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Research article

Classical solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system

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Abstract: We study solutions of the Dirichlet problem for the Brinkman system and for the Darcy-Forchheimer-Brinkman system in the spaces of functions $C^{k,\alpha}(\overline{\Omega};\mathbb{R}^m)\times C^{k-1,\alpha}(\overline{\Omega})$, where $\Omega\subset\mathbb{R}^m$ is a bounded domain.

Keywords: Brinkman system; Darcy-Forchheimer-Brinkman system; Dirichlet problem; classical solution; regularity
Mathematics Subject Classification: 35Q35

1. Introduction

The paper is devoted to classical solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system

$$\nabla p-\Delta\mathbf{v}+\lambda\mathbf{v}+a|\mathbf{v}|\mathbf{v}+b(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f},\quad \nabla\cdot\mathbf{v}=0\quad\text{in }\Omega \tag{1.1}$$

where $\Omega\subset R^m$ is a bounded domain.

Boundary value problems for the Darcy-Forchheimer-Brinkman system have been extensively studied in the recent years. This system describes flows through porous media saturated with viscous incompressible fluids, where the inertia of such fluids is not negligible. The constants $\lambda, b>0$ are determined by the physical properties of the porous medium. (For further details we refer the reader to the book [1, p.17] and the references therein.)

M. Kohr et al. studied in [2] the transmission problem, where the Darcy-Forchheimer-Brinkman system is considered in a bounded domain $\Omega_+\subset\mathbb{R}^3$ with connected Lipschitz boundary and the Stokes system is given on its complementary domain Ω_-. Solutions belong to the space $\mathcal{H}^1(\Omega_\pm)\times L^2(\Omega_\pm)$, where $\mathcal{H}^1(\Omega)=\{\mathbf{u}\in L^2_{\rm loc}(\Omega,\mathbb{R}^3);\partial_j u_i\in L^2(\Omega),(1+|\mathbf{x}|^2)^{-1/2}u_j(\mathbf{x})\in L^2(\Omega)\}$. The paper [3] is concerned with another transmission problem. A bounded domain $\Omega\subset\mathbb{R}^m$ with connected Lipschitz boundary splits into two Lipschitz domains Ω_+ and Ω_-. A solution is found satisfying the homogeneous Darcy-Forchheimer-Brinkman system is in Ω_- and the homogeneous Navier-Stokes system in Ω_+. The transmission condition on the interface $\partial\Omega_- \cap \partial\Omega_+$ is accompanied by the Robin condition on $\partial\Omega$. The paper [4] investigates the Robin problem for the Darcy-Forchheimer-Brinkman system (1.1) with $b = 0$ in the space $H^s(\Omega, \mathbb{R}^m) \times H^{s-1}(\Omega)$, where $1 < s < 3/2$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain with connected Lipschitz boundary, $m \in \{2, 3\}$. The mixed Dirichlet-Robin problem and the mixed Dirichlet-Neumann problem for the Darcy-Forchheimer-Brinkman system (1.1) with $b = 0$ are studied in $H^{3/2}(\Omega, \mathbb{R}^3) \times H^{1/2}(\Omega)$ (see [4] and [5]). Here $\Omega \subset \mathbb{R}^3$ is a bounded creased domain with connected Lipschitz boundary. M. Kohr et al. discussed in [4] the problem of Navier's type for the Darcy-Forchheimer-Brinkman system (1.1) with $b = 0$ in $H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with connected Lipschitz boundary.

Now we briefly sketch results concerning the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (1.1). It is supposed that $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary. For $\mathbf{f} \equiv 0$ and $2 \le m \le 3$ solutions of the problem are looked for in $W^{s,2}(\Omega, \mathbb{R}^m) \times W^{s-1,2}(\Omega)$ with $1 \le s < 3/2$ (see [6], [7] and [8]). The paper [9] is devoted to similar problems on compact Riemannian manifolds. [10] considers bounded solutions of the problem for $b = 0$ and a domain Ω with Ljapunov boundary.

This paper begins with the study of classical solutions of the Dirichlet problem for the generalized Brinkman system

$$\nabla p - \Delta \mathbf{v} + \lambda \mathbf{v} = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega.$$

If $\Omega \subset \mathbb{R}^m$ is a bounded open set with boundary of class $C^{k,\alpha}$ we prove the existence of a solution $(\mathbf{v}, p) \in C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times C^{k-1,\alpha}(\overline{\Omega})$. Unlike the previous papers we do not suppose that $\partial\Omega$ is connected. Using the fixed point theorems give the existence of solutions of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (1.1) in $(\mathbf{v}, p) \in C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times C^{k-1,\alpha}(\overline{\Omega})$. Here $\lambda, a, b \in C^{\max(k-2,0),\alpha}(\overline{\Omega})$. If $k \le 3$ then a can be arbitrary. If $k > 3$ then there exists $\mathbf{v} \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ with $\nabla \cdot \mathbf{v} = 0$ such that $|\mathbf{v}|\mathbf{v} \notin C^{k-2,\alpha}(\overline{\Omega}; \mathbb{R}^m)$. (See Remark 3.3.) So, for $k > 3$ we must suppose that $a \equiv 0$.

2. Dirichlet problem for the Brinkman system

Before we investigate the Dirichlet problem for the Brinkman system we need the following auxiliary lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set, $0 < \alpha < 1$ and $\lambda \in C^{0,\alpha}(\overline{\Omega})$ be non-negative. If $\mathbf{f} \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ then there exists a solution $(\mathbf{v}, p) \in C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times C^{1,\alpha}(\overline{\Omega})$ of*

$$-\Delta \mathbf{v} + \lambda \mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \quad \textit{in } \Omega. \tag{2.1}$$

Proof. Choose a bounded domain ω with smooth boundary such that $\overline{\Omega} \subset \omega$. Then we can suppose that $\mathbf{f} \in C^{0,\alpha}(\overline{\omega}, \mathbb{R}^m)$ and $\lambda \in C^{0,\alpha}(\overline{\omega})$. (See [11, Theorem 1.8.3] or [12, Chapter VI, §2].) Choose q such that $m/(1-\alpha) < q < \infty$. According to Lemma 3.6 in the Appendix there exists a solution $(\mathbf{v}, p) \in W^{1,q}(\omega, \mathbb{R}^m) \times L^q(\omega)$ of

$$-\Delta \mathbf{v} + \lambda \mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \omega, \qquad \mathbf{v} = 0 \quad \text{on } \partial\omega.$$

