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Research article

On the finite reciprocal sums of Fibonacci and Lucas polynomials

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Abstract: In this note, we consider the finite reciprocal sums of Fibonacci and Lucas polynomials and derive some identities involving these sums.

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1. Introduction

For any variable x, the Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are recursively defined by $F_{n+1}(x) = xF_n(x)+F_{n-1}(x)$, $n \geq 1$ and $L_{n+1}(x) = xL_n(x)+L_{n-1}(x)$, $n \geq 1$ with their respective initial values $F_0(x) = 0$, $F_1(x) = 1$ and $L_0(x) = 2$, $L_1(x) = x$. For $x = 1$, the Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are respectively the well known Fibonacci and Lucas numbers. The closed form expressions for $F_n(x)$ and $L_n(x)$ are indeed

$$F_n(x) = \frac{\alpha^n - \beta^n}{\sqrt{x^2+4}} \text{ and } L_n(x) = \alpha^n + \beta^n,$$

where $\alpha = \frac{1}{2}(x + \sqrt{x^2+4})$ and $\beta = \frac{1}{2}(x - \sqrt{x^2+4})$. Several authors studied extensively the Fibonacci and Lucas polynomials and deduced various important properties for both these polynomials (e.g., see [4, 8, 13]).

For n-th Fibonacci number F_n and n-th Lucas number L_n, the Fibonacci and Lucas zeta functions are respectively defined as

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \text{ and } \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},$$

which have been studied extensively in various different aspects [1, 5]. Many researchers considered the infinite sums derived from the reciprocals of different number sequences such as Fibonacci, Pell, balancing, Lucas-balancing sequence etc. and established several results concerning these sequences [2, 6, 14]. For instance, Ohtsuka and Nakamura [6] studied the partial infinite sums of reciprocal Fibonacci numbers and derived the following results, where $\lfloor . \rfloor$ denotes the floor function.

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \text{ is even and } n \geq 2; \\ F_n F_{n-1}, & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

Wu and Zhang [11] generalized the above identities by considering Fibonacci and Lucas polynomials. Later on, Wu and Zhang [12] considered the sub-series of infinite sums of reciprocal of Fibonacci and Lucas polynomials and derived various identities involving these sums.

Several authors [3, 7, 9, 10] studied the bounds for partial finite reciprocal sums involving terms from Fibonacci and other number sequences. For example, Wang and Wen [7] considered the finite reciprocal sums of Fibonacci numbers and established the following identities.

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even } (n \geq 2) \text{ and } m \geq 3; \\ F_{n-2} - 1, & \text{if } n \text{ is odd } (n > 1) \text{ and } m \geq 3. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \text{ is even } (n \geq 2) \text{ and } m \geq 2; \\ F_n F_{n-1}, & \text{if } n \text{ is odd } (n \geq 1) \text{ and } m \geq 2. \end{cases}$$

In the present study, we consider the partial finite sums of reciprocal of Fibonacci polynomials, Lucas polynomials, square of Fibonacci polynomials and square of Lucas polynomials. We derive the following results relating to the these sums that significantly improve the results of Wu and Zhang [11].

Theorem 1.1. *For any positive integers x, n and $m \geq 3$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k(x)} \right)^{-1} \right\rfloor = \begin{cases} F_n(x) - F_{n-1}(x), & \text{if } n \text{ is even}; \\ F_n(x) - F_{n-1}(x) - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.2. *For any integer $x < 0$ and any positive integers $n \geq 2$ and $m \geq 3$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k(x)} \right)^{-1} \right\rfloor = F_n(x) - F_{n-1}(x).$$

Theorem 1.3. *For any positive integers x, n and $m \geq 3$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{L_k(x)} \right)^{-1} \right\rfloor = \begin{cases} L_n(x) - L_{n-1}(x) - 1, & \text{if } n \text{ is even and } n \geq 2; \\ L_n(x) - L_{n-1}(x), & \text{if } n \text{ is odd and } n \geq 3. \end{cases}$$

Theorem 1.4. *For any integer $x < 0$ and any positive integers $n > 3$ and $m \geq 3$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{L_k(x)} \right)^{-1} \right\rfloor = L_n(x) - L_{n-1}(x).$$

Theorem 1.5. *For any integer $x \in \mathbb{Z} - \{0\}$ and any positive integers n and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} \right)^{-1} \right\rfloor = \begin{cases} xF_n(x)F_{n-1}(x) - 1, & \text{if } n \text{ is even };\\ xF_n(x)F_{n-1}(x), & \text{if } n \text{ is odd }. \end{cases}$$

Theorem 1.6. *For any integer $x \in \mathbb{Z} - \{0, \pm 1\}$ and any positive integers n and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{L_k^2(x)} \right)^{-1} \right\rfloor = \begin{cases} xL_{2n-1}(x) + 1, & \text{if } n \text{ is even and } n \geq 2;\\ xL_{2n-1}(x) - 2, & \text{if } n \text{ is odd and } n \geq 3. \end{cases}$$

2. Proof of theorems

The following two results are found in [13], which are used to prove our main theorems.

