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Research article

Fractional differential equations with coupled slit-strips type integral boundary conditions

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Abstract: In this article, we discuss the existence of solutions for coupled hybrid fractional differential equations supplemented with coupled slit-strips type boundary conditions. We make use of the standard tools of fixed point theory to obtain the desired results, which are well-illustrated with examples.

Keywords: Caputo fractional derivative; hybrid system; integral boundary conditions; existence; fixed point
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1. Introduction

Widespread applications of fractional calculus significantly contributed to the popularity of the subject. Fractional order operators are nonlocal in nature and give rise to more realistice and informative mathematical modeling of many real world phenomena, in contrast to their integer-order counterparts, for instance, see [13, 21, 31].

Nonlinear fractional order boundary value problems appear in a variety of fields such as applied mathematics, physical sciences, engineering, control theory, etc. Several aspects of these problems, such as existence, uniqueness and stability, have been explored in recent studies [5–7, 14, 22, 24, 26, 28, 32].

Coupled nonlinear fractional differential equations find their applications in various applied and technical problems such as disease models [8, 10, 29], ecological models [18], synchronization of chaotic systems [11, 33], nonlocal thermoelasticity [30], etc. Hybrid fractional differential equations also received significant attention in the recent years, for example, see [2, 3, 9, 15–17, 19, 20, 23].

The concept of slits-strips conditions introduced by Ahmad et al. in [1] is a new idea and has useful applications in imaging via strip-detectors [25] and acoustics [27].

In [1], the authors investigated the following strips-slit problem:

$$
\begin{aligned}
{}^cD^p x(t) &= f_1(t, x(t)),\ n-1 < p \le n,\ t \in [0,1] \\
x(0) = 0,\ x'(0) &= 0,\ x''(0) = 0, \ldots .\ x^{(n-2)}(0) = 0, \\
x(\xi) = a_1 \int_0^{\eta} x(s)ds &+ a_2 \int_{\xi_1}^{1} x(s)ds,\ 0 < \eta < \xi < \xi_1 < 1,
\end{aligned}
$$

where ${}^cD^p$ denotes the Caputo fractional derivative of order p, f_1 is a given continuous function and $a_1, a_2 \in \Re$.

In 2017, Ahmad et al. [4] studied a coupled system of nonlinear fractional differential equations

$$
\begin{aligned}
{}^cD^{\alpha} x(t) &= f_1(t, x(t), y(t)),\ t \in [0,1],\ 1 < \alpha \le 2, \\
{}^cD^{\beta} y(t) &= f_2(t, x(t), y(t)),\ t \in [0,1],\ 1 < \beta \le 2,
\end{aligned}
$$

supplemented with the coupled and uncoupled boundary conditions of the form:

$$
\begin{aligned}
x(0) = 0,\ x(a_1) &= d_1 \int_0^{\eta} y(s)ds + d_2 \int_{\xi_1}^{1} y(s)ds,\ 0 < \eta < a_1 < \xi_1 < 1, \\
y(0) = 0,\ y(a_1) &= d_1 \int_0^{\eta} x(s)ds + d_2 \int_{\xi_1}^{1} x(s)ds,\ 0 < \eta < a_1 < \xi_1 < 1,
\end{aligned}
$$

and

$$
\begin{aligned}
x(0) = 0,\ x(a_1) &= d_1 \int_0^{\eta} x(s)ds + d_2 \int_{\xi_1}^{1} x(s)ds,\ 0 < \eta < a_1 < \xi_1 < 1, \\
y(0) = 0,\ y(a_1) &= d_1 \int_0^{\eta} y(s)ds + d_2 \int_{\xi_1}^{1} y(s)ds,\ 0 < \eta < a_1 < \xi_1 < 1,
\end{aligned}
$$

where ${}^cD^{\alpha}$ and ${}^cD^{\beta}$ denote the Caputo fractional derivatives of orders α and β respectively, $f_1, f_2 : [0,1] \times \Re \times \Re \to \Re$ are given continuous functions and d_1, d_2 are real constants.

In this article, motivated by aforementioned works, we introduce and study the following hybrid nonlinear fractional differential equations:

$$
\begin{aligned}
{}^cD^{\gamma} \left[u(t) - h_1(t, u(t), v(t))\right] &= \theta_1(t, u(t), v(t)),\ t \in [0,1],\ 1 < \gamma \le 2, \\
{}^cD^{\delta} \left[v(t) - h_2(t, u(t), v(t))\right] &= \theta_2(t, u(t), v(t)),\ t \in [0,1],\ 1 < \delta \le 2,
\end{aligned} \tag{1.1}
$$

equipped with coupled slit-strips-type integral boundary conditions:

$$
\begin{aligned}
u(0) = 0,\ u(\eta) &= \omega_1 \int_0^{\xi_1} v(s)ds + \omega_2 \int_{\xi_2}^{1} v(s)ds,\ 0 < \xi_1 < \eta < \xi_2 < 1, \\
v(0) = 0,\ v(\eta) &= \omega_1 \int_0^{\xi_1} u(s)ds + \omega_2 \int_{\xi_2}^{1} u(s)ds,\ 0 < \xi_1 < \eta < \xi_2 < 1,
\end{aligned} \tag{1.2}
$$

where ${}^cD^{\gamma}$, ${}^cD^{\delta}$ denote the Caputo fractional derivatives of orders γ and δ respectively, $\theta_i, h_i : [0,1] \times \Re \times \Re \to \Re$ are given continuous functions with $h_i(0, u(0), v(0)) = 0,\ i = 1,2$ and ω_1, ω_2 are real constants.

We arrange the rest of the paper as follows. In section 2, we present some definitions and obtain an auxiliary result, while section 3 contains the main results for the problems (1.1) and (1.2). Section 4 is devoted to the illustrative examples for the derived results.

2. An auxiliary lemma

Let us first recall some related definitions [21].

Definition 2.1. *For a locally integrable real-valued function $g_1 : [a, \infty) \longrightarrow \mathbb{R}$, we define the Riemann-Liouville fractional integral of order $\sigma > 0$ as*

$$I^{\sigma} g_1(t) = \frac{1}{\Gamma(\sigma)} \int_0^t \frac{g_1(\tau)}{(t-\tau)^{1-\sigma}} d\tau,\ \sigma > 0,$$

where Γ is the Euler's gamma function.