Put $\mathbf{F} = \mathbf{f} - \lambda\mathbf{v}$. Since $\mathbf{v} \in W^{1,q}(\omega) \hookrightarrow C^{0,\alpha}(\overline{\omega})$ by [11, Theorem 5.7.8], we infer that $\mathbf{v}, \mathbf{F} \in C^{0,\alpha}(\overline{\omega})$ by Lemma 3.7 in the Appendix. Then

$$-\Delta\mathbf{v} + \nabla p = \mathbf{F}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \omega.$$

Choose bounded open sets ω_1 and ω_2 such that $\overline{\Omega} \subset \omega_1 \subset \overline{\omega}_1 \subset \omega_2 \subset \overline{\omega}_2 \subset \omega$. Fix $\varphi \in C^\infty(\mathbb{R}^m)$ such that $\varphi = 1$ on ω_1 and $\varphi = 0$ on $\mathbb{R}^m \setminus \omega_2$. Define $\tilde{\mathbf{F}} = \varphi\mathbf{F}$ in ω and $\tilde{\mathbf{F}} = 0$ in $\mathbb{R}^m \setminus \omega$.

For $x \in \mathbb{R}^m \setminus \{0\}$ and $i, j \in \{1, 2, \ldots, m\}$ define

$$E_{ij}(x) := \frac{1}{2\sigma_m}\left\{\frac{\delta_{ij}}{(m-2)|x|^{m-2}} + \frac{x_i x_j}{|x|^m}\right\}, \quad m \geq 3$$

$$E_{ij}(x) := \frac{1}{2\sigma_2}\left\{\delta_{ij}\ln\frac{1}{|x|} + \frac{x_i x_j}{|x|^2}\right\}, \quad m = 2,$$

$$Q_j(x) := \frac{1}{\sigma_m}\frac{x_j}{|x|^m}$$

where σ_m is the area of the unit sphere in R^m. Then $E = \{E_{ij}\}$, $Q = (Q_1, \ldots, Q_m)$ form a fundamental tensor of the Stokes system, i.e.,

$$-\Delta E_{ij} + \lambda E_{ij} + \partial_i Q_j = \delta_0\delta_{ij}, \quad i \leq m,$$

$$\partial_1 E_{1j} + \cdots + \partial_m E_{mj} = 0,$$

where δ_0 is the Dirac measure. (See for example [13].) Define $\tilde{\mathbf{v}} := E * \tilde{\mathbf{F}}$ and $\tilde{p} := Q * \tilde{\mathbf{F}}$. Then

$$-\Delta\tilde{\mathbf{v}} + \nabla\tilde{p} = \tilde{\mathbf{F}}, \quad \nabla \cdot \tilde{\mathbf{v}} = 0 \quad \text{in } \mathbb{R}^m.$$

Define

$$h_\Delta(x) := \begin{cases} \sigma_2^{-1}\ln|x|, & m = 2, \\ (2-m)^{-1}\sigma_m^{-1}|x|^{2-m}, & m > 2 \end{cases}$$

the fundamental solution for the Laplace equation. Since $F_j \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^m)$, [14, Theorem 3.14.2] gives that $h_\Delta * \tilde{F}_j \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^m)$ for $j = 1, \ldots, m$. Since $Q_j = \partial_j h_\Delta$, we infer

$$\tilde{p} = \partial_1(h_\Delta * \tilde{F}_1) + \cdots + \partial_m(h_\Delta * \tilde{F}_m) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^m).$$

Since

$$-\Delta(\mathbf{v} - \tilde{\mathbf{v}}) + \nabla(p - \tilde{p}) = \mathbf{F} - \tilde{\mathbf{F}} = 0, \quad \nabla \cdot (\mathbf{v} - \tilde{\mathbf{v}}) = 0 \quad \text{in } \omega_1,$$

we infer that

$$\Delta(p - \tilde{p}) = \nabla \cdot \nabla(p - \tilde{p}) = \nabla \cdot \Delta(\mathbf{v} - \tilde{\mathbf{v}}) = \Delta[\nabla \cdot (\mathbf{v} - \tilde{\mathbf{v}})] = 0 \quad \text{in } \omega_1$$

in the sense of distributions. Thus $p - \tilde{p} \in C^2(\omega_1)$ by [14, Theorem 2.18.2]. Since $\tilde{p} \in C^{1,\alpha}(\omega)$ we infer that $p \in C^{1,\alpha}_{loc}(\omega_1)$. Thus $\Delta\mathbf{v} = \nabla p - \mathbf{F} \in C^{0,\alpha}_{\mathrm{loc}}(\omega_1; \mathbb{R}^m)$. According to [14, Proposition 3.18.1] we obtain that $\mathbf{v} \in C^{2,\alpha}_{\mathrm{loc}}(\omega_1; \mathbb{R}^m)$. (We can prove that $\mathbf{v} \in C^{2,\alpha}_{\mathrm{loc}}(\omega_1; \mathbb{R}^m)$ also using results in [15].) □

Theorem 2.2. *Let $0 < \beta < \alpha < 1$. Suppose that $\Omega \subset \mathbb{R}^m$ is a bounded domain with boundary of class $C^{1,\alpha}$ and $\lambda \in C^{0,\beta}(\overline{\Omega})$ is non-negative. Let $\mathbf{f} \in C^{0,\beta}(\overline{\Omega}, \mathbb{R}^m)$, $\mathbf{g} \in C^{1,\beta}(\partial\Omega, \mathbb{R}^m)$. Then there exist $\mathbf{v} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)$ and $p \in C^{0,\beta}(\overline{\Omega}) \cap C^1(\Omega)$ solving*

$$-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{v} = 0 \quad in\ \Omega, \qquad \mathbf{v} = \mathbf{g} \quad on\ \partial\Omega \tag{2.2}$$

if and only if

$$\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{g}\, \mathrm{d}\sigma = 0. \tag{2.3}$$

A velocity $\mathbf{v}$ is unique and a pressure p is unique up to an additive constant. Moreover,

$$\|\mathbf{v}\|_{C^{1,\beta}(\overline{\Omega})} + \|p\|_{C^{0,\beta}(\overline{\Omega})} \le C\left(\|\mathbf{f}\|_{C^{0,\beta}(\Omega)} + \|\mathbf{g}\|_{C^{1,\beta}(\partial\Omega)} + \left|\int_\Omega p\, \mathrm{d}x\right|\right).$$

Proof. If $\mathbf{v} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)$, $p \in C^{0,\beta}(\overline{\Omega}) \cap C^1(\Omega)$ solve (2.2) then (2.3) holds by the Green formula.

