}\rho_q)(8-B_1)(b-s)+A_9\Big]h_1(s)ds+A_9\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}f_1(s)ds \\
& +A_7\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}g_1(s)ds+A_8\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}h_1(s)ds
\end{aligned}$$

$$+2(8-B_1)(\sum_{q=1}^{m}\rho_q)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds\Big)$$
$$+S_8(8-B_1)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds\Big)+4(8-B_1)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds\Big)\Big\},$$

$$c_7 = \frac{1}{\Lambda_1}\Big\{-\int_a^b\Big[2(\sum_{p=1}^{m}\zeta_p)(8-B_1)(b-s)+A_4\Big]f_1(s)ds-\int_a^b\Big[S_7(8-B_1)(b-s)$$
$$+A_5\Big]g_1(s)ds-\int_a^b\Big[4(8-B_1)(b-s)+A_6\Big]h_1(s)ds+A_6\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}f_1(s)ds$$
$$+A_4\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}g_1(s)ds+A_5\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}h_1(s)ds+4(8-B_1)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds\Big)$$
$$+2(8-B_1)(\sum_{p=1}^{m}\zeta_p)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds\Big)$$
$$+S_7(8-B_1)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds\Big)\Big\}.$$

Substituting the values of $c_1,\ c_2,\ c_4,\ c_5,\ c_7$ and c_8 in (2.16), (2.19) and (2.22), together with the notations (2.6) and (2.8) yields

$$2c_0-\Big(\sum_{j=1}^{m}\alpha_j\Big)c_3=\frac{1}{\Lambda_1}\Big\{-\int_a^b\Big[\Lambda_1\frac{(b-s)^2}{2}+((8-B_1)(b-s))(S_8E_1-4(b-a))+J_1\Big]f_1(s)ds$$
$$-\int_a^b\Big[((8-B_1)(b-s))(4E_1-2\sum_{l=1}^{m}\beta_l(b-a))+J_2\Big]g_1(s)ds$$
$$-\int_a^b\Big[((8-B_1)(b-s))(2E_1\sum_{q=1}^{m}\rho_q-S_6(b-a))+J_3\Big]h_1(s)ds$$
$$+J_3\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}f_1(s)ds+J_1\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}g_1(s)ds+J_2\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}h_1(s)ds$$
$$+(8-B_1)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)\Big[2E_1\sum_{q=1}^{m}\rho_q-S_6(b-a)\Big]f_1(s)ds\Big) \tag{2.28}$$
$$+(8-B_1)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)\Big[S_8E_1-4(b-a)\Big]g_1(s)ds\Big)$$
$$+(8-B_1)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)\Big[4E_1-2\sum_{l=1}^{m}\beta_l(b-a)\Big]h_1(s)ds\Big)$$
$$+\Lambda_1\Big(\sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}\frac{(\eta_j-s)^2}{2}g_1(s)ds\Big)\Big\},$$

$$
\begin{aligned}
2c_3 - \Big(\sum_{e=1}^{m}\delta_e\Big)c_6 = \frac{1}{\Lambda_1}\Big\{ &- \int_a^b \Big[((8-B_1)(b-s))(2E_4\sum_{p=1}^{m}\zeta_p - S_8(b-a)) + J_4\Big]f_1(s)ds \\
&- \int_a^b \Big[\Lambda_1\frac{(b-s)^2}{2} + ((8-B_1)(b-s))(S_6E_4 - 4(b-a)) + J_5\Big]g_1(s)ds \\
&- \int_a^b \Big[((8-B_1)(b-s))(4E_4 - 2\sum_{q=1}^{m}\rho_q(b-a)) + J_6\Big]h_1(s)ds \\
&+J_6\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d} f_1(s)ds + J_4\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n} g_1(s)ds + J_5\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r} h_1(s)ds \\
&+(8-B_1)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p - s)\Big[4E_4 - 2\sum_{q=1}^{m}\rho_q(b-a)\Big]f_1(s)ds\Big) \\
&+(8-B_1)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l - s)\Big[2E_4\sum_{p=1}^{m}\zeta_p - S_8(b-a)\Big]g_1(s)ds\Big) \\
&+(8-B_1)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q - s)\Big[S_6E_4 - 4(b-a)\Big]h_1(s)ds\Big) \\
&+\Lambda_1\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}\frac{(\eta_e - s)^2}{2}h_1(s)ds\Big)\Big\},
\end{aligned} \tag{2.29}
$$

$$
\begin{aligned}
-\Big(\sum_{k=1}^{m}\xi_k\Big)c_0 + 2c_6 = \frac{1}{\Lambda_1}\Big\{ &- \int_a^b \Big[((8-B_1)(b-s))(4E_7 - 2\sum_{p=1}^{m}\zeta_p(b-a)) + J_7\Big]f_1(s)ds \\
&- \int_a^b \Big[((8-B_1)(b-s))(2E_7\sum_{l=1}^{m}\beta_l - S_6(b-a)) + J_8\Big]g_1(s)ds \\
&- \int_a^b \Big[\Lambda_1\frac{(b-s)^2}{2} + ((8-B_1)(b-s))(S_6E_7 - 4(b-a)) + J_9\Big]h_1(s)ds \\
&+J_9\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d} f_1(s)ds + J_7\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n} g_1(s)ds + J_8\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r} h_1(s)ds \\
&+(8-B_1)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p - s)\Big[S_6E_7 - 4(b-a)\Big]f_1(s)ds\Big) \\
&+(8-B_1)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l - s)\Big[4E_7 - 2\sum_{p=1}^{m}\zeta_p(b-a)\Big]g_1(s)ds\Big) \\
&+(8-B_1)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q - s)\Big[2E_7\sum_{l=1}^{m}\beta_l - S_6(b-a)\Big]h_1(s)ds\Big) \\
&+\Lambda_1\Big(\sum_{k=1}^{m}\xi_k\int_a^{\eta_k}\frac{(\eta_k - s)^2}{2}f_1(s)ds\Big)\Big\}.
\end{aligned} \tag{2.30}
$$

Next, solving the system of Eqs (2.28) − (2.30) for c_0, c_3 and c_6 together with the notations (2.9), we obtain

$$
\begin{aligned}
c_0 &= \frac{1}{\Lambda}\Big\{-\int_a^b \Big[2\Lambda_1(b-s)^2+\mu_1(8-B_1)(b-s)+L_1\Big]f_1(s)ds \\
&-\int_a^b \Big[\Lambda_1(\sum_{j=1}^m \alpha_j)(b-s)^2+\mu_2(8-B_1)(b-s)+L_2\Big]g_1(s)ds \\
&-\int_a^b \Big[\Lambda_1 S_{11}\frac{(b-s)^2}{2}+\mu_3(8-B_1)(b-s)+L_3\Big]h_1(s)ds \\
&+L_3\sum_{d=1}^m \kappa_d\int_a^{\eta_d} f_1(s)ds+L_1\sum_{n=1}^m \gamma_n\int_a^{\eta_n} g_1(s)ds+L_2\sum_{r=1}^m \sigma_r\int_a^{\eta_r} h_1(s)ds \\
&+\mu_3(8-B_1)\Big(\sum_{p=1}^m \zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds\Big)+\mu_1(8-B_1)\Big(\sum_{l=1}^m \beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds\Big) \\
&+\mu_2(8-B_1)\Big(\sum_{q=1}^m \rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds\Big)+\Lambda_1 S_{11}\Big(\sum_{k=1}^m \xi_k\int_a^{\eta_k}\frac{(\eta_k-s)^2}{2}f_1(s)ds\Big) \\
&+2\Lambda_1\Big(\sum_{j=1}^m \alpha_j\int_a^{\eta_j}(\eta_j-s)^2 g_1(s)ds\Big)+\Lambda_1\Big(\sum_{j=1}^m \alpha_j\Big)\Big(\sum_{e=1}^m \delta_e\int_a^{\eta_e}(\eta_e-s)^2 h_1(s)ds\Big)\Big\}, \\
c_3 &= \frac{1}{\Lambda}\Big\{-\int_a^b \Big[\Lambda_1 S_{12}\frac{(b-s)^2}{2}+\mu_4(8-B_1)(b-s)+L_4\Big]f_1(s)ds \\
&-\int_a^b \Big[2\Lambda_1(b-s)^2+\mu_5(8-B_1)(b-s)+L_5\Big]g_1(s)ds \\
&-\int_a^b \Big[\Lambda_1(\sum_{e=1}^m \delta_e)(b-s)^2+\mu_6(8-B_1)(b-s)+L_6\Big]h_1(s)ds \\
&+L_6\sum_{d=1}^m \kappa_d\int_a^{\eta_d} f_1(s)ds+L_4\sum_{n=1}^m \gamma_n\int_a^{\eta_n} g_1(s)ds+L_5\sum_{r=1}^m \sigma_r\int_a^{\eta_r} h_1(s)ds \\
&+\mu_6(8-B_1)\Big(\sum_{p=1}^m \zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds\Big)+\mu_4(8-B_1)\Big(\sum_{l=1}^m \beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds\Big) \\
&+\mu_5(8-B_1)\Big(\sum_{q=1}^m \rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds\Big)+\Lambda_1(\sum_{e=1}^m \delta_e)\Big(\sum_{k=1}^m \xi_k\int_a^{\eta_k}(\eta_k-s)^2 f_1(s)ds\Big) \\
&+\Lambda_1 S_{12}\Big(\sum_{j=1}^m \alpha_j\int_a^{\eta_j}\frac{(\eta_j-s)^2}{2}g_1(s)ds\Big)+2\Lambda_1\Big(\sum_{e=1}^m \delta_e\int_a^{\eta_e}(\eta_e-s)^2 h_1(s)ds\Big)\Big\}, \\
c_6 &= \frac{1}{\Lambda}\Big\{-\int_a^b \Big[\Lambda_1(\sum_{k=1}^m \xi_k)(b-s)^2+\mu_7(8-B_1)(b-s)+L_7\Big]f_1(s)ds \\
&-\int_a^b \Big[\Lambda_1 S_{13}\frac{(b-s)^2}{2}+\mu_8(8-B_1)(b-s)+L_8\Big]g_1(s)ds \\
&-\int_a^b \Big[2\Lambda_1(b-s)^2+\mu_9(8-B_1)(b-s)+L_9\Big]h_1(s)ds
\end{aligned}
$$