Definition 2.2. *The Caputo derivative of order σ for an n-times continuously differentiable function $g_1 : [0, \infty) \to \Re$ is defined by*

$$^cD^{\sigma} g_1(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t (t-\tau)^{n-\sigma-1} g_1^{(n)}(\tau) d\tau,\ n-1 < \sigma < n,\ n = [\sigma] + 1$$

where $[\sigma]$ is the integer part of a real number.

Lemma 2.1. *For $\chi_i, \Phi_i \in C([0,1], \Re)$ with $\chi_i(0) = 0, i = 1, 2,$ the following linear system of equations:*

$$\begin{aligned} &^cD^{\gamma} [u(t) - \chi_1(t)] = \Phi_1(t),\ t \in [0,1],\ 1 < \gamma \leq 2, \\ &^cD^{\delta} [v(t) - \chi_2(t)] = \Phi_2(t),\ t \in [0,1],\ 1 < \delta \leq 2, \end{aligned} \tag{2.1}$$

equipped with coupled slit-strips-type integral boundary conditions (1.2), is equivalent to the integral equations:

$$\begin{aligned} u(t) = \frac{t}{\eta^2 - \Delta^2} \Bigg[&\eta \Bigg\{ \omega_1 \int_0^{\xi_1} \Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)} \Phi_2(\tau) d\tau + \chi_2(s) \Bigg) ds \\ &+ \omega_2 \int_{\xi_2}^1 \Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)} \Phi_2(\tau) d\tau + \chi_2(s) \Bigg) ds - \int_0^{\eta} \frac{(\eta - s)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(s) ds - \chi_1(\eta) \Bigg\} \\ &+ \Delta \Bigg\{ \omega_1 \int_0^{\xi_1} \Bigg(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(\tau) d\tau + \chi_1(s) \Bigg) ds \\ &+ \omega_2 \int_{\xi_2}^1 \Bigg(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(\tau) d\tau + \chi_1(s) \Bigg) ds - \int_0^{\eta} \frac{(\eta - s)^{\delta-1}}{\Gamma(\delta)} \Phi_2(s) ds - \chi_2(\eta) \Bigg\} \Bigg] \\ &+ \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(s) ds + \chi_1(t), \end{aligned} \tag{2.2}$$

$$\begin{aligned} v(t) = \frac{t}{\eta^2 - \Delta^2} \Bigg[&\Delta \Bigg\{ \omega_1 \int_0^{\xi_1} \Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)} \Phi_2(\tau) d\tau + \chi_1(s) \Bigg) ds \\ &+ \omega_2 \int_{\xi_2}^1 \Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)} \Phi_2(\tau) d\tau + \chi_1(s) \Bigg) ds - \int_0^{\eta} \frac{(\eta - \tau)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(s) ds - \chi_2(\eta) \Bigg\} \\ &+ \eta \Bigg\{ \omega_1 \int_0^{\xi_1} \Bigg(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(\tau) d\tau + \chi_2(s) \Bigg) ds \end{aligned}$$

$$+\omega_2\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau+\chi_2(s)\right)ds-\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds-\chi_2(\eta)\Bigg\}\Bigg]$$
$$+\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds+\chi_2(t),\tag{2.3}$$

where it is assumed that

$$\Delta=\frac{1}{2}\left(\omega_1\xi_1^2+\omega_2(1-\xi_2^2)\right)\neq 0.\tag{2.4}$$

Proof. Solving the fractional differential equations in (2.1), we get

$$u(t)=c_0+c_1t+\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(s)ds+\chi_1(t)\tag{2.5}$$

and

$$v(t)=c_2+c_3t+\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds+\chi_2(t),\tag{2.6}$$

where $c_0, c_1, c_2, c_3\in\Re$ are arbitrary constants.
Using the conditions $u(0)=0$ and $v(0)=0$ in (2.5) and (2.6), we find that $c_0=0$ and $c_2=0$. Thus (2.5) and (2.6) become

$$u(t)=c_1t+\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(s)ds+\chi_1(t),\tag{2.7}$$
$$v(t)=c_3t+\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds+\chi_2(t),\tag{2.8}$$

Making use of the coupled slit-strips-type integral boundary conditions given by (1.2) in (2.7) and (2.8) together with the notation (2.4), we obtain a system of equations:

$$\omega_1\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\Phi_2(\tau)d\tau+\chi_2(s)\right)ds+\omega_2\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\Phi_2(\tau)d\tau+\chi_2(s)\right)ds$$
$$-\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(s)ds-\chi_1(\eta))=c_1\eta-\Delta c_3,\tag{2.9}$$
$$\omega_1\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau+\chi_1(s)\right)ds+\omega_2\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau+\chi_1(s)\right)ds$$
$$-\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds-\chi_2(\eta)=c_3\eta-\Delta c_1.\tag{2.10}$$

Solving the systems (2.9)–(2.10) for c_1 and c_3, we find that

$$c_1=\frac{t}{\eta^2-\Delta^2}\Bigg[\eta\Bigg\{\omega_1\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\Phi_2(\tau)d\tau+\chi_2(s)\right)ds$$
$$+\omega_2\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\Phi_2(\tau)d\tau+\chi_2(s)\right)ds-\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(s)ds-\chi_1(\eta)\Bigg\}$$

$$+ \Delta\left\{\omega_1 \int_0^{\xi_1}\left(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau + \chi_1(s)\right)ds + \omega_2\int_{\xi_2}^1\left(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau + \chi_1(s)\right)ds - \int_0^\eta \frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds - \chi_2(\eta)\right\}\Bigg]$$

and

$$c_3 = \frac{t}{\eta^2-\Delta^2}\Bigg[\Delta\left\{\omega_1 \int_0^{\xi_1}\left(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\Phi_2(\tau)d\tau + \chi_1(s)\right)ds + \omega_2\int_{\xi_2}^1\left(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\Phi_2(\tau)d\tau + \chi_1(s)\right)ds - \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(s)ds - \chi_2(\eta)\right\}$$
$$+ \eta\left\{\omega_1 \int_0^{\xi_1}\left(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau + \chi_2(s)\right)ds + \omega_2\int_{\xi_2}^1\left(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1(\tau)d\tau + \chi_2(s)\right)ds - \int_0^\eta \frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\Phi_2(s)ds - \chi_1(\eta)\right\}\Bigg]$$