Lemma 2.1. *For any positive integers x, m, n, $F_{m+n+1}(x) = F_{m+1}(x)F_{n+1}(x) + F_m(x)F_n(x)$.*

Lemma 2.2. *For any positive integers x and n, $F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n$.*

Lemma 2.3. *For any positive integers x and n, $F_n(-x) = \begin{cases} F_n(x), & \text{if } n \text{ is odd};\\ -F_n(x), & \text{if } n \text{ is even}. \end{cases}$*

Proof. In order to prove the result, it suffices to show that $F_{2m-1}(-x) = F_{2m-1}(x)$ and $F_{2m}(-x) = -F_{2m}(x)$ for any positive integer $m \geq 1$. We use induction on m. For $m = 1$, $F_1(-x) = 1 = F_1(x)$ and $F_2(-x) = -x = -F_1(x)$. Let us assume the truth for $m = k$, that is, $F_{2k-1}(-x) = F_{2k-1}(x)$ and $F_{2k}(-x) = -F_{2k}(x)$ for some positive integer k. Now $F_{2(k+1)-1}(-x) = F_{2k+1}(-x) = -xF_{2k}(-x) + F_{2k-1}(-x) = xF_{2k}(x) + F_{2k-1}(x) = F_{2k+1}(x)$. Similarly, $F_{2(k+1)}(-x) = F_{2k+2}(-x) = -xF_{2k+1}(-x) + F_{2k}(-x) = -xF_{2k+1}(x) - F_{2k}(x) = -F_{2k+2}(x)$. This ends the proof the lemma. □

Using Lemma 2.3, one can easily check that the Lemmas 2.1 and 2.2 hold for any integer $x < 0$.

We prove only Theorems 1.1, 1.2 and 1.5 and the remaining theorems can be proved analogously. In order to prove these theorems, we need the following lemmas. For the sake of argument, let us consider the following auxiliary functions for any positive integer x

$$f_1(n) = \frac{1}{F_n(x) - F_{n-1}(x) + 1} - \frac{1}{F_n(x)} - \frac{1}{F_{n+1}(x) - F_n(x) + 1}, \tag{2.1}$$

$$f_2(n) = \frac{1}{F_n(x) - F_{n-1}(x)} - \frac{1}{F_n(x)} - \frac{1}{F_{n+1}(x) - F_n(x)}, \tag{2.2}$$

$$f_3(n) = \frac{1}{F_n(x) - F_{n-1}(x) - 1} - \frac{1}{F_n(x)} - \frac{1}{F_{n+1}(x) - F_n(x) - 1}, \tag{2.3}$$

$$g_1(n) = \frac{1}{xF_n(x)F_{n-1}(x) - 1} - \frac{1}{F_n^2(x)} - \frac{1}{xF_{n+1}(x)F_n(x) - 1}, \tag{2.4}$$

$$g_2(n) = \frac{1}{xF_n(x)F_{n-1}(x)} - \frac{1}{F_n^2(x)} - \frac{1}{xF_{n+1}(x)F_n(x)}, \tag{2.5}$$

and

$$g_3(n) = \frac{1}{xF_n(x)F_{n-1}(x)+1} - \frac{1}{F_n^2(x)} - \frac{1}{xF_{n+1}(x)F_n(x)+1}. \tag{2.6}$$