Suppose now that (2.3) holds. According to Lemma 2.1 there exists $(\tilde{\mathbf{v}}, \tilde{p}) \in C^{2,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{1,\beta}(\overline{\Omega})$ such that

$$-\Delta\tilde{\mathbf{v}} + \lambda\tilde{\mathbf{v}} + \nabla\tilde{p} = \mathbf{f}, \quad \nabla\cdot\tilde{\mathbf{v}} = 0 \quad \text{in } \Omega.$$

Put $\hat{\mathbf{g}} := \mathbf{g} - \tilde{\mathbf{v}}$. Then $\hat{\mathbf{g}} \in C^{1,\beta}(\partial\Omega, \mathbb{R}^m)$. Choose q such that $m/(1-\beta) < q < \infty$. According to Lemma 3.6 there exists a solution $(\hat{\mathbf{v}}, \hat{p}) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of

$$-\Delta\hat{\mathbf{v}} + \lambda\hat{\mathbf{v}} + \nabla\hat{p} = 0, \quad \nabla\cdot\hat{\mathbf{v}} = 0 \quad \text{in } \Omega, \qquad \hat{\mathbf{v}} = \hat{\mathbf{g}} \quad \text{on } \partial\Omega.$$

A velocity $\hat{\mathbf{v}}$ is unique and a pressure $\hat{p}$ is unique up to an additive constant. Define $\mathbf{v} := \tilde{\mathbf{v}} + \hat{\mathbf{v}}$, $p := \tilde{p} + \hat{p}$. Then $(\mathbf{v}, p)$ is a solution of (2.2).

If $\lambda \equiv 0$ then $\hat{\mathbf{v}} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)$, $\hat{p} \in C^{0,\beta}(\overline{\Omega})$ by [16, Theorem 5.2]. Moreover, $\hat{\mathbf{v}} \in C^\infty(\Omega; \mathbb{R}^m)$, $\hat{p} \in C^\infty(\Omega)$. (See for example [17, §1.2].) Thus $\mathbf{v} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)$, $p \in C^{0,\beta}(\overline{\Omega}) \cap C^1(\Omega)$.

Let now λ be general. Since $\mathbf{v} \in W^{1,q}(\Omega) \hookrightarrow C^{0,\beta}(\overline{\Omega})$ by [11, Theorem 5.7.8], Lemma 3.7 in the Appendix gives that $\lambda\mathbf{v} \in C^{0,\beta}(\overline{\Omega})$. Therefore

$$-\Delta\mathbf{v} + \nabla p = (\mathbf{f} - \lambda\mathbf{v}) \in C^{0,\beta}(\overline{\Omega}; \mathbb{R}^m).$$

We have proved that $\mathbf{v} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)$, $p \in C^{0,\beta}(\overline{\Omega}) \cap C^1(\Omega)$.

Denote by Y the set of all $\mathbf{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ satisfying (2.3). Define $X := \{\mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^m); \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega, \mathbf{v}|_{\partial\Omega} \in Y\}$,

$$U_\lambda(\mathbf{v}, p) := \left[-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p, \mathbf{v}, \int_\Omega p\, \mathrm{d}x\right].$$

Then $U_\lambda : X \times L^q(\Omega) \to W^{-1,q}(\Omega; \mathbb{R}^m) \times Y \times \mathbb{R}^1$ is an isomorphism by Lemma 3.6. Denote $Z = [X \cap C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)] \times C^{0,\beta}(\overline{\Omega})$, $W = C^{0,\beta}(\overline{\Omega}, \mathbb{R}^m) \times [Y \cap C^{1,\beta}(\partial\Omega, \mathbb{R}^m)] \times \mathbb{R}^1$. We have proved that $U_\lambda^{-1}(W) \subset Z$. Since $U_\lambda^{-1} : W \to Z$ is closed, it is continuous by the Closed graph theorem. □

Theorem 2.3. *Let $0 < \alpha < 1$ and $k \in \mathbb{N}_0$. Suppose that $\Omega \subset \mathbb{R}^m$ is a bounded domain with boundary of class $C^{k+2,\alpha}$ and $\lambda \in C^{k,\alpha}(\overline{\Omega})$ is non-negative. Let $\mathbf{f} \in C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^m)$, $\mathbf{g} \in C^{k+2,\alpha}(\partial\Omega, \mathbb{R}^m)$. Then there exists a solution $(\mathbf{v}, p) \in C^{k+2,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times C^{k+1,\alpha}(\overline{\Omega})$ of (2.2) if and only if (2.3) holds. A velocity $\mathbf{v}$ is unique and a pressure p is unique up to an additive constant. Moreover,*

$$\|\mathbf{v}\|_{C^{k+2,\alpha}(\overline{\Omega})} + \|p\|_{C^{k+1,\alpha}(\overline{\Omega})} \le C\left(\|\mathbf{f}\|_{C^{k,\alpha}(\overline{\Omega})} + \|\mathbf{g}\|_{C^{k+2,\alpha}(\partial\Omega)} + \left|\int_\Omega p\, \mathrm{d}x\right|\right).$$

Proof. (2.3) is a necessary condition for the solvability of the problem (2.2) by Theorem 2.2.

Denote by Y the set of all $\mathbf{g} \in C^{k+2,\alpha}(\partial\Omega; \mathbb{R}^m)$ satisfying (2.3). Define $X := \{\mathbf{v} \in C^{k+2,\alpha}(\overline{\Omega}; \mathbb{R}^m); \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega, \mathbf{v}|_{\partial\Omega} \in Y\}$,

$$U_\lambda(\mathbf{v}, p) := [-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p, \mathbf{v}, \int_\Omega p \, \mathrm{d}\sigma].$$