$$+L_9\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}f_1(s)ds+L_7\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}g_1(s)ds+L_8\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}h_1(s)ds$$
$$+\mu_9(8-B_1)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)f_1(s)ds\Big)+\mu_7(8-B_1)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)g_1(s)ds\Big)$$
$$+\mu_8(8-B_1)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)h_1(s)ds\Big)+2\Lambda_1\Big(\sum_{k=1}^{m}\xi_k\int_a^{\eta_k}\frac{(\eta_k-s)^2}{2}f_1(s)ds\Big)$$
$$+\Lambda_1\Big(\sum_{k=1}^{m}\xi_k\Big)\Big(\sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}\frac{(\eta_j-s)^2}{2}g_1(s)ds\Big)+\Lambda_1S_{13}\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}\frac{(\eta_e-s)^2}{2}h_1(s)ds\Big)\Big\}.$$

Inserting the values of c_i $(i=1,\ldots,8)$ in (2.13), (2.14) and (2.15), we get the solutions (2.2), (2.3) and (2.4). The converse follows by direct computation. This completes the proof. □

3. Main results

Let us introduce the space $\mathcal{X}=\{u(t)|u(t)\in C([a,b])\}$ equipped with norm $\|u\|=\sup\{|u(t)|,t\in[a,b]\}$. Obviously $(\mathcal{X},\|.\|)$ is a Banach space and consequently, the product space $(\mathcal{X}\times\mathcal{X}\times\mathcal{X},\|(u,v,w)\|)$ is a Banach space with norm $\|(u,v,w)\|=\|u\|+\|v\|+\|w\|$ for $(u,v,w)\in\mathcal{X}^3$. In view of Lemma 2.1, we transform the problems (1.1) and (1.2) into an equivalent fixed point problem as

$$(u,v,w)=\mathcal{H}(u,v,w), \tag{3.1}$$

where $\mathcal{H}:\mathcal{X}^3\to\mathcal{X}^3$ is defined by

$$\mathcal{H}(u,v,w)(t)=(\mathcal{H}_1(u,v,w)(t),\mathcal{H}_2(u,v,w)(t),\mathcal{H}_3(u,v,w)(t)), \tag{3.2}$$

$$\begin{aligned}
&\mathcal{H}_1(u,v,w)(t)\\
=\;&\int_a^t\frac{(t-s)^2}{2}\widehat{f}(s)ds+\frac{1}{\Lambda}\Big\{-\int_a^b\Big[2\Lambda_1(b-s)^2+G_1(t)(b-s)+P_1(t)\Big]\widehat{f}(s)ds\\
&-\int_a^b\Big[\Lambda_1\sum_{j=1}^{m}\alpha_j(b-s)^2+G_2(t)(b-s)+P_2(t)\Big]\widehat{g}(s)ds\\
&-\int_a^b\Big[\Lambda_1S_{11}\frac{(b-s)^2}{2}+G_3(t)(b-s)+P_3(t)\Big]\widehat{h}(s)ds\\
&+P_3(t)\Big(\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}\widehat{f}(s)ds\Big)+P_1(t)\Big(\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}\widehat{g}(s)ds\Big)\\
&+P_2(t)\Big(\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}\widehat{h}(s)ds\Big)+G_3(t)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)\widehat{f}(s)ds\Big)\\
&+G_1(t)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)\widehat{g}(s)ds\Big)+G_2(t)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)\widehat{h}(s)ds\Big)
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&+\Lambda_1 S_{11}\Big(\sum_{k=1}^{m}\xi_k\int_a^{\eta_k}\frac{(\eta_k-s)^2}{2}\widehat{f}(s)ds\Big)+2\Lambda_1\Big(\sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}(\eta_j-s)^2\widehat{g}(s)ds\Big)\\
&+\Lambda_1\sum_{j=1}^{m}\alpha_j\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}(\eta_e-s)^2\widehat{h}(s)ds\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{H}_2(u,v,w)(t)\\
=&\int_a^t\frac{(t-s)^2}{2}\widehat{g}(s)ds+\frac{1}{\Lambda}\Big\{-\int_a^b\Big[\Lambda_1 S_{12}\frac{(b-s)^2}{2}+G_4(t)(b-s)+P_4(t)\Big]\widehat{f}(s)ds\\
&-\int_a^b\Big[2\Lambda_1(b-s)^2+G_5(t)(b-s)+P_5(t)\Big]\widehat{g}(s)ds\\
&-\int_a^b\Big[\Lambda_1\sum_{e=1}^{m}\delta_e(b-s)^2+G_6(t)(b-s)+P_6(t)\Big]\widehat{h}(s)ds\\
&+P_6(t)\Big(\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}\widehat{f}(s)ds\Big)+P_4(t)\Big(\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}\widehat{g}(s)ds\Big) \qquad (3.4)\\
&+P_5(t)\Big(\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}\widehat{h}(s)ds\Big)+G_6(t)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)\widehat{f}(s)ds\Big)\\
&+G_4(t)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)\widehat{g}(s)ds\Big)+G_5(t)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)\widehat{h}(s)ds\Big)\\
&+\Lambda_1\sum_{e=1}^{m}\delta_e\Big(\sum_{k=1}^{m}\xi_k\int_a^{\eta_k}(\eta_k-s)^2\widehat{f}(s)ds\Big)\\
&+\Lambda_1 S_{12}\Big(\sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}\frac{(\eta_j-s)^2}{2}\widehat{g}(s)ds\Big)+2\Lambda_1\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}(\eta_e-s)^2\widehat{h}(s)ds\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{H}_3(u,v,w)(t)\\
=&\int_a^t\frac{(t-s)^2}{2}\widehat{h}(s)ds+\frac{1}{\Lambda}\Big\{-\int_a^b\Big[\Lambda_1\sum_{k=1}^{m}\xi_k\frac{(b-s)^2}{2}+G_7(t)(b-s)+P_7(t)\Big]\widehat{f}(s)ds\\
&-\int_a^b\Big[\Lambda_1 S_{13}\frac{(b-s)^2}{2}+G_8(t)(b-s)+P_8(t)\Big]\widehat{g}(s)ds\\
&-\int_a^b\Big[2\Lambda_1(b-s)^2+G_9(t)(b-s)+P_9(t)\Big]\widehat{h}(s)ds\\
&+P_9(t)\Big(\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}\widehat{f}(s)ds\Big)+P_7(t)\Big(\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}\widehat{g}(s)ds\Big) \qquad (3.5)\\
&+P_8(t)\Big(\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}\widehat{h}(s)ds\Big)+G_9(t)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)\widehat{f}(s)ds\Big)\\
&+G_7(t)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)\widehat{g}(s)ds\Big)+G_8(t)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)\widehat{h}(s)ds\Big)
\end{aligned}
$$

$$+2\Lambda_1\Big(\sum_{k=1}^{m}\xi_k\int_a^{\eta_k}(\eta_k-s)^2\widehat{f}(s)ds\Big)+\Lambda_1\sum_{k=1}^{m}\xi_k\Big(\sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}(\eta_j-s)^2\widehat{g}(s)ds\Big)$$
$$+\Lambda_1 S_{13}\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}\frac{(\eta_e-s)^2}{2}\widehat{h}(s)ds\Big)\Big\},$$

$$\widehat{f}(s)=f(s,u(s),v(s),w(s)),\ \widehat{g}(s)=g(s,u(s),v(s),w(s)),\ \widehat{h}(s)=h(s,u(s),v(s),w(s)).$$

In order to establish the main results, we need the following assumptions:

(N_1) (Linear growth conditions) There exist real constants $m_i,\bar{m}_i,\widehat{m}_i\geq 0,\ (i=1,2,3)$ and $m_0>0,\ \bar{m}_0>0,\ \widehat{m}_0>0$ such that $\forall\ u,v,w\in\mathbb{R}$, we have

$$|f(t,u,v,w)|\leq m_0+m_1|u|+m_2|v|+m_3|w|,$$
$$|g(t,u,v,w)|\leq \bar{m}_0+\bar{m}_1|u|+\bar{m}_2|v|+\bar{m}_3|w|,$$
$$|h(t,u,v,w)|\leq \widehat{m}_0+\widehat{m}_1|u|+\widehat{m}_2|v|+\widehat{m}_3|w|.$$

(N_2) (Sub-growth conditions) There exist nonnegative functions $\phi(t),\ \psi(t)$ and $\chi(t)\in L(a,b)$ and $\epsilon_i>0,\ 0<\lambda_i<1,(i=1,\dots,9)$ such that $\forall\ u,v,w\in\mathbb{R}$, we have

$$|f(t,u,v,w)|\leq \phi(t)+\epsilon_1|u|^{\lambda_1}+\epsilon_2|v|^{\lambda_2}+\epsilon_3|w|^{\lambda_3},$$
$$|g(t,u,v,w)|\leq \psi(t)+\epsilon_4|u|^{\lambda_4}+\epsilon_5|v|^{\lambda_5}+\epsilon_6|w|^{\lambda_6},$$
$$|h(t,u,v,w)|\leq \chi(t)+\epsilon_7|u|^{\lambda_7}+\epsilon_8|v|^{\lambda_8}+\epsilon_9|w|^{\lambda_9}.$$