Lemma 2.4. *If $n \geq 2$ even, then $f_1(n) + f_1(n+1) < 0$.*

Proof. For n even, by (2.1) and Lemma 2.2,

$$\begin{aligned}
& f_1(n) + f_1(n+1) \\
= & \left(\frac{1}{F_n(x) - F_{n-1}(x) + 1} - \frac{1}{F_n(x)}\right) - \left(\frac{1}{F_{n+1}(x)} + \frac{1}{F_{n+2}(x) - F_{n+1}(x) + 1}\right) \\
= & \frac{F_{n-1}(x) - 1}{F_n(x)\Big(F_n(x) - F_{n-1}(x) + 1\Big)} - \frac{F_{n+2}(x) + 1}{F_{n+1}(x)\Big(F_{n+2}(x) - F_{n+1}(x) + 1\Big)} \\
= & \frac{1}{F_n(x)\Big(\frac{F_n(x)}{F_{n-1}(x)-1} - 1\Big)} - \frac{1}{F_{n+1}(x)\Big(1 - \frac{F_{n+1}(x)}{F_{n+2}(x)+1}\Big)} \\
= & \frac{1}{\frac{F_{n+1}(x)F_{n-1}(x)-(-1)^n}{F_{n-1}(x)-1} - F_n(x)} - \frac{1}{F_{n+1}(x) - \frac{F_{n+2}(x)F_n(x)-(-1)^n}{F_{n+2}(x)+1}} \\
= & \frac{1}{\frac{F_{n+1}(x)F_{n-1}(x)-1}{F_{n-1}(x)-1} - F_n(x)} - \frac{1}{F_{n+1}(x) - \frac{F_{n+2}(x)F_n(x)-1}{F_{n+2}(x)+1}} \\
= & \frac{1}{F_{n+1}(x) - F_n(x) + \frac{F_{n+1}(x)-1}{F_{n-1}(x)-1}} - \frac{1}{F_{n+1}(x) - F_n(x) + \frac{F_n(x)+1}{F_{n+2}(x)+1}}.
\end{aligned}$$

As $\frac{F_{n+1}(x)-1}{F_{n-1}(x)-1} > \frac{F_n(x)+1}{F_{n+2}(x)+1}$, the result follows. □

Lemma 2.5. *If $m \geq 3$, and n is even, then $f_1(n) + f_1(n+1) + f_1(mn) + \frac{1}{F_{mn+1}(x) - F_{mn}(x)+1} < 0$.*

Proof. For $m \geq 3$, with the help of (2.1), we get

$$f_1(mn) + \frac{1}{F_{mn+1}(x) - F_{mn}(x) + 1} = \frac{F_{mn-1}(x) - 1}{F_{mn}(x)(F_{mn}(x) - F_{mn-1}(x) + 1)} < \frac{1}{F_{3n}(x)}.$$

Therefore,

$$\begin{aligned}
& f_1(n) + f_1(n+1) + f_1(mn) + \frac{1}{F_{mn+1}(x) - F_{mn}(x) + 1} \\
< & \frac{F_{n-1}(x) - 1}{F_n(x)\Big(F_n(x) - F_{n-1}(x) + 1\Big)} - \frac{F_{n+2}(x) + 1}{F_{n+1}(x)\Big(F_{n+2}(x) - F_{n+1}(x) + 1\Big)} + \frac{1}{F_{3n}(x)} \\
= & \frac{F_{n+2}(x) + F_n(x)F_{n-1}(x) - \Big(F_{n+2}(x)F_{n+1}(x) + F_{n+1}(x) + F_n(x) + F_{n-1}(x)\Big)}{F_n(x)F_{n+1}(x)\Big(F_n(x) - F_{n-1}(x) + 1\Big)\Big(F_{n+2}(x) - F_{n+1}(x) + 1\Big)} + \frac{1}{F_{3n}(x)}.
\end{aligned}$$

Using Lemmas 2.1 and 2.2, it can be easily verified that

$$F_{3n}(x)\Big(F_{n+2}(x) + F_n(x)F_{n-1}(x)\Big) + F_n(x)F_{n+1}(x)\Big(F_n(x) - F_{n-1}(x) + 1\Big)$$

$$\Big(F_{n+2}(x) - F_{n+1}(x) + 1\Big) < F_{3n}(x)\Big(F_{n+2}(x)F_{n+1}(x) + F_{n+1}(x) + F_n(x) + F_{n-1}(x)\Big),$$

and the result follows. □

Lemma 2.6. *If $n \geq 2$ even, then $f_2(n) + f_2(n+1) > 0$.*

Proof. Since $n \geq 2$ is even, then by 2.2 and Lemma 2.2,

$$\begin{aligned} f_2(n) &= \frac{1}{F_n(x) - F_{n-1}(x)} - \frac{1}{F_n(x)} - \frac{1}{F_{n+1}(x) - F_n(x)} \\ &= \frac{F_{n-1}(x)}{F_n(x)\Big(F_n(x) - F_{n-1}(x)\Big)} - \frac{1}{F_{n+1}(x) - F_n(x)} \\ &= \frac{(-1)^n}{F_n(x)\Big(F_n(x) - F_{n-1}(x)\Big)\Big(F_{n+1}(x) - F_n(x)\Big)} \\ &= \frac{1}{F_n(x)\Big(F_n(x) - F_{n-1}(x)\Big)\Big(F_{n+1}(x) - F_n(x)\Big)} > 0. \end{aligned}$$

Continuing as in Lemma 2.4, it can be checked that $f_2(n) + f_2(n+1)$ is positive for all even values of n. □

Lemma 2.7. *If $n \geq 1$ odd, then $f_2(n) + f_2(n+1) < 0$.*

Proof. When n is odd, $f_2(n) = \frac{-1}{F_n(x)\Big(F_n(x)-F_{n-1}(x)\Big)\Big(F_{n+1}(x)-F_n(x)\Big)} < 0$.