Inserting the values of c_1 and c_3 in (2.7) and (2.8) leads to the integral equations (2.2) and (2.3). By direct computation, one can obtain the converse of the lemma. The proof is finished. □

3. Main results

Let $\mathcal{W} = \{\tilde{w}(t) : \tilde{w}(t) \in C([0,1])\}$ be a Banach space equipped with the norm $\|\tilde{w}\| = \max\{|\tilde{w}(t)|, t \in [0,1]\}$, Then the product space $(\mathcal{W}\times\mathcal{W}, \|(u,v)\|)$ endowed with the norm $\|(u,v)\| = \|u\| + \|v\|$, $(u,v) \in \mathcal{W}\times\mathcal{W}$ is also a Banach space.

We need the following assumptions to derive the main results.

(A1) Let $\theta_1, \theta_2 : [0,1]\times\Re^2 \to \Re$ be continuous and bounded functions and there exists constants m_i, n_i such that, for all $t \in [0,1]$ and $x_i, y_i \in \Re, i = 1,2$,

$$|\theta_1(t,x_1,x_2) - \theta_1(t,y_1,y_2)| \le m_1|x_1-y_1| + m_2|x_2-y_2|,$$
$$|\theta_2(t,x_1,x_2) - \theta_2(t,y_1,y_2)| \le n_1|x_1-y_1| + n_2|x_2-y_2|.$$

(A2) For continuous and bounded functions h_i, i=1, 2, there exist real constants $\mu_i, \beta_i, \sigma_i > 0$ such that, for all $x_i, y_i \in \Re$, $|h_i(t,x,y)| \le \mu_i$ for all $(t,x,y) \in [0,1]\times\Re\times\Re$ and

$$|h_1(t,x_1,x_2) - h_1(t,y_1,y_2)| \le \beta_1|x_1-y_1| + \beta_2|x_2-y_2|,$$
$$|h_2(t,x_1,x_2) - h_2(t,y_1,y_2)| \le \sigma_1|x_1-y_1| + \sigma_2|x_2-y_2|.$$

(A3) $\sup_{t\in[0,1]} \theta_1(t,0,0) = \mathcal{N}_1 < \infty$ and $\sup_{t\in[0,1]} \theta_2(t,0,0) = \mathcal{N}_2 < \infty$.

(A4) For the sake of computational convenience, we set

$$\mathcal{M}_1 = \frac{1}{\Gamma(\gamma+1)} + \frac{1}{|\eta^2-\Delta^2|}\left[\frac{\eta^{\gamma+1}}{\Gamma(\gamma+1)} + |\Delta||\omega_1|\frac{\xi_1^{\gamma+1}}{\Gamma(\gamma+2)} + \Delta|\omega_2|\frac{1-\xi_2^{\gamma+1}}{\Gamma(\gamma+2)}\right],$$

$$\mathcal{M}_2 = \frac{1}{|\eta^2-\Delta^2|}\left[\eta|\omega_1|\frac{\xi_1^{\delta+1}}{\Gamma(\delta+2)} + \eta|\omega_2|\frac{1-\xi_2^{\delta+1}}{\Gamma(\delta+2)} + |\Delta|\frac{\eta^{\delta}}{\Gamma(\delta+1)}\right],$$
$$\mathcal{M}_3 = \frac{1}{|\eta^2-\Delta^2|}\left[\eta|\omega_1|\frac{\xi_1^{\gamma+1}}{\Gamma(\gamma+2)} + \eta|\omega_2|\frac{1-\xi_2^{\gamma+1}}{\Gamma(\gamma+2)} + |\Delta|\frac{\eta^{\gamma}}{\Gamma(\gamma+1)}\right],$$
$$\mathcal{M}_4 = \frac{1}{\Gamma(\delta+1)} + \frac{1}{|\eta^2-\Delta^2|}\left[\frac{\eta^{\delta+1}}{\Gamma(\delta+1)} + |\Delta||\omega_1|\frac{\xi_1^{\delta+1}}{\Gamma(\delta+2)} + \Delta|\omega_2|\frac{1-\xi_2^{\delta+1}}{\Gamma(\delta+2)}\right],$$
$$\mathcal{N}_3 = \frac{\eta}{|\eta^2-\Delta^2|}\left[|\omega_1|\xi_1\mu_2 + |\omega_2|\mu_2(1-\xi_2) + \mu_1\right] + |\Delta|\left[|\omega_1|\mu_1\xi_1 + \mu_1|\omega_2|(1-\xi_2) + \mu_2\right] + \mu_1,$$
$$\mathcal{N}_4 = \frac{1}{|\eta^2-\Delta^2|}\left[|\Delta||\omega_1|\mu_1\xi_1 + |\omega_2|(1-\xi_2)\mu_1 + \mu_2\right] + |\eta|\left[|\omega_1|\mu_2\xi_1 + |\omega_2|(1-\xi_2)\mu_2 + \mu_1\right] + \mu_2,$$
$$\mathcal{N}_5 = \frac{1}{|\eta^2-\Delta^2|}\left[\eta|\omega_1|\xi_1 + |\omega_2|\eta(1-\xi_2) + |\Delta|\right],$$
$$\mathcal{N}_6 = \frac{1}{|\eta^2-\Delta^2|}\left[|\Delta|\xi_1|\omega_1| + |\Delta||\omega_2|(1-\xi_2) + \eta\right] + 1,$$
$$\mathcal{N}_7 = \frac{1}{|\eta^2-\Delta^2|}\left[|\Delta| + |\eta||\omega_1|\xi_1 + |\eta||\omega_2|(1-\xi_2) + |\omega_2| + |\Delta|\right],$$

and

$$\begin{aligned}\mathcal{M}_k = \min\Big\{&1-(\mathcal{M}_1+\mathcal{M}_3)k_1-(\mathcal{M}_2+\mathcal{M}_4)\lambda_1,\\ &1-(\mathcal{M}_1+\mathcal{M}_3)k_2-(\mathcal{M}_2+\mathcal{M}_4)\lambda_2\Big\}, k_i,\lambda_i \geq 0,\ i=1,2.\end{aligned} \tag{3.1}$$