(N_3) (Lipschitz conditions) For all $t\in[a,b]$ and $u_i,v_i,w_i\in\mathbb{R},\ i=1,2$ there exist $\ell_i>0\ (i=1,2,3)$ such that

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|\leq \ell_1(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$
$$|g(t,u_1,v_1,w_1)-g(t,u_2,v_2,w_2)|\leq \ell_2(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$
$$|h(t,u_1,v_1,w_1)-h(t,u_2,v_2,w_2)|\leq \ell_3(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).$$

For the sake of computational convenience, we set

$$\Theta_1=\Delta_1+\Delta_4+\Delta_7,\quad \Theta_2=\Delta_2+\Delta_5+\Delta_8,\quad \Theta_3=\Delta_3+\Delta_6+\Delta_9, \tag{3.6}$$

where

$$\begin{aligned}\Delta_1 &= \frac{(b-a)^3}{6}+\frac{1}{3|8-B_3|}\Big[2(b-a)^3+S_{11}\Big(\sum_{k=1}^{m}\xi_k\frac{(\eta_k-a)^3}{2}\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_1\frac{(b-a)^2}{2}\\ &\quad+\Upsilon_1(b-a)+\Upsilon_3\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)+Q_3\Big(\sum_{p=1}^{m}\zeta_p\frac{(\eta_p-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.7}$$

$$\Delta_2 = \frac{\sum_{j=1}^{m}\alpha_j}{3|8-B_3|}\Big[(b-a)^3+2(\eta_j-a)^3\Big]+\frac{1}{|\Lambda|}\Big[Q_2\frac{(b-a)^2}{2}+\Upsilon_2(b-a)$$

$$+\Upsilon_1\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+Q_1\Big(\sum_{l=1}^{m}\beta_l\frac{(\eta_l-a)^2}{2}\Big)\Big], \tag{3.8}$$

$$\begin{aligned}\Delta_3 &= \frac{1}{3|8-B_3|}\Big[S_{11}\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^{m}\alpha_j\Big)\Big(\sum_{e=1}^{m}\delta_e(\eta_e-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_3\frac{(b-a)^2}{2}\\ &\quad +\Upsilon_3(b-a)+\Upsilon_2\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)+Q_2\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Delta_4 &= \frac{1}{3|8-B_3|}\Big[S_{12}\frac{(b-a)^3}{2}+\Big(\sum_{e=1}^{m}\delta_e\Big)\Big(\sum_{k=1}^{m}\xi_k(\eta_k-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_4\frac{(b-a)^2}{2}\\ &\quad +\Upsilon_4(b-a)+\Upsilon_6\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)+Q_6\Big(\sum_{p=1}^{m}\zeta_p\frac{(\eta_p-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Delta_5 &= \frac{(b-a)^3}{6}+\frac{1}{3|8-B_3|}\Big[2(b-a)^3+S_{12}\Big(\sum_{j=1}^{m}\alpha_j\frac{(\eta_j-a)^3}{2}\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_5\frac{(b-a)^2}{2}\\ &\quad +\Upsilon_5(b-a)+\Upsilon_4\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+Q_4\Big(\sum_{l=1}^{m}\beta_l\frac{(\eta_l-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.11}$$

$$\begin{aligned}\Delta_6 &= \frac{\sum_{e=1}^{m}\delta_e}{3|8-B_3|}\Big[(b-a)^3+2(\eta_e-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_6\frac{(b-a)^2}{2}+\Upsilon_6(b-a)\\ &\quad +\Upsilon_5\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)+Q_5\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.12}$$

$$\begin{aligned}\Delta_7 &= \frac{\sum_{k=1}^{m}\xi_k}{3|8-B_3|}\Big[\frac{(b-a)^3}{2}+2(\eta_k-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_7\frac{(b-a)^2}{2}+\Upsilon_7(b-a)\\ &\quad +\Upsilon_9\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)+Q_9\Big(\sum_{p=1}^{m}\zeta_p\frac{(\eta_p-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.13}$$

$$\begin{aligned}\Delta_8 &= \frac{1}{3|8-B_3|}\Big[S_{13}\frac{(b-a)^3}{2}+\Big(\sum_{k=1}^{m}\xi_k\Big)\Big(\sum_{j=1}^{m}\alpha_j(\eta_j-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_8\frac{(b-a)^2}{2}\\ &\quad +\Upsilon_8(b-a)+\Upsilon_7\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+Q_7\Big(\sum_{l=1}^{m}\beta_l\frac{(\eta_l-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Delta_9 &= \frac{(b-a)^3}{6}+\frac{1}{3|8-B_3|}\Big[2(b-a)^3+S_{13}\Big(\sum_{e=1}^{m}\delta_e\frac{(\eta_e-a)^3}{2}\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_9\frac{(b-a)^2}{2}\\ &\quad +\Upsilon_9(b-a)+\Upsilon_8\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)+Q_8\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big],\end{aligned} \tag{3.15}$$

$Q_i = \max_{t\in[a,b]}|G_i(t)|$, and $\Upsilon_i = \max_{t\in[a,b]}|P_i(t)|$, $(i = 1,\dots,9)$. Also, we set

$$\begin{aligned}\Theta = \ &\min\{1-(\Theta_1 m_1+\Theta_2\bar{m}_1+\Theta_3\widehat{m}_1),1-(\Theta_1 m_2+\Theta_2\bar{m}_2+\Theta_3\widehat{m}_2),\\ &1-(\Theta_1 m_3+\Theta_2\bar{m}_3+\Theta_3\widehat{m}_3)\},\end{aligned} \tag{3.16}$$

where $m_i, \bar{m}_i, \widehat{m}_i$ are given in (N_1).

3.1. Existence results

Firstly, we apply Leray-Schauder alternative [22] to prove the existence of solutions for the problems (1.1) and (1.2).

Lemma 3.1. *(Leray-Schauder alternative). Let $\mathcal{Y}$ be a Banach space, and $T : \mathcal{Y} \to \mathcal{Y}$ be a completely continuous operator (i.e., a map restricted to any bounded set in $\mathcal{Y}$ is compact). Let $\Xi(T) = \{x \in \mathcal{Y} : x = \varphi T(x)$ for some $0 < \varphi < 1\}$. Then either the set $\Xi(T)$ is unbounded, or T has at least one fixed point.*

Theorem 3.1. *Assume that the condition (N_1) holds and that*

$$\Theta_1 m_1 + \Theta_2 \bar{m}_1 + \Theta_3 \widehat{m}_1 < 1,\ \Theta_1 m_2 + \Theta_2 \bar{m}_2 + \Theta_3 \widehat{m}_2 < 1 \text{ and } \Theta_1 m_3 + \Theta_2 \bar{m}_3 + \Theta_3 \widehat{m}_3 < 1, \tag{3.17}$$

where Θ_1, Θ_2 and Θ_3 are given by (3.6). Then there exists at least one solution for the problem (1.1) and (1.2) on $[a, b]$.

Proof. First of all, we show that the operator $\mathcal{H} : \mathcal{X}^3 \to \mathcal{X}^3$ defined by (3.2) is completely continuous. Notice that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are continuous in view of continuity of the functions f, g and h. So the operator $\mathcal{H}$ is continuous. Let $\Phi \subset \mathcal{X}^3$ be a bounded set. Then there exist positive constants ϱ_f, ϱ_g and ϱ_h such that $|\widehat{f}(t)| = |f(t, u(t), v(t), w(t))| \le \varrho_f$, $|\widehat{g}(t)| = |g(t, u(t), v(t), w(t))| \le \varrho_g$ and $|\widehat{h}(t)| = |h(t, u(t), v(t), w(t))| \le \varrho_h$, $\forall (u, v, w) \in \Phi$. Then, for any $(u, v, w) \in \Phi$, we obtain