Now,

$$\begin{aligned} f_2(n) + f_2(n+1) &= \Big(\frac{1}{F_n(x) - F_{n-1}(x)} - \frac{1}{F_n(x)}\Big) - \Big(\frac{1}{F_{n+1}(x)} + \frac{1}{F_{n+2}(x) - F_{n+1}(x)}\Big) \\ &= \frac{F_{n-1}(x)}{F_n(x)\Big(F_n(x) - F_{n-1}(x)\Big)} - \frac{F_{n+2}(x)}{F_{n+1}(x)\Big(F_{n+2}(x) - F_{n+1}(x)\Big)} \\ &= \frac{1}{F_n(x)\Big(\frac{F_n(x)}{F_{n-1}(x)} - 1\Big)} - \frac{1}{F_{n+1}(x)\Big(1 - \frac{F_{n+1}(x)}{F_{n+2}(x)}\Big)} \\ &= \frac{1}{\frac{F_{n+1}(x)F_{n-1}(x)-(-1)^n}{F_{n-1}(x)} - F_n(x)} - \frac{1}{F_{n+1}(x) - \frac{F_{n+2}(x)F_n(x)-(-1)^n}{F_{n+2}(x)}} \\ &= \frac{1}{\frac{F_{n+1}(x)F_{n-1}(x)-1}{F_{n-1}(x)} - F_n(x)} - \frac{1}{F_{n+1}(x) - \frac{F_{n+2}(x)F_n(x)-1}{F_{n+2}(x)}} \\ &= \frac{1}{F_{n+1}(x) - F_n(x) + \frac{1}{F_{n-1}(x)}} - \frac{1}{F_{n+1}(x) - F_n(x) + \frac{1}{F_{n+2}(x)}}. \end{aligned}$$

Since $\frac{1}{F_{n-1}(x)} > \frac{1}{F_{n+2}(x)}$, which follows $f_2(n) + f_2(n+1) < 0$. □

Lemma 2.8. *For any positive integers n, m with $n \geq 1$ odd, $m \geq 3$ and mn odd, $f_2(n) + f_2(n+1) + f_2(mn) + \frac{1}{F_{mn+1}(x)-F_{mn}(x)} < 0$.*

Proof. The proof of this result is similar to Lemma 2.5. □

Lemma 2.9. *For any odd positive integer n,* $f_3(n) + f_3(n+1) > 0$.

Proof. The proof of this result is analogous to Lemma 2.4. □

Lemma 2.10. *For any even positive integer n,* $g_1(n) + g_1(n+1) > 0$.

Proof. From (2.4),

$$g_1(n) = \frac{1}{xF_n(x)F_{n-1}(x) - 1} - \frac{1}{F_n^2(x)} - \frac{1}{xF_{n+1}(x)F_n(x) - 1}.$$

Now, for even n,

$$\begin{aligned} g_1(n) + g_1(n+1) &= \frac{1}{xF_n(x)F_{n-1}(x) - 1} - \frac{1}{F_n^2(x)} - \frac{1}{F_{n+1}^2(x)} - \frac{1}{xF_{n+2}(x)F_{n+1}(x) - 1} \\ &= \frac{F_n(x)F_{n-2}(x) + 1}{F_n^2(x)(xF_n(x)F_{n-1}(x) - 1)} - \frac{F_{n+1}(x)F_{n+3}(x) - 1}{F_{n+1}^2(x)(xF_{n+2}(x)F_{n+1}(x) - 1)}. \end{aligned}$$

Using Lemmas 2.1 and 2.2, it can be seen that

$$\begin{aligned} &F_{n+1}^2(x)(F_n(x)F_{n-2}(x) + 1)(xF_{n+2}(x)F_{n+1}(x) - 1) \\ &\qquad > F_n^2(x)(F_{n+1}(x)F_{n+3}(x) - 1)(xF_n(x)F_{n-1}(x) - 1), \end{aligned}$$

and hence the result follows. □

The following lemmas can be analogously proved and hence their proofs are omitted.