(A5) $(\mathcal{M}_1+\mathcal{M}_3)(m_1+m_2)+(\mathcal{M}_2+\mathcal{M}_4)(n_1+n_2)+(\mathcal{N}_5+\mathcal{N}_6)(\sigma_1+\sigma_2)+(\mathcal{N}_7+\mathcal{N}_8)(\beta_1+\beta_2)<1.$

In view of Lemma 2.1, we define an operator $\mathcal{T}:\mathcal{W}\times\mathcal{W}\to\mathcal{W}\times\mathcal{W}$ associated with the problems (1.1) and (1.2) as follows:

$$\mathcal{T}(u,v)(t) = \begin{pmatrix}\mathcal{T}_1(u,v)(t)\\ \mathcal{T}_2(u,v)(t)\end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned}\mathcal{T}_1(u,v)(t) = \frac{t}{\eta^2-\Delta^2}\Bigg[&\eta\Bigg\{\omega_1\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\theta_2(\tau,u(\tau),v(\tau))d\tau + h_2(s,u(s),v(s)\right)ds\\ &+\omega_2\int_{\xi_2}^1\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\theta_2(\tau,u(\tau),v(\tau))d\tau + h_2(s,u(s),v(s)\right)ds\\ &-\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(s,u(s),v(s))ds - h_1(\eta,u(\eta),v(\eta))\Bigg\}\\ &+\Delta\Bigg\{\omega_1\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(\tau,u(\tau),v(\tau))d\tau + h_1(s,u(s),v(s)\right)ds\\ &+\omega_2\int_{\xi_2}^1\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(\tau,u(\tau),v(\tau))d\tau + h_1(s,u(s),v(s)\right)ds\\ &-\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\theta_2(s,u(s),v(s))ds - h_2(\eta,u(\eta),v(\eta))\Bigg\}\Bigg]\end{aligned}$$

$$+\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(s,u(s),v(s))ds + h_1(t,u(t),v(t))$$

and

$$\begin{aligned}\mathcal{T}_2(u,v)(t) = \frac{t}{\eta^2-\Delta^2}\Bigg[\Delta\Bigg\{\omega_1\int_0^{\xi_1}\Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\theta_2(\tau,u(\tau),v(\tau))d\tau + h_1(s,u(s),v(s)\Bigg)ds \\ + \omega_2\int_{\xi_2}^1\Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\theta_2(\tau,u(\tau),v(\tau))d\tau + h_1(s,u(s),v(s)\Bigg)ds \\ - \int_0^\eta \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(s,u(s),v(s))ds - h_2(\eta,u(\eta),v(\eta))\Bigg\} \\ + \eta\Bigg\{\omega_1\int_0^{\xi_1}\Bigg(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(\tau,u(\tau),v(\tau))d\tau + h_2(s,u(s),v(s)\Bigg)ds \\ + \omega_2\int_{\xi_2}^1\Bigg(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\theta_1(\tau,u(\tau),v(\tau))d\tau + h_2(s,u(s),v(s)\Bigg)ds \\ - \int_0^\eta \frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\theta_2(s,u(s),v(s))ds - h_1(\eta,u(\eta),v(\eta))\Bigg\}\Bigg] \\ + \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\theta_2(s,u(s),v(s))ds + h_2(t,u(t),v(t)).\end{aligned}$$

Theorem 3.1. *Assume that conditions (A1) to (A5) are satisfied. Then there exists a unique solution for the problems* (1.1) *and* (1.2) *on* $[0,1]$.

Proof. In the first step, we establish that $\mathcal{T}\bar{B}_r \subset \bar{B}_r$, where $\bar{B}_r = \{(u,v) \in \mathcal{W}\times\mathcal{W} : \|(u,v)\| \le r\}$ is a closed ball with

$$r \ge \frac{(\mathcal{M}_1+\mathcal{M}_3)\mathcal{N}_1 + (\mathcal{M}_2+\mathcal{M}_4)\mathcal{N}_2 + \mathcal{N}_3\mu}{1-[(\mathcal{M}_1+\mathcal{M}_3)(m_1+m_2)+(\mathcal{M}_2+\mathcal{M}_4)(n_1+n_2)+\mathcal{N}_3(\beta_1+\beta_2)]},$$

and the operator $\mathcal{T} : \mathcal{W}\times\mathcal{W} \to \mathcal{W}\times\mathcal{W}$ is defined by (3.2). For $(u,v) \in \bar{B}_r$ and $t \in [0,1]$, it follows by (A1) that

$$|\theta_1(t,u(t),v(t))| \le |\theta_1(t,u(t),v(t)) - \theta_1(t,0,0)| \le m_1\|u\| + m_2\|v\|.$$

Similarly one can find that $|\theta_2(t,u(t),v(t))| \le n_1\|u\| + n_2\|v\|$. Then we have

$$\begin{aligned}&|\mathcal{T}_1(u,v)(t)| \\ &\le \max_{t\in[0,1]}\Bigg[\frac{t}{|\eta^2-\Delta^2|}\Bigg[\eta\Bigg\{|\omega_1|\int_0^{\xi_1}\Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}|\theta_2(\tau,u(\tau),v(\tau))|d\tau + |h_2(s,u(s),v(s)|\Bigg)ds \\ &+ |\omega_2|\int_{\xi_2}^1\Bigg(\int_0^s \frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}|\theta_2(\tau,u(\tau),v(\tau))|d\tau + |h_2(s,u(s),v(s)|\Bigg)ds \\ &+ \int_0^\eta \frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}|\theta_1(s,u(s),v(s))|ds + |h_1(\eta,u(\eta),v(\eta))|\Bigg\} \\ &+ |\Delta|\Bigg\{|\omega_1|\int_0^{\xi_1}\Bigg(\int_0^s \frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}|\theta_1(\tau,u(\tau),v(\tau))|d\tau + |h_1(s,u(s),v(s)|\Bigg)ds\end{aligned}$$