$$\begin{aligned}
&|\mathcal{H}_1(u, v, w)(t)| \\
= &\ \Big|\int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds + \frac{1}{\Lambda}\Big\{-\int_a^b \Big[2\Lambda_1(b-s)^2 + G_1(t)(b-s) + P_1(t)\Big]\widehat{f}(s)ds \\
&-\int_a^b \Big[\Lambda_1 \sum_{j=1}^m \alpha_j (b-s)^2 + G_2(t)(b-s) + P_2(t)\Big]\widehat{g}(s)ds \\
&-\int_a^b \Big[\Lambda_1 S_{11}\frac{(b-s)^2}{2} + G_3(t)(b-s) + P_3(t)\Big]\widehat{h}(s)ds \\
&+P_3(t)\Big(\sum_{d=1}^m \kappa_d \int_a^{\eta_d} \widehat{f}(s)ds\Big) + P_1(t)\Big(\sum_{n=1}^m \gamma_n \int_a^{\eta_n} \widehat{g}(s)ds\Big) \\
&+P_2(t)\Big(\sum_{r=1}^m \sigma_r \int_a^{\eta_r} \widehat{h}(s)ds\Big) + G_3(t)\Big(\sum_{p=1}^m \zeta_p \int_a^{\eta_p} (\eta_p - s)\widehat{f}(s)ds\Big) \\
&+G_1(t)\Big(\sum_{l=1}^m \beta_l \int_a^{\eta_l} (\eta_l - s)\widehat{g}(s)ds\Big) + G_2(t)\Big(\sum_{q=1}^m \rho_q \int_a^{\eta_q} (\eta_q - s)\widehat{h}(s)ds\Big) \\
&+\Lambda_1 S_{11}\Big(\sum_{k=1}^m \xi_k \int_a^{\eta_k} \frac{(\eta_k - s)^2}{2}\widehat{f}(s)ds\Big) + 2\Lambda_1\Big(\sum_{j=1}^m \alpha_j \int_a^{\eta_j} (\eta_j - s)^2 \widehat{g}(s)ds\Big) \\
&+\Lambda_1 \sum_{j=1}^m \alpha_j\Big(\sum_{e=1}^m \delta_e \int_a^{\eta_e} (\eta_e - s)^2 \widehat{h}(s)ds\Big)\Big\}\Big| \\
\le &\ \varrho_f\Big\{\frac{(b-a)^3}{6} + \frac{1}{3|8 - B_3|}\Big[2(b-a)^3 + S_{11}\Big(\sum_{k=1}^m \xi_k \frac{(\eta_k - a)^3}{2}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}\Big[Q_1\frac{(b-a)^2}{2}+\Upsilon_1(b-a)+\Upsilon_3\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)+Q_3\Big(\sum_{p=1}^{m}\zeta_p\frac{(\eta_p-a)^2}{2}\Big)\Big]\Big\}\\
&+\varrho_g\Big\{\frac{\sum_{j=1}^{m}\alpha_j}{3|8-B_3|}\Big[(b-a)^3+2(\eta_j-a)^3\Big]+\frac{1}{|\Lambda|}\Big[Q_2\frac{(b-a)^2}{2}+\Upsilon_2(b-a)\\
&+\Upsilon_1\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+Q_1\Big(\sum_{l=1}^{m}\beta_l\frac{(\eta_l-a)^2}{2}\Big)\Big]\Big\}\\
&+\varrho_h\Big\{\frac{1}{3|8-B_3|}\Big[S_{11}\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^{m}\alpha_j\Big)\Big(\sum_{e=1}^{m}\delta_e(\eta_e-a)^3\Big)\Big]\\
&+\frac{1}{|\Lambda|}\Big[Q_3\frac{(b-a)^2}{2}+\Upsilon_3(b-a)+\Upsilon_2\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)+Q_2\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big]\Big\}\\
\le\ &\varrho_f\Delta_1+\varrho_g\Delta_2+\varrho_h\Delta_3,
\end{aligned}
$$

which implies that

$$\|\mathcal{H}_1(u,v,w)\|\le\varrho_f\Delta_1+\varrho_g\Delta_2+\varrho_h\Delta_3,$$

where we have used the notations (3.7), (3.8) and (3.9). In a similar manner, it can be shown that

$$\|\mathcal{H}_2(u,v,w)\|\le\varrho_f\Delta_4+\varrho_g\Delta_5+\varrho_h\Delta_6,$$

and

$$\|\mathcal{H}_3(u,v,w)\|\le\varrho_f\Delta_7+\varrho_g\Delta_8+\varrho_h\Delta_9,$$

where Δ_i $(i=4,\dots,9)$ are given by (3.10) − (3.15). In consequence, we get

$$\|\mathcal{H}(u,v,w)\|\le\varrho_f\Theta_1+\varrho_g\Theta_2+\varrho_h\Theta_3,$$

where Θ_1, Θ_2 and Θ_3 are given by (3.6). From the foregoing arguments, it follows that the operator $\mathcal{H}$ is uniformly bounded. Next, we prove that $\mathcal{H}$ is equicontinuous. For $a<t<\tau<b$, and $(u,v,w)\in\Phi$, we have

$$
\begin{aligned}
&|\mathcal{H}_1(u,v,w)(\tau)-\mathcal{H}_1(u,v,w)(t)|\\
\le\ &\Big|\int_a^t\Big[\frac{(\tau-s)^2}{2}-\frac{(t-s)^2}{2}\Big]\widehat{f}(s)ds+\int_t^\tau\frac{(\tau-s)^2}{2}\widehat{f}(s)ds\\
&-\int_a^b\Big[(\tau-t)\Big(\frac{4(b-s)}{(8-B_2)}+\frac{A_1}{\Lambda_1}\Big)+\frac{2}{(8-B_1)}(\tau^2-t^2)\Big]\widehat{f}(s)ds\\
&-\int_a^b\Big[(\tau-t)\Big(\frac{2\sum_{l=1}^{m}\beta_l}{(8-B_2)}(b-s)+\frac{A_2}{\Lambda_1}\Big)+\frac{\sum_{n=1}^{m}\gamma_n}{(8-B_1)}(\tau^2-t^2)\Big]\widehat{g}(s)ds\\
&-\int_a^b\Big[(\tau-t)\Big(\frac{S_6}{(8-B_2)}(b-s)+\frac{A_3}{\Lambda_1}\Big)+\frac{S_1}{2(8-B_1)}(\tau^2-t^2)\Big]\widehat{h}(s)ds\\
&+\sum_{d=1}^{m}\kappa_d\int_a^{\eta_d}\Big[\frac{A_3}{\Lambda_1}(\tau-t)+\frac{S_1}{2(8-B_1)}(\tau^2-t^2)\Big]\widehat{f}(s)ds\\
&+\sum_{n=1}^{m}\gamma_n\int_a^{\eta_n}\Big[\frac{A_1}{\Lambda_1}(\tau-t)+\frac{2}{(8-B_1)}(\tau^2-t^2)\Big]\widehat{g}(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{r=1}^{m}\sigma_r\int_a^{\eta_r}\Big[\frac{A_2}{\Lambda_1}(\tau-t)+\frac{\sum_{n=1}^{m}\gamma_n}{(8-B_1)}(\tau^2-t^2)\Big]\widehat{h}(s)ds\\
&+\frac{S_6}{(8-B_2)}(\tau-t)\Big(\sum_{p=1}^{m}\zeta_p\int_a^{\eta_p}(\eta_p-s)\widehat{f}(s)ds\Big)\\
&+\frac{4}{(8-B_2)}(\tau-t)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)\widehat{g}(s)ds\Big)\\
&+\frac{2\sum_{l=1}^{m}\beta_l}{(8-B_2)}(\tau-t)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)\widehat{h}(s)ds\Big)\Big|\\
\leq\ & \varrho_f\Big[\frac{(\tau-t)^3}{3}+\frac{1}{3!}\Big|(\tau-a)^3-(t-a)^3\Big|\Big]+\frac{(\tau-t)}{|8-B_2|}\Big[(b-a)^2\Big(2\varrho_f+\varrho_g\sum_{l=1}^{m}\beta_l+\frac{1}{2}\varrho_h S_6\Big)\\
&+\varrho_f S_6\Big(\sum_{p=1}^{m}\zeta_p\frac{(\eta_p-a)^2}{2}\Big)+2\varrho_g\Big(\sum_{l=1}^{m}\beta_l(\eta_l-a)^2\Big)+\varrho_h\Big(\sum_{l=1}^{m}\beta_l\Big)\Big(\sum_{q=1}^{m}\rho_q(\eta_q-a)^2\Big)\Big]\\
&+\frac{(\tau-t)}{|\Lambda_1|}\Big[(b-a)\Big(\varrho_f|A_1|+\varrho_g|A_2|+\varrho_h|A_3|\Big)+\varrho_f|A_3|\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)\\
&+\varrho_g|A_1|\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+\varrho_h|A_2|\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\Big]\\
&+\frac{(\tau^2-t^2)}{|8-B_1|}\Big[(b-a)\Big(2\varrho_f+\varrho_g\sum_{n=1}^{m}\gamma_n+\frac{1}{2}\varrho_h S_1\Big)+\frac{1}{2}\varrho_f S_1\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)\\
&+2\varrho_g\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+\varrho_h\Big(\sum_{n=1}^{m}\gamma_n\Big)\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\Big]\\
\to\ & 0 \text{ independent of } (u,v,w)\in\Phi \text{ as } \tau-t\to 0.
\end{aligned}
$$