Lemma 2.11. *For any even positive integer n,* $g_2(n) + g_2(n+1) < 0$.

Lemma 2.12. *For any even integer n and any integer* $m \geq 2$,

$$g_2(n) + g_2(n+1) + g_2(mn) + \frac{1}{xF_{mn+1}(x)F_{mn}(x)} < 0.$$

Lemma 2.13. *For any odd positive integer n,* $g_2(n) + g_2(n+1) > 0$.

Lemma 2.14. *For any odd integer n and any integer* $m \geq 2$,

$$g_2(n) + g_2(n+1) + g_2(mn) + \frac{1}{xF_{mn+1}(x)F_{mn}(x)} > 0.$$

Lemma 2.15. *For any odd positive integer n,* $g_3(n) + g_3(n+1) < 0$.

Lemma 2.16. *For any even integer n and any integer* $m \geq 2$,

$$g_3(n) + g_3(n+1) + \frac{1}{xF_{mn+1}(x)F_{mn}(x) + 1} < 0.$$

Lemma 2.17. *For any even integer n and any integer* $m \geq 2$,

$$g_3(n) + g_3(n+1) + g_3(mn) + \frac{1}{xF_{mn+1}(x)F_{mn}(x) + 1} < 0.$$

Now, we are in a position to derive our main results.

Proof of Theorem 1.1. We first consider the case where n is even. Taking summation over k from n to mn in (2.1) and using the Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k(x)} &= \sum_{k=n}^{mn}\Big(\frac{1}{F_k(x)-F_{k-1}(x)+1}-\frac{1}{F_{k+1}(x)-F_k(x)+1}\Big)-\sum_{k=n}^{mn}f_1(k)\\ &= \frac{1}{F_n(x)-F_{n-1}(x)+1}-\frac{1}{F_{mn+1}(x)-F_{mn}(x)+1}-\sum_{k=n}^{mn}f_1(k)\\ &= \frac{1}{F_n(x)-F_{n-1}(x)+1}-\Big(f_1(n)+f_1(n+1)+f_1(mn)\\ &\quad +\frac{1}{F_{mn+1}(x)-F_{mn}(x)+1}\Big)-\sum_{k=n+2}^{mn-1}f_1(k)\\ &> \frac{1}{F_n(x)-F_{n-1}(x)+1}.\end{aligned}$$

On the other hand, using (2.2) and Lemma 2.6, we get

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k(x)} &= \sum_{k=n}^{mn}\Big(\frac{1}{F_k(x)-F_{k-1}(x)}-\frac{1}{F_{k+1}(x)-F_k(x)}\Big)-\sum_{k=n}^{mn}f_2(k)\\ &= \frac{1}{F_n(x)-F_{n-1}(x)}-\frac{1}{F_{mn+1}(x)-F_{mn}(x)}-\sum_{k=n}^{mn}f_2(k)\\ &< \frac{1}{F_n(x)-F_{n-1}(x)}.\end{aligned}$$

In order prove the theorem for odd n, we proceed as follows. With the help of (2.2), we have

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k(x)} &= \sum_{k=n}^{mn}\Big(\frac{1}{F_k(x)-F_{k-1}(x)}-\frac{1}{F_{k+1}(x)-F_k(x)}\Big)-\sum_{k=n}^{mn}f_2(k)\\ &= \frac{1}{F_n(x)-F_{n-1}(x)}-\frac{1}{F_{mn+1}(x)-F_{mn}(x)}-\sum_{k=n}^{mn}f_2(k).\end{aligned}$$

For odd mn, by virtue of Lemmas 2.7 and 2.8, we get

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k(x)} &= \frac{1}{F_n(x)-F_{n-1}(x)}-\frac{1}{F_{mn+1}(x)-F_{mn}(x)}-\sum_{k=n}^{mn}f_2(k)\\ &= \frac{1}{F_n(x)-F_{n-1}(x)}-\Big(f_2(n)+f_2(n+1)+f_2(mn)+\frac{1}{F_{mn+1}(x)-F_{mn}(x)}\Big)\\ &\quad -\sum_{k=n+2}^{mn-1}f_2(k)>\frac{1}{F_n(x)-F_{n-1}(x)}.\end{aligned}$$

For even mn, using Lemma 2.7, we obtain

$$\sum_{k=n}^{mn}\frac{1}{F_k(x)} = \frac{1}{F_n(x)-F_{n-1}(x)}-\frac{1}{F_{mn+1}(x)-F_{mn}(x)}-\sum_{k=n}^{mn}f_2(k)$$

$$= \frac{1}{F_n(x) - F_{n-1}(x)} - \Big(f_2(n) + f_2(n+1) + \frac{1}{F_{mn+1}(x) - F_{mn}(x)}\Big) - \sum_{k=n+2}^{mn} f_2(k) > \frac{1}{F_n(x) - F_{n-1}(x)}.$$