Then $U_0 : X \times C^{k+1,\alpha}(\overline{\Omega}) \to C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times Y \times \mathbb{R}^1$ is an isomorphism by Theorem 2.2 and [18, Theorem IV.7.1]. Since $\mathbf{u} \mapsto \lambda\mathbf{u}$ is a bounded operator on $C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ by Lemma 3.7 and $C^{k+2,\alpha}(\overline{\Omega}, \mathbb{R}^m) \hookrightarrow C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ is compact by [19, Lemma 6.36], the operator $U_\lambda - U_0 : X \times C^{k+1,\alpha}(\overline{\Omega}) \to C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times Y \times \mathbb{R}^1$ is compact. Hence the operator $U_\lambda : X \times C^{k+1,\alpha}(\overline{\Omega}) \to C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times Y \times \mathbb{R}^1$ is Fredholm with index 0. The kernel of U_λ is trivial by Theorem 2.2. Therefore, $U_\lambda : X \times C^{k+1,\alpha}(\overline{\Omega}) \to C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m) \times Y \times \mathbb{R}^1$ is an isomorphism. □

3. Darcy-Forchheimer-Brinkman system

We prove the existence of a classical solution of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system using fixed point theorems. The following two lemmas are crucial for it.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^m$ be open, $0 < \alpha \le 1$ and $k \in \mathbb{N}$. Put $l = \max(k-2, 0)$. Let $b \in C^{l,\alpha}(\overline{\Omega}; \mathbb{R}^m)$. Define*

$$L_b(\mathbf{u}, \mathbf{v}) := b(\mathbf{u} \cdot \nabla)\mathbf{v}. \tag{3.1}$$

Then there exists $C_1 \in (0, \infty)$ such that if $\mathbf{u}, \mathbf{v} \in C^{k,\alpha}(\overline{\Omega}; \mathbb{R}^m)$, then $L_b(\mathbf{u}, \mathbf{v}) \in C^{l,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ and

$$\|L_b(\mathbf{u}, \mathbf{v})\|_{C^{l,\alpha}(\overline{\Omega})} \le C_1 \|\mathbf{u}\|_{C^{k,\alpha}(\overline{\Omega})} \|\mathbf{v}\|_{C^{k,\alpha}(\overline{\Omega})}, \tag{3.2}$$

$$\|L_b(\mathbf{u}, \mathbf{u}) - L_b(\mathbf{v}, \mathbf{v})\|_{C^{l,\alpha}(\overline{\Omega})} \le C_1 \|\mathbf{u} - \mathbf{v}\|_{C^{k,\alpha}(\overline{\Omega})} \left(\|\mathbf{v}\|_{C^{k,\alpha}(\overline{\Omega})} + \|\mathbf{u}\|_{C^{k,\alpha}(\overline{\Omega})} \right). \tag{3.3}$$

Proof. Lemma 3.7 in the Appendix forces that $L_b(\mathbf{u}, \mathbf{v}) \in C^{l,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ and the estimate (3.2) holds true. Since

$$L_b(\mathbf{u}, \mathbf{u}) - L_b(\mathbf{v}, \mathbf{v}) = L_b(\mathbf{u} - \mathbf{v}, \mathbf{u}) + L_b(\mathbf{v}, \mathbf{u} - \mathbf{v}),$$

the estimate (3.3) is a consequence of the estimate (3.2). □

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class $C^{1,\alpha}$ and $0 < \beta \le \alpha < 1$. Suppose that $a \in C^{0,\beta}(\overline{\Omega})$. For $\mathbf{v} \in C(\overline{\Omega}; \mathbb{R}^m)$ define*

$$A_a\mathbf{v} := a|\mathbf{v}|\mathbf{v}. \tag{3.4}$$

1. *Then there exists a constant C_1 such that for $\mathbf{u}, \mathbf{v} \in C^{1,0}(\overline{\Omega}; \mathbb{R}^m)$ it holds*

$$\|A_a\mathbf{v}\|_{C^{0,\beta}(\overline{\Omega})} \le C_1 \|\mathbf{v}\|^2_{C^{1,0}(\overline{\Omega})}, \tag{3.5}$$

$$\|A_a\mathbf{v} - A_a\mathbf{u}\|_{C^{0,\beta}(\overline{\Omega})} \le C_1 \|\mathbf{v} - \mathbf{u}\|_{C^{1,0}(\overline{\Omega})} \left[\|\mathbf{v}\|_{C^{1,0}(\overline{\Omega})} + \|\mathbf{u}\|_{C^{1,0}(\overline{\Omega})} \right]. \tag{3.6}$$

2. *If $a \in C^{1,\beta}(\overline{\Omega})$, then there exists a positive constant C_2 such that $A_a : C^{2,0}(\overline{\Omega}; \mathbb{R}^m) \to C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)$ is a compact continuous mapping and*

$$\|A_a\mathbf{v}\|_{C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)} \le C_2 \|\mathbf{v}\|^2_{C^{2,0}(\overline{\Omega}; \mathbb{R}^m)}. \tag{3.7}$$

Proof. Easy calculation yields that

$$\| \, |\mathbf{v}| \, \|_{C^{0,\beta}(\overline{\Omega})} \leq \|\mathbf{v}\|_{C^{0,\beta}(\overline{\Omega};\mathbb{R}^m)}.$$

So, according to Lemma 3.7 in the Appendix

$$\|A_a \mathbf{v}\|_{C^{0,\beta}(\overline{\Omega};\mathbb{R}^m)} \leq 4\|a\|_{C^{0,\beta}(\overline{\Omega})}\|\mathbf{v}\|^2_{C^{0,\beta}(\overline{\Omega};\mathbb{R}^m)}.$$

Since $C^{1,0}(\overline{\Omega};\mathbb{R}^m) \hookrightarrow C^{0,\beta}(\overline{\Omega};\mathbb{R}^m)$ by [19, Lemma 6.36], we obtain the estimate (3.5).