Similarly, it can be established that

$$
\begin{aligned}
&|\mathcal{H}_2(u,v,w)(\tau)-\mathcal{H}_2(u,v,w)(t)|\\
\leq\ & \varrho_g\Big[\frac{(\tau-t)^3}{3}+\frac{1}{3!}\Big|(\tau-a)^3-(t-a)^3\Big|\Big]+\frac{(\tau-t)}{|8-B_2|}\Big[(b-a)^2\Big(\frac{1}{2}\varrho_f S_8+2\varrho_g+\varrho_h\sum_{q=1}^{m}\rho_q\Big)\\
&+\varrho_f\Big(\sum_{q=1}^{m}\rho_q\Big)\Big(\sum_{p=1}^{m}\zeta_p(\eta_p-a)^2\Big)+\varrho_g S_8\Big(\sum_{l=1}^{m}\beta_l\frac{(\eta_l-a)^2}{2}\Big)+2\varrho_h\Big(\sum_{q=1}^{m}\rho_q(\eta_q-a)^2\Big)\Big]\\
&+\frac{(\tau-t)}{|\Lambda_1|}\Big[(b-a)\Big(\varrho_f|A_7|+\varrho_g|A_8|+\varrho_h|A_9|\Big)+\varrho_f|A_9|\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)\\
&+\varrho_g|A_7|\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+\varrho_h|A_8|\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\Big]\\
&+\frac{(\tau^2-t^2)}{|8-B_1|}\Big[(b-a)\Big(\frac{1}{2}\varrho_f S_3+2\varrho_g+\varrho_h\sum_{r=1}^{m}\sigma_r\Big)+\varrho_f\Big(\sum_{r=1}^{m}\sigma_r\Big)\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\varrho_g S_3\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+2\varrho_h\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\Big]\\
\to\ & 0 \text{ independent of } (u,v,w)\in\Phi \text{ as } \tau-t\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& |\mathcal{H}_3(u,v,w)(\tau)-\mathcal{H}_3(u,v,w)(t)|\\
\leq\ & \varrho_h\Big[\frac{(\tau-t)^3}{3}+\frac{1}{3!}\Big|(\tau-a)^3-(t-a)^3\Big|\Big]+\frac{(\tau-t)}{|8-B_2|}\Big[(b-a)^2\Big(\varrho_f\sum_{p=1}^{m}\zeta_p+\frac{1}{2}\varrho_g S_7+2\varrho_h\Big)\\
& +2\varrho_f\Big(\sum_{p=1}^{m}\zeta_p(\eta_p-a)^2\Big)+\varrho_g\Big(\sum_{p=1}^{m}\zeta_p\Big)\Big(\sum_{l=1}^{m}\beta_l(\eta_l-a)^2\Big)+\varrho_h S_7\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big]\\
& +\frac{(\tau-t)}{|\Lambda_1|}\Big[(b-a)\Big(\varrho_f|A_4|+\varrho_g|A_5|+\varrho_h|A_6|\Big)+\varrho_f|A_6|\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)\\
& +\varrho_g|A_4|\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+\varrho_h|A_5|\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\Big]\\
& +\frac{(\tau^2-t^2)}{|8-B_1|}\Big[(b-a)\Big(\varrho_f\sum_{d=1}^{m}\kappa_d+\frac{1}{2}\varrho_g S_2+2\varrho_h\Big)+2\varrho_f\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big)\\
& +\varrho_g\Big(\sum_{d=1}^{m}\kappa_d\Big)\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big)+\frac{1}{2}\varrho_h S_2\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\Big]\\
\to\ & 0 \text{ independent of } (u,v,w)\in\Phi \text{ as } \tau-t\to 0.
\end{aligned}
$$

In view of the foregoing steps, the Arzelá-Ascoli theorem applies and hence the operator $\mathcal{H}$ is completely continuous. Finally, it will be verified that the set $\Xi=\{(u,v,w)\in\mathcal{X}^3|(u,v,w)=\varphi\mathcal{H}(u,v,w),0<\varphi<1\}$ is bounded. Let $(u,v,w)\in\Xi$. Then $(u,v,w)=\varphi\mathcal{H}(u,v,w)$ and for any $t\in[a,b]$, we have

$$
u(t)=\varphi\mathcal{H}_1(u,v,w)(t),\quad v(t)=\varphi\mathcal{H}_2(u,v,w)(t),\quad w(t)=\varphi\mathcal{H}_3(u,v,w)(t).
$$

Thus, we get

$$
\begin{aligned}
|u(t)| \leq\ & \Delta_1(m_0+m_1\|u\|+m_2\|v\|+m_3\|w\|)+\Delta_2(\bar{m}_0+\bar{m}_1\|u\|+\bar{m}_2\|v\|+\bar{m}_3\|w\|)\\
& +\Delta_3(\widehat{m}_0+\widehat{m}_1\|u\|+\widehat{m}_2\|v\|+\widehat{m}_3\|w\|)\\
\leq\ & \Delta_1 m_0+\Delta_2\bar{m}_0+\Delta_3\widehat{m}_0+(\Delta_1 m_1+\Delta_2\bar{m}_1+\Delta_3\widehat{m}_1)\|u\|\\
& +(\Delta_1 m_2+\Delta_2\bar{m}_2+\Delta_3\widehat{m}_2)\|v\|+(\Delta_1 m_3+\Delta_2\bar{m}_3+\Delta_3\widehat{m}_3)\|w\|,
\end{aligned}
$$

$$
\begin{aligned}
|v(t)| \leq\ & \Delta_4(m_0+m_1\|u\|+m_2\|v\|+m_3\|w\|)+\Delta_5(\bar{m}_0+\bar{m}_1\|u\|+\bar{m}_2\|v\|+\bar{m}_3\|w\|)\\
& +\Delta_6(\widehat{m}_0+\widehat{m}_1\|u\|+\widehat{m}_2\|v\|+\widehat{m}_3\|w\|)\\
\leq\ & \Delta_4 m_0+\Delta_5\bar{m}_0+\Delta_6\widehat{m}_0+(\Delta_4 m_1+\Delta_5\bar{m}_1+\Delta_6\widehat{m}_1)\|u\|\\
& +(\Delta_4 m_2+\Delta_5\bar{m}_2+\Delta_6\widehat{m}_2)\|v\|+(\Delta_4 m_3+\Delta_5\bar{m}_3+\Delta_6\widehat{m}_3)\|w\|,
\end{aligned}
$$

and

$$
\begin{aligned}
|w(t)| \;\leq\;& \Delta_7 m_0 + \Delta_8 \bar{m}_0 + \Delta_9 \widehat{m}_0 + (\Delta_7 m_1 + \Delta_8 \bar{m}_1 + \Delta_9 \widehat{m}_1)\|u\| \\
&+(\Delta_7 m_2 + \Delta_8 \bar{m}_2 + \Delta_9 \widehat{m}_2)\|v\| + (\Delta_7 m_3 + \Delta_8 \bar{m}_3 + \Delta_9 \widehat{m}_3)\|w\|.
\end{aligned}
$$

Therefore, we can deduce that

$$
\begin{aligned}
\|u\| + \|v\| + \|w\| \;\leq\;& \Theta_1 m_0 + \Theta_2 \bar{m}_0 + \Theta_3 \widehat{m}_0 + (\Theta_1 m_1 + \Theta_2 \bar{m}_1 + \Theta_3 \widehat{m}_1)\|u\| \\
&+(\Theta_1 m_2 + \Theta_2 \bar{m}_2 + \Theta_3 \widehat{m}_2)\|v\| + (\Theta_1 m_3 + \Theta_2 \bar{m}_3 + \Theta_3 \widehat{m}_3)\|w\|.
\end{aligned}
$$

Using (3.17) together with the value of Θ given by (3.16), we find that

$$
\|(u, v, w)\| \leq \frac{\Theta_1 m_0 + \Theta_2 \bar{m}_0 + \Theta_3 \widehat{m}_0}{\Theta},
$$

which shows that the set Ξ is bounded. Hence, by Lemma 3.1, the operator $\mathcal{H}$ has at least one fixed point. Therefore, the problems (1.1) and (1.2) have at least one solution on [a,b]. This completes the proof. □

Secondly, we apply the sub-growth condition (N_2) under Schauder's fixed point theorem to show the existence of solutions for the problems (1.1) and (1.2).

Theorem 3.2. *Assume that* (N_2) *holds. Then there exists at least one solution for the problems* (1.1) *and* (1.2) *on* $[a, b]$.

Proof. Define a set Γ in the Banach space $\mathcal{X}^3$ as follows $\Gamma = \{(u, v, w) \in \mathcal{X}^3 : \|(u, v, w)\| \leq x\}$, where

$$
\begin{aligned}
x \;\geqslant\;& \max\{12\Theta_1\|\phi\|, 12\Theta_2\|\psi\|, 12\Theta_3\|\chi\|, (12\Theta_1\epsilon_1)^{\frac{1}{1-\lambda_1}}, (12\Theta_1\epsilon_2)^{\frac{1}{1-\lambda_2}}, (12\Theta_1\epsilon_3)^{\frac{1}{1-\lambda_3}}, \\
&(12\Theta_2\epsilon_4)^{\frac{1}{1-\lambda_4}}, (12\Theta_2\epsilon_5)^{\frac{1}{1-\lambda_5}}, (12\Theta_2\epsilon_6)^{\frac{1}{1-\lambda_6}}, (12\Theta_3\epsilon_7)^{\frac{1}{1-\lambda_7}}, (12\Theta_3\epsilon_8)^{\frac{1}{1-\lambda_8}}, (12\Theta_3\epsilon_9)^{\frac{1}{1-\lambda_9}}\}
\end{aligned}
$$