$$
\begin{aligned}
&+|\omega_2|\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}|\theta_1(\tau,u(\tau),v(\tau))|d\tau+|h_1(s,u(s),v(s)|\right)ds\\
&+\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}|\theta_2(s,u(s),v(s))|ds+|h_2(\eta,u(\eta),v(\eta))|\Bigg\}\Bigg]\\
&+\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}|\theta_1(s,u(s),v(s))|ds+|h_1(t,u(t),v(t))|\Bigg]\\
&\le\frac{1}{|\eta^2-\Delta^2|}\Bigg[\eta\Bigg\{|\omega_1|\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}(n_1\|u\|+n_2\|v\|+\mathcal{N}_2)\,d\tau+\mu_2\right)ds\\
&+|\omega_2|\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}|\,(n_1\|u\|+n_2\|v\|+\mathcal{N}_2)\,d\tau+\mu_2\right)ds\\
&+\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}(m_1\|u\|+m_2\|v\|+\mathcal{N}_1)\,d\tau+\mu_1\Bigg\}\\
&+|\Delta|\Bigg\{|\omega_1|\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}(m_1\|u\|+m_2\|v\|+\mathcal{N}_1)\,d\tau+\mu_1\right)ds\\
&+|\omega_2|\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}(m_1\|u\|+m_2\|v\|+\mathcal{N}_1)\,d\tau+\mu_1\right)ds\\
&+\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}(n_1\|u\|+n_2\|v\|+\mathcal{N}_2)\,ds+\mu_2\Bigg\}\Bigg]\\
&+\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}(m_1\|u\|+m_2\|v\|+\mathcal{N}_1)\,ds+\mu_1\\
&\le\frac{1}{|\eta^2-\Delta^2|}\left[\eta|\omega_1|\frac{\xi_1^{\delta+1}}{\Gamma(\delta+2)}+\eta|\omega_2|\frac{1-\xi_2^{\delta+1}}{\Gamma(\delta+2)}+|\Delta|\frac{\eta^{\delta}}{\Gamma(\delta+1)}\right](n_1\|u\|+n_2\|v\|+\mathcal{N}_2)\\
&+\left[\frac{1}{|\eta^2-\Delta^2|}\left(\frac{\eta^{\gamma+1}}{\Gamma(\gamma+1)}+|\Delta||\omega_1|\frac{\xi_1^{\gamma+1}}{\Gamma(\gamma+2)}+|\Delta||\omega_2|\frac{1-\xi_2^{\gamma+1}}{\Gamma(\gamma+2)}\right)+\frac{1}{\Gamma(\gamma+1)}\right](m_1\|u\|+m_2\|v\|+\mathcal{N}_1)\\
&+\frac{\eta}{|\eta^2-\Delta^2|}(|\omega_1|\mu_2\xi_1+|\omega_2|\mu_2(1-\xi_2)+\mu_1)+|\Delta|\,(|\omega_1|\mu_1\xi_1+|\omega_2|\mu_1(1-\xi_2)+\mu_2)+\mu_1\\
&\le(\mathcal{M}_2n_1+\mathcal{M}_1m_1+\mathcal{M}_2n_2+\mathcal{M}_1m_2)\,r+\mathcal{M}_2\mathcal{N}_2+\mathcal{M}_1\mathcal{N}_1+\mathcal{N}_3.
\end{aligned}
$$

Analogously, one can find that

$$|\mathcal{T}_2(u,v)(t)|\le(\mathcal{M}_4n_1+\mathcal{M}_3m_1+\mathcal{M}_4n_2+\mathcal{M}_3m_2)\,r+\mathcal{M}_4\mathcal{N}_2+\mathcal{M}_3\mathcal{N}_1+\mathcal{N}_4.$$

From the foregoing estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$, we obtain $\|\mathcal{T}(u,v)(t)\|\le r$.
Next, for $(u_1,v_1),(u_2,v_2)\in\mathcal{W}\times\mathcal{W}$ and $t\in[0,1]$, we get

$$
\begin{aligned}
&|\mathcal{T}_1(u_2,v_2)(t)-\mathcal{T}_1(u_1,v_1)(t)|\\
&\le\frac{1}{|\eta^2-\Delta^2|}\Bigg[\eta\Bigg\{|\omega_1|\Bigg(\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}|\theta_2(\tau,u_2(\tau),v_2(\tau))-\theta_2(\tau,u_1(\tau),v_1(\tau))|\right)d\tau\\
&\qquad\qquad+|h_2(s,u_2(s),v_2(s))-h_2(s,u_1(s),v_1(s))|\,ds\Bigg)
\end{aligned}
$$

$$+|\omega_2|\left(\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\delta-1}}{\Gamma(\delta)}\left|\theta_2(\tau,u_2(\tau),v_2(\tau))-\theta_2(\tau,u_1(\tau),v_1(\tau))\right|\right)d\tau\right.$$

$$\left.+\left|h_2(s,u_2(s),v_2(s))-h_2(s,u_1(s),v_1(s))\right|ds\right)$$

$$+\int_0^{\eta}\frac{(\eta-s)^{\gamma-1}}{\Gamma(\gamma)}\left|\theta_1(\tau,u_2(\tau),v_2(\tau))-\theta_1(\tau,u_1(\tau),v_1(\tau))\right|ds$$

$$\left.+\left|h_1(\eta,u_2(\eta),v_2(\eta))-h_1(\eta,u_1(\eta),v_1(\eta))\right|\right\}$$

$$+|\Delta|\left\{|\omega_1|\left(\int_0^{\xi_1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\left|\theta_1(\tau,u_2(\tau),v_2(\tau))-\theta_1(\tau,u_1(\tau),v_1(\tau))\right|\right)d\tau\right.\right.$$

$$\left.+\left|h_1(s,u_2(s),v_2(s))-h_1(s,u_1(s),v_1(s))\right|ds\right)$$

$$+|\omega_2|\left(\int_{\xi_2}^{1}\left(\int_0^s\frac{(s-\tau)^{\gamma-1}}{\Gamma(\gamma)}\left|\theta_1(\tau,u_2(\tau),v_2(\tau))-\theta_1(\tau,u_1(\tau),v_1(\tau))\right|\right)d\tau\right.$$