On the other hand, using (2.3), we get

$$\sum_{k=n}^{mn} \frac{1}{F_k(x)} = \frac{1}{F_n(x) - F_{n-1}(x) - 1} - \frac{1}{F_{mn+1}(x) - F_{mn}(x) - 1} - \sum_{k=n}^{mn} f_3(k).$$

For even mn, from Lemma 2.9, we conclude $\sum_{k=n}^{mn} f_3(k) > 0$ and therefore

$$\sum_{k=n}^{mn} \frac{1}{F_k(x)} < \frac{1}{F_n(x) - F_{n-1}(x) - 1}.$$

For odd mn, using Lemma 2.9, we obtain

$$\begin{aligned}\sum_{k=n}^{mn} \frac{1}{F_k(x)} &= \frac{1}{F_n(x) - F_{n-1}(x) - 1} - \Big[f_3(mn) + \frac{1}{F_{mn+1}(x) - F_{mn}(x) - 1}\Big] - \sum_{k=n}^{mn-1} f_3(k) \\ &= \frac{1}{F_n(x) - F_{n-1}(x) - 1} - \frac{F_{mn-1}(x) + 1}{F_{mn}(x)\Big(F_{mn+1}(x) - F_{mn}(x) - 1\Big)} - \sum_{k=n}^{mn-1} f_3(k) \\ &< \frac{1}{F_n(x) - F_{n-1}(x) - 1}.\end{aligned}$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since $x < 0$ is an integer, let $x = -y, y \in \mathbb{Z}_{>0}$. Thus, the theorem is equivalent to

$$\frac{1}{F_n(-y) - F_{n-1}(-y) + 1} < \sum_{k=n}^{mn} \frac{1}{F_k(-y)} < \frac{1}{F_n(-y) - F_{n-1}(-y)}.$$

Let us consider

$$f_1^*(n) = \frac{1}{F_n(-y) - F_{n-1}(-y) + 1} - \frac{1}{F_n(-y)} - \frac{1}{F_{n+1}(-y) - F_n(-y) + 1}, \tag{2.7}$$

and

$$f_2^*(n) = \frac{1}{F_n(-y) - F_{n-1}(-y)} - \frac{1}{F_n(-y)} - \frac{1}{F_{n+1}(-y) - F_n(-y)}. \tag{2.8}$$

Using Lemmas 2.2 and 2.3,

$$\begin{aligned} f_1^*(n) &= \frac{1}{F_n(-y) - F_{n-1}(-y) + 1} - \frac{1}{F_n(-y)} - \frac{1}{F_{n+1}(-y) - F_n(-y) + 1} \\ &= \frac{F_{n-1}(-y) - 1}{F_n(-y)\Big(F_n(-y) - F_{n-1}(-y) + 1\Big)} - \frac{1}{F_{n+1}(-y) - F_n(-y)}\end{aligned}$$

$$= \frac{(-1)^n - 1 - (F_{n+1}(-y) - F_{n-1}(-y))}{F_n(-y)\Big(F_n(-y) - F_{n-1}(-y) + 1\Big)\Big(F_{n+1}(-y) - F_n(-y) + 1\Big)}.$$

When n is even, using Lemma 2.3,

$$f_1^*(n) = \frac{-(F_{n+1}(y) - F_{n-1}(y))}{F_n(y)\Big(F_n(y) + F_{n-1}(y) - 1\Big)\Big(F_{n+1}(y) + F_n(y) + 1\Big)} < 0, \tag{2.9}$$

and when $n > 1$ is odd, using Lemma 2.3,

$$f_1^*(n) = \frac{-(F_{n+1}(y) - F_{n-1}(y)) + 2}{F_n(y)\Big(F_n(y) + F_{n-1}(y) + 1\Big)\Big(F_{n+1}(y) + F_n(y) - 1\Big)} < 0. \tag{2.10}$$

Similarly, using (2.8), Lemmas 2.2 and 2.3, we have

$$f_2^*(n) = \frac{(-1)^n}{F_n(-y)\Big(F_n(-y) - F_{n-1}(-y)\Big)\Big(F_{n+1}(-y) - F_n(-y)\Big)}.$$

For any positive integer n, using Lemma 2.3,

$$f_2^*(n) = \frac{1}{F_n(y)\Big(F_n(y) + F_{n-1}(y)\Big)\Big(F_{n+1}(y) + F_n(y)\Big)} > 0. \tag{2.11}$$