Firstly, we prove that $\mathcal{H} : \Gamma \to \Gamma$. For that, we consider

$$
\begin{aligned}
&|\mathcal{H}_1(u, v, w)(t)| \\
=\;& \Big| \int_a^t \frac{(t-s)^2}{2} \widehat{f}(s)ds + \frac{1}{\Lambda}\Big\{ - \int_a^b \Big[2\Lambda_1(b-s)^2 + G_1(t)(b-s) + P_1(t)\Big]\widehat{f}(s)ds \\
&- \int_a^b \Big[\Lambda_1 \sum_{j=1}^m \alpha_j (b-s)^2 + G_2(t)(b-s) + P_2(t)\Big]\widehat{g}(s)ds \\
&- \int_a^b \Big[\Lambda_1 S_{11}\frac{(b-s)^2}{2} + G_3(t)(b-s) + P_3(t)\Big]\widehat{h}(s)ds \\
&+P_3(t)\Big(\sum_{d=1}^m \kappa_d \int_a^{\eta_d} \widehat{f}(s)ds\Big) + P_1(t)\Big(\sum_{n=1}^m \gamma_n \int_a^{\eta_n} \widehat{g}(s)ds\Big) \\
&+P_2(t)\Big(\sum_{r=1}^m \sigma_r \int_a^{\eta_r} \widehat{h}(s)ds\Big) + G_3(t)\Big(\sum_{p=1}^m \zeta_p \int_a^{\eta_p} (\eta_p - s)\widehat{f}(s)ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+G_1(t)\Big(\sum_{l=1}^{m}\beta_l\int_a^{\eta_l}(\eta_l-s)\widehat{g}(s)ds\Big)+G_2(t)\Big(\sum_{q=1}^{m}\rho_q\int_a^{\eta_q}(\eta_q-s)\widehat{h}(s)ds\Big)\\
&+\Lambda_1 S_{11}\Big(\sum_{k=1}^{m}\xi_k\int_a^{\eta_k}\frac{(\eta_k-s)^2}{2}\widehat{f}(s)ds\Big)+2\Lambda_1\Big(\sum_{j=1}^{m}\alpha_j\int_a^{\eta_j}(\eta_j-s)^2\widehat{g}(s)ds\Big)\\
&+\Lambda_1\sum_{j=1}^{m}\alpha_j\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}(\eta_e-s)^2\widehat{h}(s)ds\Big)\Big\}\Big|\\
\leq\ &\Big(\phi(t)+\epsilon_1|u|^{\lambda_1}+\epsilon_2|v|^{\lambda_2}+\epsilon_3|w|^{\lambda_3}\Big)\Delta_1+\Big(\psi(t)+\epsilon_4|u|^{\lambda_4}+\epsilon_5|v|^{\lambda_5}+\epsilon_6|w|^{\lambda_6}\Big)\Delta_2\\
&+\Big(\chi(t)+\epsilon_7|u|^{\lambda_7}+\epsilon_8|v|^{\lambda_8}+\epsilon_9|w|^{\lambda_9}\Big)\Delta_3,
\end{aligned}
$$

which, on taking the norm

$$
\begin{aligned}
\|\mathcal{H}_1(u,v,w)\| \leq\ &\Big(\phi+\epsilon_1|u|^{\lambda_1}+\epsilon_2|v|^{\lambda_2}+\epsilon_3|w|^{\lambda_3}\Big)\Delta_1\\
&+\Big(\psi+\epsilon_4|u|^{\lambda_4}+\epsilon_5|v|^{\lambda_5}+\epsilon_6|w|^{\lambda_6}\Big)\Delta_2\\
&+\Big(\chi+\epsilon_7|u|^{\lambda_7}+\epsilon_8|v|^{\lambda_8}+\epsilon_9|w|^{\lambda_9}\Big)\Delta_3,
\end{aligned}
$$

where we have used the notations (3.7) − (3.9). Analogously, we have

$$
\begin{aligned}
\|\mathcal{H}_2(u,v,w)\| \leq\ &\Big(\phi+\epsilon_1|u|^{\lambda_1}+\epsilon_2|v|^{\lambda_2}+\epsilon_3|w|^{\lambda_3}\Big)\Delta_4\\
&+\Big(\psi+\epsilon_4|u|^{\lambda_4}+\epsilon_5|v|^{\lambda_5}+\epsilon_6|w|^{\lambda_6}\Big)\Delta_5\\
&+\Big(\chi+\epsilon_7|u|^{\lambda_7}+\epsilon_8|v|^{\lambda_8}+\epsilon_9|w|^{\lambda_9}\Big)\Delta_6,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{H}_3(u,v,w)\| \leq\ &\Big(\phi+\epsilon_1|u|^{\lambda_1}+\epsilon_2|v|^{\lambda_2}+\epsilon_3|w|^{\lambda_3}\Big)\Delta_7\\
&+\Big(\psi+\epsilon_4|u|^{\lambda_4}+\epsilon_5|v|^{\lambda_5}+\epsilon_6|w|^{\lambda_6}\Big)\Delta_8\\
&+\Big(\chi+\epsilon_7|u|^{\lambda_7}+\epsilon_8|v|^{\lambda_8}+\epsilon_9|w|^{\lambda_9}\Big)\Delta_9,
\end{aligned}
$$

where Δ_i $(i = 4, \dots, 9)$ are given by (3.10) − (3.15). Consequently,

$$
\begin{aligned}
\|\mathcal{H}(u,v,w)\| \leq\ &\Big(\phi+\epsilon_1|u|^{\lambda_1}+\epsilon_2|v|^{\lambda_2}+\epsilon_3|w|^{\lambda_3}\Big)\Theta_1\\
&+\Big(\psi+\epsilon_4|u|^{\lambda_4}+\epsilon_5|v|^{\lambda_5}+\epsilon_6|w|^{\lambda_6}\Big)\Theta_2\\
&+\Big(\chi+\epsilon_7|u|^{\lambda_7}+\epsilon_8|v|^{\lambda_8}+\epsilon_9|w|^{\lambda_9}\Big)\Theta_3 \leq x,
\end{aligned}
$$

where Θ_1, Θ_2 and Θ_3 are given by (3.6). Therefore, we conclude that $\mathcal{H} : \Gamma \to \Gamma$, where $\mathcal{H}_1(u,v,w)(t)$, $\mathcal{H}_2(u,v,w)(t)$ and $\mathcal{H}_3(u,v,w)(t)$ are continuous on $[a,b]$.

As in Theorem 3.1, one can show that the operator $\mathcal{H}$ is completely continuous. So, by Schauder's fixed point theorem, there exists a solution for the problems (1.1) and (1.2) on $[a,b]$. □

3.2. Existence of a unique solution

Here we apply Banach's contraction mapping principle to show the existence of a unique solution for the problems (1.1) and (1.2).

Theorem 3.3. *Assume that* (N_3) *holds. In addition, we suppose that*

$$\Theta_1\ell_1 + \Theta_2\ell_2 + \Theta_3\ell_3 < 1, \tag{3.18}$$

where Θ_1, Θ_2 *and* Θ_3 *are given by* (3.6). *Then the problems* (1.1) *and* (1.2) *have a unique solution on* $[a, b]$.

Proof. Let us set $\sup_{t\in[a,b]} |f(t, 0, 0, 0)| = M_1$, $\sup_{t\in[a,b]} |g(t, 0, 0, 0)| = M_2$, $\sup_{t\in[a,b]} |h(t, 0, 0, 0)| = M_3$, and show that $\mathcal{H}B_\varsigma \subset B_\varsigma$, where $B_\varsigma = \{(u, v, w) \in \mathcal{X}^3 : \|(u, v, w)\| \le \varsigma\}$ with

$$\varsigma \ge \frac{\Theta_1 M_1 + \Theta_2 M_2 + \Theta_3 M_3}{1 - (\Theta_1\ell_1 + \Theta_2\ell_2 + \Theta_3\ell_3)}.$$

For any $(u, v, w) \in B_\varsigma$, $t \in [a, b]$, we find that

$$\begin{aligned} |f(s, u(s), v(s), w(s))| &= |f(s, u(s), v(s), w(s)) - f(s, 0, 0, 0) + f(s, 0, 0, 0)| \\ &\le |f(s, u(s), v(s), w(s)) - f(s, 0, 0, 0)| + |f(s, 0, 0, 0)| \\ &\le \ell_1(\|u\| + \|v\| + \|w\|) + M_1 \le \ell_1\|(u, v, w)\| + M_1 \le \ell_1\varsigma + M_1. \end{aligned}$$

In a similar manner, we have

$$|g(s, u(s), v(s), w(s))| \le \ell_2\varsigma + M_2, \quad |h(s, u(s), v(s), w(s)| \le \ell_3\varsigma + M_3.$$

Then, for $(u, v, w) \in B_\varsigma$, we obtain

$$\begin{aligned} &|\mathcal{H}_1(u, v, w)(t)| \\ = \; & \Big| \int_a^t \frac{(t-s)^2}{2}\widehat{f}(s)ds + \frac{1}{\Lambda}\Big\{ - \int_a^b \Big[2\Lambda_1(b-s)^2 + G_1(t)(b-s) + P_1(t)\Big]\widehat{f}(s)ds \\ & - \int_a^b \Big[\Lambda_1 \sum_{j=1}^m \alpha_j(b-s)^2 + G_2(t)(b-s) + P_2(t)\Big]\widehat{g}(s)ds \\ & - \int_a^b \Big[\Lambda_1 S_{11}\frac{(b-s)^2}{2} + G_3(t)(b-s) + P_3(t)\Big]\widehat{h}(s)ds \\ & + P_3(t)\Big(\sum_{d=1}^m \kappa_d \int_a^{\eta_d} \widehat{f}(s)ds\Big) + P_1(t)\Big(\sum_{n=1}^m \gamma_n \int_a^{\eta_n} \widehat{g}(s)ds\Big) \\ & + P_2(t)\Big(\sum_{r=1}^m \sigma_r \int_a^{\eta_r} \widehat{h}(s)ds\Big) + G_3(t)\Big(\sum_{p=1}^m \zeta_p \int_a^{\eta_p} (\eta_p - s)\widehat{f}(s)ds\Big) \\ & + G_1(t)\Big(\sum_{l=1}^m \beta_l \int_a^{\eta_l} (\eta_l - s)\widehat{g}(s)ds\Big) + G_2(t)\Big(\sum_{q=1}^m \rho_q \int_a^{\eta_q} (\eta_q - s)\widehat{h}(s)ds\Big) \\ & + \Lambda_1 S_{11}\Big(\sum_{k=1}^m \xi_k \int_a^{\eta_k} \frac{(\eta_k - s)^2}{2}\widehat{f}(s)ds\Big) + 2\Lambda_1\Big(\sum_{j=1}^m \alpha_j \int_a^{\eta_j} (\eta_j - s)^2\widehat{g}(s)ds\Big) \end{aligned}$$