$$\left.+\left|h_1(s,u_2(s),v_2(s))-h_1(s,u_1(s),v_1(s))\right|ds\right)$$

$$+\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}\left|\theta_1(\tau,u_2(\tau),v_2(\tau))-\theta_1(\tau,u_2(\tau),v_2(\tau)\right|ds$$

$$\left.\left.+\left|h_2(\eta,u_2(\eta),v_2(\eta))-h_2(\eta,u_1(\eta),v_1(\eta))\right|ds\right\}\right]$$

$$+\int_0^{t}\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\left(\left|\theta_1(s,u_2(s),v_2(s))-\theta_1(s,u_1(s),v_1(s))\right|\right)ds$$

$$+\left|h_1(t,u_2(t),v_2(t))-h_1(t,u_1(t),v_1(t))\right|$$

$$\le\frac{1}{|\eta^2-\Delta^2|}\left[\eta|\omega_1|\frac{\xi_1^{\delta+1}}{\Gamma(\delta+2)}+\eta|\omega_2|\frac{1-\eta^{\delta+1}}{\Gamma(\delta+2)}+|\Delta|\frac{\eta^{\delta}}{\Gamma(\delta+1)}\right]\times(n_1\|u_2-u_1\|+n_2\|v_2-v_1\|)$$

$$+\left(\frac{1}{|\eta^2-\Delta^2|}\left[\frac{\eta^{\gamma+1}}{\Gamma(\gamma+1)}+|\Delta||\omega_1|\frac{\xi_1^{\gamma+1}}{\Gamma(\gamma+2)}+|\Delta||\omega_2|\frac{1-\xi_2^{\gamma+1}}{\Gamma(\gamma+2)}\right]\right.$$

$$\left.+\frac{1}{\Gamma(\gamma+1)}\right)\times(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|)$$

$$+\frac{1}{|\eta^2-\Delta^2|}\Big[(\eta|\omega_1|\xi_1+\eta|\omega_2|(1-\xi_2)+|\Delta|)(\sigma_1\|u_2-u_1\|+\sigma_2\|v_2-v_1\|)$$

$$+(\eta+|\Delta||\omega_1|\xi_1+|\Delta||\omega_2|(1-\xi_2)+1)(\beta_1\|u_2-u_1\|+\beta_2\|v_2-v_1\|)\Big]$$

$$\le\mathcal{M}_2\left(n_1\|u_2-u_1\|+n_2\|v_2-v_1\|\right)+\mathcal{M}_1\left(m_1\|u_2-u_1\|+m_2\|v_2-v_1\|\right)$$
$$+\mathcal{N}_5\left(\sigma_1\|u_2-u_1\|+\sigma_2\|v_2-v_1\|\right)+\mathcal{N}_6\left(\beta_1\|u_2-u_1\|+\beta_2\|v_2-v_1\|\right)$$
$$=(\mathcal{M}_2n_1+\mathcal{M}_1m_1+\mathcal{N}_5\sigma_1+\mathcal{N}_6\beta_1)\|u_2-u_1\|+(\mathcal{M}_2n_2+\mathcal{M}_1m_2+\mathcal{N}_5\sigma_2)\|v_2-v_1\|)$$

which implies that

$$\|\mathcal{T}_1(u_2, v_2)(t) - \mathcal{T}_1(u_1, v_1)(t)\|$$
$$\leq (\mathcal{M}_2 n_1 + \mathcal{M}_1 m_1 + \mathcal{N}_5\sigma_1 + \mathcal{N}_6\beta_1 + \mathcal{M}_2 n_2 + \mathcal{M}_1 m_2 + \mathcal{N}_5\sigma_2 + \mathcal{N}_6\beta_2)(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (3.3)$$

Likewise, we have

$$\|\mathcal{T}_2(u_2, v_2)(t) - \mathcal{T}_2(u_1, v_1)(t)\|$$
$$\leq (\mathcal{M}_4 n_1 + \mathcal{M}_3 m_1 + \mathcal{N}_6\sigma_1 + \mathcal{N}_7\beta_1 + \mathcal{M}_4 n_2 + \mathcal{M}_3 m_2 + \mathcal{N}_6\sigma_2 + \mathcal{N}_7\beta_2)(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (3.4)$$

From (3.3) and (3.4), we deduce that

$$\|\mathcal{T}(u_2, v_2)(t) - \mathcal{T}(u_1, v_1)(t)\|$$
$$\leq [(\mathcal{M}_1 + \mathcal{M}_3)(m_1 + m_2) + (\mathcal{M}_2 + \mathcal{M}_4)(n_1 + n_2) + (\mathcal{N}_7 + \mathcal{N}_8)(\beta_1 + \beta_2) + (\mathcal{N}_5 + \mathcal{N}_6)(\sigma_1 + \sigma_2)]$$
$$\times (\|u_2 - u_1\| + \|v_2 - v_1\|),$$

which shows that $\mathcal{T}$ is a contraction by the assumption (A5) and hence it has a unique fixed point by Banach fixed point theorem. This leads to the conclusion that there exists a unique solution for the problems (1.1) and (1.2) on $[0, 1]$. The proof is complete. □

Now, we discuss the existence of solutions for the problems (1.1) and (1.2) by means of Leray-Schauder alternative ([12], p. 4).

Theorem 3.2. *Assume that there exists real constants* $\tilde{k}_0 > 0$, $\tilde{\lambda}_0 > 0$ *and* $\tilde{k}_i, \tilde{\lambda}_i \geq 0,\ i = 1, 2$ *such that, for any* $u_i \in \Re,\ i = 1, 2$

$$|\theta_1(t, u_1, u_2)| \leq \tilde{k}_0 + \tilde{k}_1|u_1| + \tilde{k}_2|u_2|,\ \ |\theta_2(t, u_1, u_2)| \leq \tilde{\lambda}_0 + \tilde{\lambda}_1|u_1| + \tilde{\lambda}_2|u_2|.$$

In addition,

$$(\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_1 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_1 < 1,\ \ (\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_2 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_2 < 1,$$

where $\mathcal{M}_i,\ i = 1, 2, 3, 4$ *are given in (A4). Then the problems* (1.1) *and* (1.2) *have at least one solution on* $[0, 1]$.