Clearly

$$|A_a \mathbf{v}(x) - A_a \mathbf{v}(x)| \leq |a(x)| \, ||\mathbf{v}(x)| - |\mathbf{u}(x)|| \, |\mathbf{v}(x)| + |a(x)||\mathbf{u}(x)||\mathbf{v}(x) - \mathbf{u}(x)|.$$

Hence there exists a constant c_1 such that

$$\|A_a \mathbf{v} - A_a \mathbf{u}\|_{C^0(\overline{\Omega})} \leq c_1 \|\mathbf{v} - \mathbf{u}\|_{C^0(\overline{\Omega})} \left(\|\mathbf{v}\|_{C^0(\overline{\Omega})} + \|\mathbf{u}\|_{C^0(\overline{\Omega})} \right). \tag{3.8}$$

We now calculate derivatives of $A_1 \mathbf{v}$. If $x \in \Omega$ and $\mathbf{v}(x) \neq 0$ then

$$\partial_j [|\mathbf{v}(x)| v_k(x)] = |\mathbf{v}(x)| \partial_j v_k(x) + \frac{v_k(x)[\mathbf{v}(x) \cdot \partial_j \mathbf{v}(x)]}{|\mathbf{v}(x)|}. \tag{3.9}$$

Hence

$$|\nabla A_1 \mathbf{v}(x)| \leq (m+1)^2 |\mathbf{v}(x)| \, \|\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}. \tag{3.10}$$

Let now $x \in \Omega$ and $\mathbf{v}(x) = 0$. Denote $\mathbf{e}_j = (\delta_{1j}, \ldots, \delta_{mj})$. Then

$$\partial_j [|\mathbf{v}(x)| v_k(x)] = \lim_{t \to 0} \frac{|\mathbf{v}(x + t e_j)| [v_k(x + t e_j) - v_k(x)]}{t} = |\mathbf{v}(x)| \partial_j v_k(x) = 0$$

and (3.10) holds too. Suppose now that $x \in \partial\Omega$. If $\mathbf{v}(x) \neq 0$ then $\partial_j [|\mathbf{v}(x)| v_k(x)]$ can be continuously extended to x by (3.9). If $\mathbf{v}(x) = 0$ then (3.10) gives that $\nabla A_1 \mathbf{v}$ can be continuously extended to x by $\nabla A_1 \mathbf{v}(x) = 0$. The estimates (3.5) and (3.10) give that there exists a constant c_2 such that

$$\|A_1 \mathbf{v}\|_{C^{1,0}(\overline{\Omega})} \leq c_2 \|\mathbf{v}\|^2_{C^{1,0}(\overline{\Omega})}. \tag{3.11}$$

Suppose that $\mathbf{u}, \mathbf{v} \in C^{1,0}(\overline{\Omega};\mathbb{R}^m)$. Let $x \in \Omega$. Suppose first that $\mathbf{u}(x) = 0$. Since $\nabla A_1 \mathbf{u}(x) = 0$, (3.10) gives

$$|\nabla A_1 \mathbf{v}(x) - \nabla A_1 \mathbf{u}(x)| = |\nabla A_1 \mathbf{v}(x)| \leq (m+1)^2 |\mathbf{v}(x) - \mathbf{u}(x)| \, \|\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}. \tag{3.12}$$

Let now $|\mathbf{v}(x)| \geq |\mathbf{u}(x)| > 0$. According to (3.9)

$$\begin{aligned} |\partial_j [|\mathbf{v}(x)| v_k(x)] - \partial_j [|\mathbf{u}(x)| u_k(x)]| &\leq |\mathbf{v}(x) - \mathbf{u}(x)| |\partial_j v_k(x)| + |\mathbf{u}(x)| |\partial_j v_k(x) - \partial_j u_k(x)| \\ &+ \frac{|\mathbf{u}(x)| |v_k(x) - u_k(x)| |\mathbf{v}(x) \cdot \partial_j \mathbf{v}(x)| + |\mathbf{u}(x)| |u_k(x)| |\mathbf{v}(x) - \mathbf{u}(x)| |\partial_j \mathbf{v}(x)|}{|\mathbf{v}(x)| \, |\mathbf{u}(x)|} \\ &+ \frac{|u_k(x)| |\mathbf{u}(x)|^2 |\partial_j \mathbf{v}(x) - \partial_j \mathbf{u}(x)| + |\mathbf{u}(x) - \mathbf{v}(x)| |u_k(x)| |\mathbf{u}(x) \cdot \partial_j \mathbf{u}(x)|}{|\mathbf{v}(x)| \, |\mathbf{u}(x)|}. \end{aligned}$$

$$\leq 6\|\mathbf{u}-\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}\left(\|\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}+\|\mathbf{u}\|_{C^{1,0}(\overline{\Omega})}\right).$$

This inequality, (3.12) and (3.8) give that there exists a constant c_3 such that

$$\|A_1\mathbf{u}-A_1\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}\leq c_3\|\mathbf{u}-\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}\left(\|\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}+\|\mathbf{u}\|_{C^{1,0}(\overline{\Omega})}\right).$$

Since $C^{1,0}(\overline{\Omega};\mathbb{R}^m)\hookrightarrow C^{0,\beta}(\overline{\Omega};\mathbb{R}^m)$ by [19, Lemma 6.36], there exists a constant c_4 such that

$$\|A_1\mathbf{u}-A_1\mathbf{v}\|_{C^{0,\beta}(\overline{\Omega})}\leq c_4\|\mathbf{u}-\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}\left(\|\mathbf{v}\|_{C^{1,0}(\overline{\Omega})}+\|\mathbf{u}\|_{C^{1,0}(\overline{\Omega})}\right).$$

Since $A_a\mathbf{v}=aA_1\mathbf{v}$, Lemma 3.7 gives that there exists a constant C_1 such that (3.6) holds.

Let now $\mathbf{v}\in C^{2,0}(\overline{\Omega};\mathbb{R}^m)$. We are going to calculate $\nabla^2 A_1\mathbf{v}$. If $x\in\Omega$ and $\mathbf{v}(x)\neq 0$, then we obtain from (3.9)

$$\begin{aligned}\partial_l\partial_j[|\mathbf{v}(x)|v_k(x)] &= |\mathbf{v}(x)|\partial_l\partial_j v_k(x)+\frac{\partial_j v_k(x)[\mathbf{v}(x)\cdot\partial_l\mathbf{v}(x)]}{|\mathbf{v}(x)|}\\ &+\frac{\partial_l v_k(x)[\mathbf{v}(x)\cdot\partial_j\mathbf{v}(x)]+v_k(x)[\partial_l\mathbf{v}(x)\cdot\partial_j\mathbf{v}(x)]+v_k(x)[\mathbf{v}(x)\cdot\partial_l\partial_j\mathbf{v}(x)]}{|\mathbf{v}(x)|}\\ &-\frac{v_k(x)[\mathbf{v}(x)\cdot\partial_j\mathbf{v}(x)][\mathbf{v}(x)\cdot\partial_l\mathbf{v}(x)]}{|\mathbf{v}(x)|^3}.\end{aligned}\tag{3.13}$$

So,

$$|\partial_l\partial_j[|\mathbf{v}(x)|v_k(x)]|\leq 6\|\mathbf{v}\|^2_{C^{2,0}(\overline{\Omega})}.\tag{3.14}$$