$$
\begin{aligned}
&+\Lambda_1\sum_{j=1}^{m}\alpha_j\Big(\sum_{e=1}^{m}\delta_e\int_a^{\eta_e}(\eta_e-s)^2\widehat{h}(s)ds\Big)\Big\}\Big| \\
\leq\ & (\ell_1\varsigma+M_1)\Big\{\frac{(b-a)^3}{6}+\frac{1}{3|8-B_3|}\Big[2(b-a)^3+S_{11}\Big(\sum_{k=1}^{m}\xi_k\frac{(\eta_k-a)^3}{2}\Big)\Big] \\
&+\frac{1}{|\Lambda|}\Big[Q_1\frac{(b-a)^2}{2}+\Upsilon_1(b-a)+\Upsilon_3\Big(\sum_{d=1}^{m}\kappa_d(\eta_d-a)\Big) \\
&+Q_3\Big(\sum_{p=1}^{m}\zeta_p\frac{(\eta_p-a)^2}{2}\Big)\Big]\Big\}+(\ell_2\varsigma+M_2)\Big\{\frac{\sum_{j=1}^{m}\alpha_j}{3|8-B_3|}\Big[(b-a)^3+2(\eta_j-a)^3\Big] \\
&+\frac{1}{|\Lambda|}\Big[Q_2\frac{(b-a)^2}{2}+\Upsilon_2(b-a)+\Upsilon_1\Big(\sum_{n=1}^{m}\gamma_n(\eta_n-a)\Big) \\
&+Q_1\Big(\sum_{l=1}^{m}\beta_l\frac{(\eta_l-a)^2}{2}\Big)\Big]\Big\}+(\ell_3\varsigma+M_3)\Big\{\frac{1}{3|8-B_3|}\Big[S_{11}\frac{(b-a)^3}{2} \\
&+\Big(\sum_{j=1}^{m}\alpha_j\Big)\Big(\sum_{e=1}^{m}\delta_e(\eta_e-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_3\frac{(b-a)^2}{2}+\Upsilon_3(b-a) \\
&+\Upsilon_2\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)+Q_2\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big]\Big\} \\
\leq\ & (\ell_1\varsigma+M_1)\Delta_1+(\ell_2\varsigma+M_2)\Delta_2+(\ell_3\varsigma+M_3)\Delta_3,
\end{aligned}
$$

which, on taking the norm for $t\in[a,b]$, yields

$$\|\mathcal{H}_1(u,v,w)\|\leq(\ell_1\varsigma+M_1)\Delta_1+(\ell_2\varsigma+M_2)\Delta_2+(\ell_3\varsigma+M_3)\Delta_3.$$

Similarly, we can find that

$$\|\mathcal{H}_2(u,v,w)\|\leq(\ell_1\varsigma+M_1)\Delta_4+(\ell_2\varsigma+M_2)\Delta_5+(\ell_3\varsigma+M_3)\Delta_6,$$

and

$$\|\mathcal{H}_3(u,v,w)\|\leq(\ell_1\varsigma+M_1)\Delta_7+(\ell_2\varsigma+M_2)\Delta_8+(\ell_3\varsigma+M_3)\Delta_9,$$

where Δ_i $(i=1,\ldots,9)$ are defined in (3.7) – (3.15). In consequence, it follows that

$$\|\mathcal{H}(u,v,w)\|\leq(\ell_1\varsigma+M_1)\Theta_1+(\ell_2\varsigma+M_2)\Theta_2+(\ell_3\varsigma+M_3)\Theta_3\leq\varsigma.$$

Next we show that the operator $\mathcal{H}$ is a contraction. For $(u_1,v_1,w_1),\ (u_2,v_2,w_2)\in\mathcal{X}^3$, we have

$$
\begin{aligned}
&\Big|\mathcal{H}_1(u_1,v_1,w_1)(t)-\mathcal{H}_1(u_2,v_2,w_2)(t)\Big| \\
\leq\ & \int_a^t\frac{(t-s)^2}{2}\Big|f(s,u_1(s),v_1(s),w_1(s))-f(s,u_2(s),v_2(s),w_2(s))\Big|ds \\
&+\frac{1}{|\Lambda|}\Big\{\int_a^b\Big[2|\Lambda_1|(b-s)^2+|G_1(t)|(b-s)+|P_1(t)|\Big] \\
&\times\Big|f(s,u_1(s),v_1(s),w_1(s))-f(s,u_2(s),v_2(s),w_2(s))\Big|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_a^b \Big[|\Lambda_1|\sum_{j=1}^m \alpha_j(b-s)^2+|G_2(t)|(b-s)+|P_2(t)|\Big]\\
&\times\Big|g(s,u_1(s),v_1(s),w_1(s))-g(s,u_2(s),v_2(s),w_2(s))\Big|ds\\
&+\int_a^b \Big[|\Lambda_1|S_{11}\frac{(b-s)^2}{2}+|G_3(t)|(b-s)+|P_3(t)|\Big]\\
&\times\Big|h(s,u_1(s),v_1(s),w_1(s))-h(s,u_2(s),v_2(s),w_2(s))\Big|ds\\
&+|P_3(t)|\Big(\sum_{d=1}^m \kappa_d\int_a^{\eta_d}\Big|f(s,u_1(s),v_1(s),w_1(s))-f(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|P_1(t)|\Big(\sum_{n=1}^m \gamma_n\int_a^{\eta_n}\Big|g(s,u_1(s),v_1(s),w_1(s))-g(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|P_2(t)|\Big(\sum_{r=1}^m \sigma_r\int_a^{\eta_r}\Big|h(s,u_1(s),v_1(s),w_1(s))-h(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|G_3(t)|\Big(\sum_{p=1}^m \zeta_p\int_a^{\eta_p}(\eta_p-s)\Big|f(s,u_1(s),v_1(s),w_1(s))-f(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|G_1(t)|\Big(\sum_{l=1}^m \beta_l\int_a^{\eta_l}(\eta_l-s)\Big|g(s,u_1(s),v_1(s),w_1(s))-g(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|G_2(t)|\Big(\sum_{q=1}^m \rho_q\int_a^{\eta_q}(\eta_q-s)\Big|h(s,u_1(s),v_1(s),w_1(s))-h(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|\Lambda_1|S_{11}\Big(\sum_{k=1}^m \xi_k\int_a^{\eta_k}\frac{(\eta_k-s)^2}{2}\Big|f(s,u_1(s),v_1(s),w_1(s))-f(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+2|\Lambda_1|\Big(\sum_{j=1}^m \alpha_j\int_a^{\eta_j}(\eta_j-s)^2\Big|g(s,u_1(s),v_1(s),w_1(s))-g(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\\
&+|\Lambda_1|\sum_{j=1}^m \alpha_j\Big(\sum_{e=1}^m \delta_e\int_a^{\eta_e}(\eta_e-s)^2\Big|h(s,u_1(s),v_1(s),w_1(s))-h(s,u_2(s),v_2(s),w_2(s))\Big|ds\Big)\Big\}\\
\leq\ & \ell_1(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|)\Big\{\frac{(b-a)^3}{6}+\frac{1}{3|8-B_3|}\Big[2(b-a)^3\\
&+S_{11}\Big(\sum_{k=1}^m \xi_k\frac{(\eta_k-a)^3}{2}\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_1\frac{(b-a)^2}{2}+\Upsilon_1(b-a)+\Upsilon_3\Big(\sum_{d=1}^m \kappa_d(\eta_d-a)\Big)\\
&+Q_3\Big(\sum_{p=1}^m \zeta_p\frac{(\eta_p-a)^2}{2}\Big)\Big]\Big\}+\ell_2(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|)\Big\{\frac{\sum_{j=1}^m \alpha_j}{3|8-B_3|}\Big[(b-a)^3\\
&+2(\eta_j-a)^3\Big]+\frac{1}{|\Lambda|}\Big[Q_2\frac{(b-a)^2}{2}+\Upsilon_2(b-a)+\Upsilon_1\Big(\sum_{n=1}^m \gamma_n(\eta_n-a)\Big)\\
&+Q_1\Big(\sum_{l=1}^m \beta_l\frac{(\eta_l-a)^2}{2}\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\ell_3(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|)\Big\{\frac{1}{3|8-B_3|}\Big[S_{11}\frac{(b-a)^3}{2}\\
&+\Big(\sum_{j=1}^{m}\alpha_j\Big)\Big(\sum_{e=1}^{m}\delta_e(\eta_e-a)^3\Big)\Big]+\frac{1}{|\Lambda|}\Big[Q_3\frac{(b-a)^2}{2}+\Upsilon_3(b-a)+\Upsilon_2\Big(\sum_{r=1}^{m}\sigma_r(\eta_r-a)\Big)\\
&+Q_2\Big(\sum_{q=1}^{m}\rho_q\frac{(\eta_q-a)^2}{2}\Big)\Big]\Big\}\\
\leq\ &(\ell_1\Delta_1+\ell_2\Delta_2+\ell_3\Delta_3)(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),
\end{aligned}
$$

which implies that

$$\Big\|\mathcal{H}_1(u_1,v_1,w_1)-\mathcal{H}_1(u_2,v_2,w_2)\Big\|\leq(\ell_1\Delta_1+\ell_2\Delta_2+\ell_3\Delta_3)(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$

where Δ_1 Δ_2 and Δ_3 are given by (3.7), (3.8) and (3.9) respectively. In a similar fashion, one can find that

$$\Big\|\mathcal{H}_2(u_1,v_1,w_1)-\mathcal{H}_2(u_2,v_2,w_2)\Big\|\leq(\ell_1\Delta_4+\ell_2\Delta_5+\ell_3\Delta_6)(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$

and

$$\Big\|\mathcal{H}_3(u_1,v_1,w_1)-\mathcal{H}_3(u_2,v_2,w_2)\Big\|\leq(\ell_1\Delta_7+\ell_2\Delta_8+\ell_3\Delta_9)(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$

where $\Delta_i,\ (i=4,\ldots,9)$ are given by (3.10) – (3.15). Thus we have

$$\|\mathcal{H}(u_1,v_1,w_1)-\mathcal{H}(u_2,v_2,w_2)\Big\|\leq(\Theta_1\ell_1+\Theta_2\ell_2+\Theta_3\ell_3)(\|u_1-u_2\|+\|v_1-v_2\|+\|w_1-w_2\|),\quad(3.19)$$

where Θ_1, Θ_2 and Θ_3 are given by (3.6). By the assumption (3.18) it follows from (3.19) that the operator $\mathcal{H}$ is a contraction. Thus, by Banach contraction mapping principle, we deduce that the operator $\mathcal{H}$ has a fixed point, which corresponds to a unique solution of the problems (1.1) and (1.2) on $[a,b]$. □