Proof. The proof consists of two steps. First we show that the operator $\mathcal{T} : \mathcal{W} \times \mathcal{W} \to \mathcal{W} \times \mathcal{W}$ defined by (3.2) is completely continuous. Observe that continuity of the operator $\mathcal{T}$ follows from that of θ_1 and θ_2. Consider a bounded set $\Omega \subset \mathcal{W} \times \mathcal{W}$ so that we can find positive constants l_1 and l_2 such that $|\theta_1(t, u(t), v(t))| \leq l_1$ and $|\theta_2(t, u(t), v(t))| \leq l_2$ for every $(u, v) \in \Omega$. Hence, for any $(u, v) \in \Omega$, we find that

$$|\mathcal{T}_1(u, v)(t)| \leq \frac{1}{|\eta^2 - \Delta^2|}\left[\eta|\omega_1|\frac{\xi_1^{\delta+1}}{\Gamma(\delta + 2)} + \eta|\omega_2|\frac{1 - \xi_2^{\delta+1}}{\Gamma(\delta + 2)} + |\Delta|\frac{\eta^{\delta}}{\Gamma(\delta + 1)}\right]l_2$$
$$+ \left\{\frac{1}{|\eta^2 - \Delta^2|}\left[\frac{\eta^{\gamma}}{\Gamma(\gamma + 1)} + |\Delta||\omega_1|\frac{\xi_1^{\gamma+1}}{\Gamma(\gamma + 2)} + |\Delta||\omega_2|\frac{1 - \xi_2^{\gamma+1}}{\Gamma(\gamma + 2)} + \frac{1}{\Gamma(\gamma + 1)}\right]\right\}l_1$$
$$+ \frac{1}{|\eta^2 - \Delta^2|}\Big\{\eta\left[|\omega_1|\xi_1\mu_2 + |\omega_2|\mu_2(1 - \xi_2) + \mu_1\right] + \eta\mu_1 + |\Delta|\left[\mu_1\xi_1|\omega_1| + \mu_1|\omega_2|(1 - \xi_2) + \mu_2\right]\Big\} + \mu_1$$

$$= \mathcal{M}_2 l_2 + \mathcal{M}_1 l_1 + \mathcal{N}_3.$$

Thus we deduce that $\|\mathcal{T}_1(u,v)\| \leq \mathcal{M}_2 l_2 + \mathcal{M}_1 l_1 + \mathcal{N}_3$. In a similar fashion, it can be found that $\|\mathcal{T}_2(u,v)\| \leq \mathcal{M}_4 l_2 + \mathcal{M}_3 l_1 + \mathcal{N}_4$. Hence, it follows from the foregoing inequalities that $\mathcal{T}_1$ and $\mathcal{T}_2$ are uniformly bounded and hence the operator $\mathcal{T}$ is uniformly bounded. In order to show that $\mathcal{T}$ is equicontinuous, we take $0 < r_1 < r_2 < 1$. Then, for any $(u,v) \in \Omega$, we obtain

$$\begin{aligned}
&|\mathcal{T}_1(u(r_2),v(r_2)) - \mathcal{T}_1(u(r_1),v(r_1))| \\
&\leq \frac{l_1}{\Gamma(\gamma)} \int_0^{r_1} \left[(r_2-s)^{\gamma-1} - (r_1-s)^{\gamma-1}\right] ds + \frac{l_1}{\Gamma(\gamma)} \int_{r_1}^{r_2} (r_2-s)^{\gamma-1} ds \\
&+ \frac{r_2-r_1}{|\eta^2-\Delta^2|} \left\{ \left[\eta|\omega_1| \frac{\xi_1^{\delta+1}}{\Gamma(\delta+2)} + \eta|\omega_2| \frac{1-\xi_2^{\delta+1}}{\Gamma(\delta+2)} + |\Delta| \frac{\eta^{\delta}}{\Gamma(\delta+1)} \right] l_2 \right. \\
&\left. + \left[\frac{\eta^{\gamma+1}}{\Gamma(\gamma+1)} + |\Delta||\omega_1| \frac{\xi_1^{\gamma+1}}{\Gamma(\gamma+2)} + |\Delta||\omega_2| \frac{1-\xi_2^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{1}{\Gamma(\gamma+2)} \right] l_1 + N_3 \right\},
\end{aligned}$$

$$\begin{aligned}
&|\mathcal{T}_2(u(r_2),v(r_2)) - \mathcal{T}_2(u(r_1),v(r_1))| \\
&\leq \frac{l_2}{\Gamma(\delta)} \int_0^{r_1} \left[(r_2-s)^{\delta-1} - (r_1-s)^{\delta-1}\right] ds + \frac{l_2}{\Gamma(\delta)} \int_{r_1}^{r_2} (r_2-s)^{\delta-1} ds \\
&+ \frac{r_2-r_1}{|\eta^2-\Delta^2|} \left\{ \left[\eta|\omega_1| \frac{\xi_1^{\delta+1}}{\Gamma(\delta+2)} + \eta|\omega_2| \frac{1-\xi_2^{\delta+1}}{\Gamma(\delta+2)} + \frac{\eta.\eta^{\delta+1}}{\Gamma(\delta+1)} \right] l_2 \right. \\
&\left. + \left[|\Delta| \frac{\eta^{\gamma}}{\Gamma(\gamma+1)} + \eta|\omega_1| \frac{\xi_1^{\gamma+1}}{\Gamma(\gamma+2)} + \eta|\omega_2| \frac{1-\xi_2^{\gamma+1}}{\Gamma(\gamma+2)} \right] l_1 + \mathcal{N}_4 \right\},
\end{aligned}$$

which imply that the operator $\mathcal{T}(u,v)$ is equicontinuous. In view of the foregoing arguments, we deduce that operator $\mathcal{T}(u,v)$ is completely continuous.