Taking summation over k from n to mn in (2.7), we obtain

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{F_k(-y)} &= \sum_{k=n}^{mn} \Big(\frac{1}{F_k(-y) - F_{k-1}(-y) + 1} - \frac{1}{F_{k+1}(-y) - F_k(-y) + 1}\Big) - \sum_{k=n}^{mn} f_1^*(k) \\
&= \frac{1}{F_n(-y) - F_{n-1}(-y) + 1} - \frac{1}{F_{mn+1}(-y) - F_{mn}(-y) + 1} - \sum_{k=n}^{mn} f_1^*(k) \\
&= \frac{1}{F_n(-y) - F_{n-1}(-y) + 1} - \Big(f_1^*(n) + f_1^*(n+1) + f_1^*(mn) \\
&\quad + \frac{1}{F_{mn+1}(-y) - F_{mn}(-y) + 1}\Big) - \sum_{k=n+2}^{mn-1} f_1^*(k) \\
&> \frac{1}{F_n(-y) - F_{n-1}(-y) + 1},
\end{aligned}$$

since $f_1^*(n) + f_1^*(n+1) + f_1^*(mn) + \frac{1}{F_{mn+1}(-y) - F_{mn}(-y) + 1} < 0$ for $m \geq 3$, which can be easily checked by using (2.9), (2.10) and Lemma 2.3 as similar process to Lemma 2.5. On the other hand, using (2.8), we get

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{F_k(-y)} &= \sum_{k=n}^{mn} \Big(\frac{1}{F_k(-y) - F_{k-1}(-y)} - \frac{1}{F_{k+1}(-y) - F_k(-y)}\Big) - \sum_{k=n}^{mn} f_2^*(k) \\
&= \frac{1}{F_n(-y) - F_{n-1}(-y)} - \frac{1}{F_{mn+1}(-y) - F_{mn}(-y)} - \sum_{k=n}^{mn} f_2^*(k).
\end{aligned}$$

For even mn, by virtue of (2.11) and Lemma 2.3, we get

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k(-y)} &= \frac{1}{F_n(-y)-F_{n-1}(-y)}-\frac{1}{F_{mn+1}(y)+F_{mn}(y)}-\sum_{k=n}^{mn}f_2^*(k)\\ &< \frac{1}{F_n(-y)-F_{n-1}(-y)}.\end{aligned}$$

Using (2.11) and Lemma 2.3, it can be easily verified that $f_2^*(n)+f_2^*(n+1)+f_2^*(mn)+\frac{1}{F_{mn+1}(-y)-F_{mn}(-y)}>0$ for even mn, $m\geq 3$. Therefore, using (2.11), we obtain

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k(-y)} &= \frac{1}{F_n(-y)-F_{n-1}(-y)}-\frac{1}{F_{mn+1}(-y)-F_{mn}(-y)}-\sum_{k=n}^{mn}f_2^*(k)\\ &= \frac{1}{F_n(-y)-F_{n-1}(-y)}-\Big(f_2*(n)+f_2*(n+1)+f_2^*(mn)\\ &\quad+\frac{1}{F_{mn+1}(-y)-F_{mn}(-y)}\Big)-\sum_{k=n+2}^{mn}f_2*(k)<\frac{1}{F_n(-y)-F_{n-1}(-y)}.\end{aligned}$$

This finishes the proof of Theorem 1.2.

Proof of Theorem 1.5. First we consider the case when n is even. For this case, the theorem is equivalent to

$$\frac{1}{xF_n(x)F_{n-1}(x)}<\sum_{k=n}^{mn}\frac{1}{F_k^2(x)}<\frac{1}{xF_n(x)F_{n-1}(x)-1}.$$

Taking summation over k from n to mn in (2.4), we obtain

$$\begin{aligned}\sum_{k=n}^{mn}\frac{1}{F_k^2(x)} &= \sum_{k=n}^{mn}\Big(\frac{1}{xF_k(x)F_{k-1}(x)-1}-\frac{1}{xF_{k+1}(x)F_k(x)-1}\Big)-\sum_{k=n}^{mn}g_1(k)\\ &= \frac{1}{xF_n(x)F_{n-1}(x)-1}-\Big[g_1(mn)+\frac{1}{xF_{mn+1}(x)F_{mn}(x)-1}\Big]-\sum_{k=n}^{mn-1}g_1(k)\\ &= \frac{1}{xF_n(x)F_{n-1}(x)-1}-\frac{F_{mn}(x)F_{mn-2}(x)+1}{F_{mn}^2(x)\big(xF_{mn}(x)F_{mn-1}(x)-1\big)}-\sum_{k=n}^{mn-1}g_1(k).\end{aligned}$$