Now we calculate $\partial_l\partial_j[|\mathbf{v}(x)|v_k(x)]$ in the sense of distributions. For $\epsilon\geq 0$ denote $\Omega(\epsilon):=\{x\in\Omega;|\mathbf{v}(x)|>\epsilon\}$, $V(\epsilon):=\Omega\setminus\overline{\Omega(\epsilon)}$. Suppose that $\varphi\in C^\infty(\mathbb{R}^m)$ has compact support in Ω. We have proved that if $\mathbf{v}(x)=0$, then $\partial_j[|\mathbf{v}(x)|v_k(x)]=0$. Thus

$$\begin{aligned}\langle\partial_l\partial_j[|\mathbf{v}|v_k],\varphi\rangle &= -\int_\Omega[\partial_l\varphi(x)]\partial_j[|\mathbf{v}(x)|v_k(x)]\,\mathrm{d}x\\ &= -\lim_{\epsilon\downarrow 0}\int_{\Omega(\epsilon)}[\partial_l\varphi(x)]\partial_j[|\mathbf{v}(x)|v_k(x)]\,\mathrm{d}x.\end{aligned}$$

According to the Green formula

$$\langle\partial_l\partial_j[|\mathbf{v}|v_k],\varphi\rangle=\lim_{\epsilon\downarrow 0}\left[\int_{\Omega(\epsilon)}\varphi\partial_l\partial_j[|\mathbf{v}|v_k]\,\mathrm{d}x-\int_{\partial\Omega(\epsilon)}\varphi n_l\partial_j[|\mathbf{v}|v_k]\,\mathrm{d}\sigma\right].$$

If $x\in\partial\Omega(\epsilon)\cap\Omega$ then $|\mathbf{v}(x)|=\epsilon$. If $x\in\partial\Omega(\epsilon)\setminus\Omega$ then $\varphi(x)=0$. Thus we obtain by (3.14) and (3.9)

$$\langle\partial_l\partial_j[|\mathbf{v}|v_k],\varphi\rangle=\int_{\Omega(0)}\varphi\partial_l\partial_j[|\mathbf{v}|v_k]-\lim_{\epsilon\downarrow 0}\int_{\partial\Omega(\epsilon)}\varphi n_l\Big[\epsilon\partial_j v_k+\frac{v_k\mathbf{v}\cdot\partial_j\mathbf{v}}{\epsilon}\Big].\tag{3.15}$$

According to the Green formula

$$\left|\lim_{\epsilon\downarrow 0}\int_{\partial\Omega(\epsilon)}\varphi n_l\Big[\epsilon\partial_j v_k+\frac{v_k\mathbf{v}\cdot\partial_j\mathbf{v}}{\epsilon}\Big]\right|\leq\lim_{\epsilon\downarrow 0}\left|\epsilon\int_{\partial\Omega(\epsilon)}\varphi n_l\partial_j v_k\,\mathrm{d}\sigma\right|$$

$$+\lim_{\epsilon\downarrow 0}\left|\int_{\partial V(\epsilon)}\varphi n_l\frac{v_k\mathbf{v}\cdot\partial_j\mathbf{v}}{\epsilon}\,\mathrm{d}\sigma\right| = \lim_{\epsilon\downarrow 0}\epsilon\left|\int_{\Omega(\epsilon)}\partial_l[\varphi\partial_j v_k]\,\mathrm{d}x\right|$$

$$+\lim_{\epsilon\downarrow 0}\left|\int_{V(\epsilon)}\frac{1}{\epsilon}[\partial_l\varphi v_k\mathbf{v}\cdot\partial_j\mathbf{v}+\varphi\partial_l v_k\mathbf{v}\cdot\partial_j\mathbf{v}+\varphi v_k\partial_l\mathbf{v}\cdot\partial_j\mathbf{v}+\varphi v_k\mathbf{v}\cdot\partial_l\partial_j\mathbf{v}]\,\mathrm{d}x\right|$$

$$\le 0+\lim_{\epsilon\downarrow 0}\int_{V(\epsilon)\setminus V(0)}\|\varphi\|_{C^1(\overline{\Omega})}\|\mathbf{v}\|^2_{C^2(\overline{\Omega})}\,\mathrm{d}x = 0.$$

This and (3.15) give

$$\langle\partial_l\partial_j[|\mathbf{v}|v_k],\varphi\rangle = \int_{\Omega(0)}\varphi\partial_l\partial_j[|\mathbf{v}|v_k].$$

So, if we define $\partial_l\partial_j[|\mathbf{v}|v_k](x)$ by (3.13) for $\mathbf{v}(x)\neq 0$, and $\partial_l\partial_j[|\mathbf{v}|v_k](x) = 0$ for $\mathbf{v}(x) = 0$, then this function is $\partial_l\partial_j[|\mathbf{v}|v_k]$ in sense of distributions. (3.14) forces $A_1\mathbf{v}\in W^{2,\infty}(\Omega;\mathbb{R}^m)$ and

$$\|A_1\mathbf{v}\|_{W^{2,\infty}(\Omega)}\le c_5\|\mathbf{v}\|^2_{C^{2,0}(\overline{\Omega})}.$$

$W^{2,\infty}(\Omega)\hookrightarrow C^{1,\beta}(\overline{\Omega})$ compactly by [20, Theorem 6.3]. So,

$$\|A_1\mathbf{v}\|_{C^{1,\beta}(\overline{\Omega})}\le c_6\|\mathbf{v}\|^2_{C^{2,0}(\overline{\Omega})}. \tag{3.16}$$

Since A_1 maps bounded subsets of $C^{2,0}(\overline{\Omega};\mathbb{R}^m)$ to bounded subsets of $W^{2,\infty}(\Omega;\mathbb{R}^m)$ and $W^{2,\infty}(\Omega)\hookrightarrow C^{1,\beta}(\overline{\Omega})$ compactly, the mapping $A_1 : C^{2,0}(\overline{\Omega};\mathbb{R}^m)\to C^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ is compact. We now show that the mapping $A_1 : C^{2,0}(\overline{\Omega};\mathbb{R}^m)\to C^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ is continuous. Suppose the opposite. Then there exist $\{\mathbf{v}_k\}\subset C^{2,0}(\overline{\Omega};\mathbb{R}^m)$ and $\epsilon > 0$ such that $\mathbf{v}_k\to\mathbf{v}$ in $C^{2,0}(\overline{\Omega};\mathbb{R}^m)$ and $\|A_1\mathbf{v}_k - A_1\mathbf{v}\|_{C^{1,\beta}(\overline{\Omega})} > \epsilon$. Since $A_1 : C^{2,0}(\overline{\Omega};\mathbb{R}^m)\to C^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ is compact, there exists a sub-sequence $\{\mathbf{v}_{k(n)}\}$ of $\{\mathbf{v}_k\}$ and $\mathbf{w}\in C^{1,\beta}(\overline{\Omega})$ such that $A_1\mathbf{v}_{k(n)}\to\mathbf{w}$ in $C^{1,\beta}(\overline{\Omega})$. Since $A_1 : C^{2,0}(\overline{\Omega};\mathbb{R}^m)\to C^{0,\beta}(\overline{\Omega};\mathbb{R}^m)$ is continuous, it must be $\mathbf{w} = A_1\mathbf{v}$. That is a contradiction.