Next, we consider a set $\mathcal{P} = \{(u,v) \in \mathcal{W} \times \mathcal{W} : (u,v) = \lambda \mathcal{T}(u,v),\ 0 \leq \lambda \leq 1\}$ and show that it is bounded. Let us take $(u,v) \in \mathcal{P}$ and $t \in [0,1]$. Then it follows from $u(t) = \lambda \mathcal{T}_1(u,v)(t)$ and $v(t) = \lambda \mathcal{T}_2(u,v)(t)$, together with the given assumptions that

$$\begin{aligned}
\|u\| &\leq \mathcal{M}_1 \left(\tilde{k}_0 + \tilde{k}_1\|u\| + \tilde{k}_2\|v\|\right) + \mathcal{M}_2 \left(\tilde{\lambda}_0 + \tilde{\lambda}_1\|u\| + \tilde{\lambda}_2\|v\|\right) + \mathcal{N}_3, \\
\|v\| &\leq \mathcal{M}_3 \left(\tilde{k}_0 + \tilde{k}_1\|u\| + \tilde{k}_2\|v\|\right) + \mathcal{M}_4 \left(\tilde{\lambda}_0 + \tilde{\lambda}_1\|u\| + \tilde{\lambda}_2\|v\|\right) + \mathcal{N}_4,
\end{aligned}$$

which lead to

$$\begin{aligned}
\|u\| + \|v\| \leq &\left[(\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_0 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_0 + \mathcal{N}_3 + \mathcal{N}_4\right] \\
&+ \left[(\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_1 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_1\right]\|u\| + \left[(\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_2 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_2\right]\|v\|.
\end{aligned}$$

Thus

$$\|(u,v)\| \leq \frac{(\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_0 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_0 + \mathcal{N}_3 + \mathcal{N}_4}{\mathcal{M}_k},$$

where $\mathcal{M}_k$ is defined by (3.1). Consequently the set $\mathcal{P}$ is bounded. Hence, it follows by Leray-Schauder alternative ([12], p. 4) that the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problems (1.1) and (1.2) have at least one solution on [0, 1]. This finishes the proof. □

4. Examples

Example 4.1. *Consider a coupled boundary value problem of fractional differential equations with slit-strips-type conditions given by*

$$
\begin{aligned}
&{}^{c}D^{3/2}\left(u(t) - \frac{\sin t|u(t)|}{2(2+|u(t)|)}\right) = \frac{1}{56}u(t) + \frac{2}{7}\frac{v(t)}{1+v(t)} + \frac{5}{7},\\
&{}^{c}D^{5/4}\left(v(t) - \frac{\sin t|v(t)|}{2(2+|v(t)|)}\right) = \frac{1}{39}\frac{|\cos u(t)|}{1+|\cos u(t)|} + \frac{1}{28}\sin v(t) + \frac{3}{7},
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
&u(0) = 0,\ u\left(\frac{1}{2}\right) = \int_0^{1/5} v(s)ds + \int_{4/5}^{1} v(s)ds,\\
&v(0) = 0,\ v\left(\frac{1}{2}\right) = \int_0^{1/5} u(s)ds + \int_{4/5}^{1} u(s)ds.
\end{aligned} \tag{4.2}
$$

Here $\gamma = \frac{3}{2}$, $\delta = \frac{5}{4}$, $\omega_1 = 1$, $\omega_2 = 1$, $\eta = \frac{1}{2}$, $\xi_1 = \frac{1}{5}$, $\xi_2 = \frac{4}{5}$. *From the given data, we find that* $\Delta = -0.11$, $m_1 = \frac{1}{56}$, $m_2 = \frac{2}{71}$, $n_1 = \frac{1}{39}$, $n_2 = \frac{1}{28}$, $\mathcal{M}_1 \simeq 1.44716$, $\mathcal{M}_2 \simeq 0.51905$, $\mathcal{M}_3 \simeq 0.4046$, $\mathcal{M}_4 \simeq 2.51887$, $\mathcal{N}_5 \simeq 2.94238$, $\mathcal{N}_6 \simeq 7.3223$, $\mathcal{N}_7 \simeq 5.6164$, $\mathcal{N}_8 \simeq 5.2206$, *and* $(\mathcal{M}_1 + \mathcal{M}_3)(m_1 + m_2) + (\mathcal{M}_2 + \mathcal{M}_4)(n_1 + n_2) + (\mathcal{N}_5 + \mathcal{N}_6)(\sigma_1 + \sigma_2) + (\mathcal{N}_7 + \mathcal{N}_8)(\beta_1 + \beta_2) \simeq 0.8030305 < 1$.
Clearly all the conditions of Theorem 3.1 are satisfied. In consequence, the conclusion of Theorem 3.1 applies to the problems (4.1)–(4.2).

Example 4.2. *We consider the problems* (4.1)–(4.2) *with*

$$
\theta_1(t, u(t), v(t)) = \frac{1}{2} + \frac{2}{39}tanu(t) + \frac{2}{41}v(t),\ \ \theta_2(t, u(t), v(t)) = \frac{2}{5} + \frac{1}{9}sinu(t) + \frac{1}{17}v(t). \tag{4.3}
$$

Observe that

$$
|\theta_1(t, u, v)| \leq \tilde{k}_0 + \tilde{k}_1|u| + \tilde{k}_2|v|,\ \ |\theta_2(t, u, v)| \leq \tilde{\lambda}_0 + \tilde{\lambda}_1|u| + \tilde{\lambda}_2|v|
$$

with $\tilde{k}_0 = \frac{1}{2}$, $\tilde{k}_1 = \frac{2}{39}$, $\tilde{k}_2 = \frac{2}{41}$ $\tilde{\lambda}_0 = \frac{2}{5}$, $\tilde{\lambda}_1 = \frac{1}{9}$, $\tilde{\lambda}_2 = \frac{1}{17}$. *Furthermore,*

$$
(\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_1 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_1 \simeq 0.432507777 < 1, (\mathcal{M}_1 + \mathcal{M}_3)\tilde{k}_2 + (\mathcal{M}_2 + \mathcal{M}_4)\tilde{\lambda}_2 \simeq 0.269030756 < 1.
$$

Thus all the conditions of Theorem 3.2 hold true and hence there exists at least one solution for the problems (4.1)–(4.2) *with* $\theta_1(t, u, v)$ *and* $\theta_2(t, u, v)$ *given by* (4.3).

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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