From Lemma 2.10, it follows that $\sum_{k=n}^{mn-1}g_1(k)>0$ and therefore,

$$\sum_{k=n}^{mn}\frac{1}{F_k^2(x)}<\frac{1}{xF_n(x)F_{n-1}(x)-1}.$$

Using (2.5) and Lemmas 2.11 and 2.12, we get

$$\sum_{k=n}^{mn}\frac{1}{F_k^2(x)} = \sum_{k=n}^{mn}\Big(\frac{1}{xF_k(x)F_{k-1}(x)}-\frac{1}{xF_{k+1}(x)F_k(x)}\Big)-\sum_{k=n}^{mn}g_2(k)$$

$$
\begin{aligned}
&= \frac{1}{xF_n(x)F_{n-1}(x)} - \frac{1}{xF_{mn+1}(x)F_{mn}(x)} - \sum_{k=n}^{mn} g_2(k) \\
&= \frac{1}{xF_n(x)F_{n-1}(x)} - \left[g_2(n) + g_2(n+1) + g_2(mn) + \frac{1}{xF_{mn+1}(x)F_{mn}(x)}\right] \\
&\quad - \sum_{k=n+2}^{mn-1} g_2(k) > \frac{1}{xF_n(x)F_{n-1}(x)}.
\end{aligned}
$$

This completes the proof of the theorem for even n. Now, it remains to prove the case when n is odd. In order to prove this case, it suffices to show that

$$\frac{1}{xF_n(x)F_{n-1}(x)+1} < \sum_{k=n}^{mn} \frac{1}{F_k^2(x)} < \frac{1}{xF_n(x)F_{n-1}(x)}.$$

Using (2.5), we obtain

$$\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} = \frac{1}{xF_n(x)F_{n-1}(x)} - \frac{1}{xF_{mn+1}(x)F_{mn}(x)} - \sum_{k=n}^{mn} g_2(k).$$

For even mn, from Lemma 2.13, it is clear that $\sum_{k=n}^{mn} g_2(k)$ is positive and therefore,

$$\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} < \frac{1}{xF_n(x)F_{n-1}(x)}.$$

For odd mn, using Lemmas 2.13 and 2.14, we get

$$
\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} &= \frac{1}{xF_n(x)F_{n-1}(x)} - \left[g_2(n) + g_2(n+1) + g_2(mn) + \frac{1}{xF_{mn+1}(x)F_{mn}(x)}\right] \\
&\quad - \sum_{k=n+2}^{mn-1} g_2(k) < \frac{1}{xF_n(x)F_{n-1}(x)}.
\end{aligned}
$$

On the other hand, taking summation over k from n to mn in (2.6), we obtain

$$\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} = \frac{1}{xF_n(x)F_{n-1}(x)+1} - \frac{1}{xF_{mn+1}(x)F_{mn}(x)+1} - \sum_{k=n}^{mn} g_3(k).$$

For even mn, we can write

$$
\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} &= \frac{1}{xF_n(x)F_{n-1}(x)+1} - \Big[g_3(n) + g_3(n+1) \\
&\quad + \frac{1}{xF_{mn+1}(x)F_{mn}(x)+1}\Big] - \sum_{k=n+2}^{mn} g_3(k).
\end{aligned}
$$

Using Lemmas 2.15 and 2.16, from the above equation, we conclude

$$\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} > \frac{1}{xF_n(x)F_{n-1}(x)+1}.$$

For odd mn, clearly $\sum_{k=n+2}^{mn-1} g_3(k) < 0$ from Lemma 2.15. Now, we can write

$$\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} = \frac{1}{xF_n(x)F_{n-1}(x)+1} - \Big[g_3(n) + g_3(n+1) + g_3(mn) + \frac{1}{xF_{mn+1}(x)F_{mn}(x)+1}\Big] - \sum_{k=n+2}^{mn-1} g_3(k).$$

From the above equation and Lemma 2.17, it follows that

$$\sum_{k=n}^{mn} \frac{1}{F_k^2(x)} > \frac{1}{xF_n(x)F_{n-1}(x)+1}.$$

Using Lemma 2.3, it is cleared that $F_n^2(-x) = F_n^2(x)$. Therefore the theorem also holds for any integer $x < 0$. This ends the proof of Theorem 1.5.

Conflict of interest

The author declares there is no conflicts of interest in this paper.

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