Since $A_a\mathbf{v} = aA_1\mathbf{v}$, Lemma 3.7 and (3.16) give that $A_a : C^{2,0}(\overline{\Omega};\mathbb{R}^m)\to C^{1,\beta}(\overline{\Omega};\mathbb{R}^m)$ is a compact continuous mapping and (3.7) holds. □

Remark 3.3. Let A_a be from Lemma 3.2 with $a\in C^\infty(\overline{\Omega})$. We cannot show that $A_a : C^{k,\beta}(\overline{\Omega};\mathbb{R}^m)\to C^{k-2,\beta}(\overline{\Omega};\mathbb{R}^m)$ for $k\ge 4$ as shows the following example. Fix $z\in\Omega$ and define $\mathbf{v}(x) := (x_2 - z_2, 0,\dots,0)$. Then $\mathbf{v}\in C^\infty(R^m;R^m)$ but $|\mathbf{v}|\mathbf{v}\notin C^2(\Omega;R^m)$.

Theorem 3.4. *Let $0<\beta\le\alpha<1$ and $k\in\mathbb{N}$. If $k=1$ suppose that $\beta<\alpha$. Put $l=\max(k-2,0)$. Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with boundary of class $C^{k,\alpha}$. Let $a,b,\lambda\in C^{l,\beta}(\overline{\Omega})$ and $\lambda\ge 0$. If $k\ge 3$ suppose that $a\equiv 0$. Then there exist $\delta,\epsilon,C\in(0,\infty)$ such that the following holds: If $\mathbf{g}\in C^{k,\beta}(\partial\Omega;\mathbb{R}^m)$ satisfying (2.3), $\mathbf{F}\in C^{l,\beta}(\overline{\Omega};\mathbb{R}^m)$ and*

$$\|\mathbf{g}\|_{C^{k,\beta}(\partial\Omega)}+\|\mathbf{F}\|_{C^{l,\beta}(\overline{\Omega})}<\delta, \tag{3.17}$$

then there exists a unique solution $(\mathbf{u},p)\in[C^{k,\beta}(\overline{\Omega};\mathbb{R}^m)\cap C^2(\Omega;\mathbb{R}^m)]\times[C^{k-1,\beta}(\overline{\Omega})\cap C^1(\Omega)]$ of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system

$$\nabla p-\Delta\mathbf{u}+\lambda\mathbf{u}+a|\mathbf{u}|\mathbf{u}+b(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{F},\quad \nabla\cdot\mathbf{u}=0\quad \textit{in } \Omega, \tag{3.18a}$$

$$\mathbf{u}=\mathbf{g}\qquad \textit{on } \partial\Omega \tag{3.18b}$$

such that

$$\int_\Omega p \, \mathrm{d}x = 0 \tag{3.19}$$

and

$$\|\mathbf{u}\|_{C^{k,\beta}(\overline{\Omega})} < \epsilon. \tag{3.20}$$

Moreover,

$$\|\mathbf{u}\|_{C^{k,\beta}(\overline{\Omega})} + \|p\|_{C^{k-1,\beta}(\overline{\Omega})} \le C\left(\|\mathbf{g}\|_{C^{k,\beta}(\partial\Omega)} + \|\mathbf{F}\|_{C^{l,\beta}(\overline{\Omega})}\right). \tag{3.21}$$

If $\tilde{\mathbf{g}} \in C^{k,\beta}(\partial\Omega;\mathbb{R}^m)$, $\tilde{\mathbf{F}} \in C^{l,\beta}(\overline{\Omega};\mathbb{R}^m)$, $\tilde{\mathbf{u}} \in C^{k,\beta}(\overline{\Omega};\mathbb{R}^m) \cap C^2(\Omega;\mathbb{R}^m)$ *and* $\tilde{p} \in C^{k-1,\beta}(\overline{\Omega}) \cap C^1(\Omega)$*,*

$$\nabla\tilde{p} - \Delta\tilde{\mathbf{u}} + a|\tilde{\mathbf{u}}|\tilde{\mathbf{u}} + \lambda\tilde{\mathbf{u}} + b(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} = \tilde{\mathbf{F}}, \quad \nabla\cdot\tilde{\mathbf{u}} = 0 \quad \textit{in } \Omega, \tag{3.22a}$$

$$\tilde{\mathbf{u}} = \tilde{\mathbf{g}} \qquad \textit{on } \partial\Omega, \qquad \int_\Omega \tilde{p} \, \mathrm{d}x = 0 \tag{3.22b}$$

and $\|\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})} < \epsilon$*, then*

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})} + \|p-\tilde{p}\|_{C^{k-1,\beta}(\overline{\Omega})} \le C\left(\|\mathbf{g}-\tilde{\mathbf{g}}\|_{C^{k,\beta}(\partial\Omega)} + \|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^{l,\beta}(\overline{\Omega})}\right). \tag{3.23}$$

Proof. Let L_b be defined by (3.1), A_a be given by (3.4). Put $D_{ab}\mathbf{u} := L_b(\mathbf{u},\mathbf{u}) + A_a\mathbf{u}$. According to Lemma 3.1 and Lemma 3.2 there exists a constant C_1 such that

$$\|D_{ab}\mathbf{v}\|_{C^{l,\beta}(\overline{\Omega})} \le C_1\|\mathbf{v}\|^2_{C^{k,\beta}(\overline{\Omega})}, \tag{3.24}$$

$$\|D_{ab}\mathbf{v} - D_{ab}\mathbf{u}\|_{C^{l,\beta}(\overline{\Omega})} \le C_1\|\mathbf{v}-\mathbf{u}\|_{C^{k,\beta}(\overline{\Omega})}\left[\|\mathbf{v}\|_{C^{k,\beta}(\overline{\Omega})} + \|\mathbf{u}\|_{C^{k,\beta}(\overline{\Omega})}\right]. \tag{3.25}$$