3.3. Examples

Example 3.1. *Consider the following coupled system of third-order ordinary differential equations*

$$
\begin{aligned}
u'''(t)&=\frac{5}{31\sqrt{t^3+24}}+\frac{|u(t)|^2}{204(1+|u(t)|)}+\frac{3}{342}\sin v(t)+\frac{1}{t^2+97}w(t),\ t\in[1,3],\\
v'''(t)&=\frac{e^{-(t-1)}}{12(15+t)}+\frac{1}{798\pi}\sin(7\pi u)+\frac{|v(t)|^3}{96(1+|v(t)|^2)}+\frac{4}{(t+7)^3}w(t),\ \ t\in[1,3],\qquad(3.20)\\
w'''(t)&=\frac{1}{2(4+t)^2}\cos t+\frac{2}{6\sqrt{4356t}}u(t)+\frac{w(t)|v(t)|}{810(1+|v(t)|)},\ t\in[1,3],
\end{aligned}
$$

supplemented to the following boundary conditions

$$\begin{aligned} u(1)+u(3) &= \sum_{j=1}^{4}\alpha_j v(\eta_j), \;\; u'(1)+u'(3) = \sum_{l=1}^{4}\beta_l v'(\eta_l), \;\; u''(1)+u''(3) = \sum_{n=1}^{4}\gamma_n v''(\eta_n), \\ v(1)+v(3) &= \sum_{e=1}^{4}\delta_e w(\eta_e), \;\; v'(1)+v'(3) = \sum_{q=1}^{4}\rho_q w'(\eta_q), \;\; v''(1)+v''(3) = \sum_{r=1}^{4}\sigma_r w''(\eta_r), \\ w(1)+w(3) &= \sum_{k=1}^{4}\xi_k u(\eta_k), \;\; w'(1)+w'(3) = \sum_{p=1}^{4}\zeta_p u'(\eta_p), \;\; w''(1)+w''(3) = \sum_{d=1}^{4}\kappa_d u''(\eta_d), \end{aligned} \tag{3.21}$$

where
$a = 1,\ b = 3,\ m = 4,\ \eta_1 = 4/3,\ \eta_2 = 5/3,\ \eta_3 = 2,\ \eta_4 = 7/3,\ \alpha_1 = 1/4,\ \alpha_2 = 1/2,\ \alpha_3 = 3/4,\ \alpha_4 = 1,\ \beta_1 = 0.2,\ \beta_2 = 8/15,\ \beta_3 = 13/15,\ \beta_4 = 6/5,\ \gamma_1 = 1/8,\ \gamma_2 = 9/40,\ \gamma_3 = 13/40,\ \gamma_4 = 17/40,\ \delta_1 = 2/11,\ \delta_2 = 3/11,\ \delta_3 = 4/11,\ \delta_4 = 5/11,\ \rho_1 = 1/6,\ \rho_2 = 7/24,\ \rho_3 = 5/12,\ \rho_4 = 13/24,\ \sigma_1 = 1/9,\ \sigma_2 = 2/9,\ \sigma_3 = 1/3,\ \sigma_4 = 4/9,\ \xi_1 = 1/7,\ \xi_2 = 2/7,\ \xi_3 = 3/7,\ \xi_4 = 4/7,\ \zeta_1 = 2/15,\ \zeta_2 = 1/3,\ \zeta_3 = 8/15,\ \zeta_4 = 11/15,\ \kappa_1 = 1/3,\ \kappa_2 = 4/9,\ \kappa_3 = 5/9,\ \kappa_4 = 2/3.$
By direct substitution, we get $B_1 \approx 2.444444 \neq 8,\ \ B_2 \approx 6.875556 \neq 8,\ \ B_3 \approx 4.545452 \neq 8,$ *and* $\Lambda \approx 21.580256$ *(*Λ *is given by* (2.11)*). Also,* $\Delta_1 \approx 21.294227,\ \ \Delta_2 \approx 22.603176,\ \ \Delta_3 \approx 11.800813,\ \ \Delta_4 \approx 7.983258,\ \ \Delta_5 \approx 12.996835,\ \ \Delta_6 \approx 8.497948,\ \ \Delta_7 \approx 10.977544,\ \ \Delta_8 \approx 14.165941$ *and* $\Delta_9 \approx 12.745457$ *(*Δ_i $(i = 1,\dots,9)$ *are defined in* (3.7) − (3.15)*). Furthermore we obtain* $\Theta_1 \approx 40.255029,\ \ \Theta_2 \approx 49.765952$ *and* $\Theta_3 \approx 33.044218$ ($\Theta_1,\ \Theta_2$ *and* Θ_3 *are given by* (3.6)*). Evidently,*

$$\begin{aligned} |f(t,u,v,w)| &\le \frac{1}{31} + \frac{1}{204}\|u\| + \frac{1}{114}\|v\| + \frac{1}{98}\|w\|, \\ |g(t,u,v,w)| &\le \frac{1}{192} + \frac{1}{114}\|u\| + \frac{1}{96}\|v\| + \frac{1}{128}\|w\|, \\ |h(t,u,v,w)| &\le \frac{1}{50} + \frac{1}{198}\|u\| + \frac{1}{810}\|w\|. \end{aligned}$$

Clearly, $m_0 = 1/31,\ m_1 = 1/204,\ m_2 = 1/114,\ m_3 = 1/98,\ \bar{m}_0 = 1/192,\ \bar{m}_1 = 1/114,\ \bar{m}_2 = 1/96,\ \bar{m}_3 = 1/128,$ *and* $\widehat{m}_0 = 1/50,\ \widehat{m}_1 = 1/198,\ \widehat{m}_2 = 0,\ \widehat{m}_3 = 1/810.$ *Using* (3.17), *we find that* $\Theta_1 m_1 + \Theta_2\bar{m}_1 + \Theta_3\widehat{m}_1 \approx 0.800762 < 1,\ \ \Theta_1 m_2 + \Theta_2\bar{m}_2 + \Theta_3\widehat{m}_2 \approx 0.871509 < 1$ *and* $\Theta_1 m_3 + \Theta_2\bar{m}_3 + \Theta_3\widehat{m}_3 \approx 0.840357 < 1.$ *Also, from* (3.16) *we obtain* $\Theta = 0.128491$. *Hence, all the conditions of Theorem* 3.1 *are satisfied and consequently the problems* (3.20) *and* (3.21) *has at least one solution on* $[1,3]$.

Example 3.2. *Consider the following system*

$$\begin{aligned} u'''(t) &= \frac{3}{9(t^3+72)}\Big(\tan^{-1}(u(t)) + v(t) + \frac{|w|}{1+|w|}\Big) + e^{-(t-1)}, \qquad t \in [1,3], \\ v'''(t) &= \frac{1}{610\pi}\sin(2\pi u) + \frac{4}{2t+1218}\sin(v(t)) + \frac{7}{3} + \frac{1}{305}w(t),\ t \in [1,3], \\ w'''(t) &= \frac{3}{22\sqrt{999+90t}}\Big(u(t) + \frac{|v(t)|}{1+|v(t)|} + \tan^{-1}(w(t))\Big) + \cos(t-1),\ t \in [1,3], \end{aligned} \tag{3.22}$$

subject to the coupled boundary conditions (3.21). *It is easy to see that* $\ell_1 = 1/219, \ell_2 = 1/305$ *and* $\ell_3 = 1/242$ *as*

$$|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)| \le \frac{1}{219}(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|),$$

$$|g(t,u_1,v_1,w_1)-g(t,u_2,v_2,w_2)| \le \frac{1}{305}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),$$
$$|h(t,u_1,v_1,w_1)-h(t,u_2,v_2,w_2)| \le \frac{1}{242}(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|).$$

Using the values obtained in Example 3.1*, we find that* $\Theta_1\ell_1+\Theta_2\ell_2+\Theta_3\ell_3 \approx 0.483526 < 1$*, where* Θ_1*,* Θ_2 *and* Θ_3 *are given by* (3.6)*. Therefore, by Theorem* 3.3*, the system* (3.22) *equipped with the boundary conditions* (3.21) *has a unique solution on* $[1,3]$*.*

4. Conclusions

In this paper, we discussed the existence and uniqueness of solutions for a coupled system of nonlinear third order ordinary differential equations supplemented with nonlocal multi-point anti-periodic type boundary conditions on an arbitrary domain with the aid of modern fixed point theorems. Our results are new and enrich the literature on third-order boundary value problems. As a special case, our results correspond to the ones for an anti-periodic boundary value problem of nonlinear third order ordinary differential equations by fixing all $\alpha_j=\beta_l=\gamma_n=\delta_e=\rho_q=\sigma_r=\xi_k=\zeta_p=\kappa_d=0$ in (1.2).

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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