By Theorem 2.2 and Theorem 2.3 there exists a constant C_2 such that for each $\mathbf{g} \in C^{k,\beta}(\partial\Omega;\mathbb{R}^m)$ satisfying (2.3) and $\mathbf{f} \in C^{l,\beta}(\overline{\Omega};\mathbb{R}^m)$ there exists a unique solution $(\mathbf{u},p) \in [C^{k,\beta}(\overline{\Omega};\mathbb{R}^m) \cap C^2(\Omega;\mathbb{R}^m)] \times [C^{k-1,\beta}(\overline{\Omega}) \cap C^1(\Omega)]$ of the Dirichlet problem (2.2), (3.19). Moreover,

$$\|\mathbf{u}\|_{C^{k,\beta}(\overline{\Omega})} + \|p\|_{C^{k-1,\beta}(\overline{\Omega})} \le C_2\left(\|\mathbf{g}\|_{C^{k,\beta}(\partial\Omega)} + \|\mathbf{f}\|_{C^{l,\beta}(\overline{\Omega})}\right). \tag{3.26}$$

Remark that $(\mathbf{u},p)$ is a solution of (3.18) if $(\mathbf{u},p)$ is a solution of (2.2) with $\mathbf{f} = \mathbf{F} - D_{ab}\mathbf{u}$. Put

$$\epsilon := \frac{1}{4(C_1+1)(C_2+1)}, \quad \delta := \frac{\epsilon}{2(C_2+1)}.$$

If $(\mathbf{u},p),(\tilde{\mathbf{u}},\tilde{p}) \in [C^{k,\beta}(\overline{\Omega};\mathbb{R}^m) \cap C^2(\Omega;\mathbb{R}^m)] \times [C^{k-1,\beta}(\overline{\Omega}) \cap C^1(\Omega)]$ are solutions of (3.18), (3.19) and (3.22) with (3.20) and $\|\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})} < \epsilon$, then

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})} + \|p-\tilde{p}\|_{C^{k-1,\beta}(\overline{\Omega})} \le C_2[\|\mathbf{g}-\tilde{\mathbf{g}}\|_{C^{k,\beta}(\partial\Omega)} + \|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^{l,\beta}(\overline{\Omega})}$$
$$+\|D_{ab}(\mathbf{u},\mathbf{u}) - D_{ab}(\tilde{\mathbf{u}},\tilde{\mathbf{u}})\|_{C^{l,\beta}(\overline{\Omega})}] \le C_2[\|\mathbf{g}-\tilde{\mathbf{g}}\|_{C^{k,\beta}(\partial\Omega)}$$
$$+\|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^{l,\beta}(\overline{\Omega})} + 2\epsilon C_1\|\mathbf{u}-\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})}].$$

Since $2C_1C_2\epsilon < 1/2$ we get subtracting $2\epsilon C_1C_2\|\mathbf{u}-\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})}$ from the both sides

$$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{C^{k,\beta}(\overline{\Omega})} + \|p-\tilde{p}\|_{C^{k-1,\beta}(\overline{\Omega})} \le 2C_2[\|\mathbf{g}-\tilde{\mathbf{g}}\|_{C^{k,\beta}(\partial\Omega)} + \|\mathbf{F}-\tilde{\mathbf{F}}\|_{C^{l,\beta}(\overline{\Omega})}].$$

Therefore a solution of (3.18) satisfying (3.19) and (3.20) is unique. Putting $\tilde{p} \equiv 0$, $\tilde{\mathbf{u}} \equiv 0$, $\tilde{\mathbf{F}} \equiv 0$ and $\tilde{\mathbf{g}} \equiv 0$, we obtain (3.21) with $C = 2C_2$.

Denote $X := \{\mathbf{v} \in C^{k,\beta}(\overline{\Omega}, \mathbb{R}^m); \|\mathbf{v}\|_{C^{k,\beta}(\overline{\Omega})} \leq \epsilon\}$. Fix $\mathbf{g} \in C^{k,\beta}(\partial\Omega; \mathbb{R}^m)$ and $\mathbf{F} \in C^{l,\beta}(\overline{\Omega}; \mathbb{R}^m)$ satisfying (2.3) and (3.17). For $\mathbf{v} \in X$ there exists a unique solution $(\mathbf{u}^{\mathbf{v}}, p^{\mathbf{v}}) \in [C^{k,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)] \times [C^{k-1,\beta}(\overline{\Omega}) \cap C^1(\Omega)]$ of the Dirichlet problem (2.2), (3.19) with $\mathbf{f} = \mathbf{F} - D_{ab}\mathbf{v}$. Remember that $(\mathbf{u}^{\mathbf{v}}, p^{\mathbf{v}})$ is a solution of (3.18) if and only if $\mathbf{u}^{\mathbf{v}} = \mathbf{v}$. According to (3.26), (3.17) and (3.24)

$$\|\mathbf{u}^{\mathbf{v}}\|_{C^{k,\beta}(\overline{\Omega})} \leq C_2 \left[\|\mathbf{g}\|_{C^{k,\beta}(\partial\Omega)} + \|\mathbf{F}\|_{C^{l,\beta}(\overline{\Omega})} + \|D_{ab}\mathbf{v}\|_{C^{l,\beta}(\overline{\Omega})}\right] \leq C_2\delta + C_2C_1\epsilon^2.$$

As $C_2\delta + C_2C_1\epsilon^2 < \epsilon$, we infer $\mathbf{u}^{\mathbf{v}} \in X$. If $\mathbf{w} \in X$ then

$$\|\mathbf{u}^{\mathbf{v}} - \mathbf{u}^{\mathbf{w}}\|_{C^{k,\beta}(\overline{\Omega})} \leq C_2\|D_{ab}\mathbf{v} - D_{ab}\mathbf{w}\|_{C^{l,\beta}(\overline{\Omega})} \leq C_2C_1 2\epsilon\|\mathbf{w} - \mathbf{v}\|_{C^{k,\beta}(\overline{\Omega})}$$

by (3.26) and (3.25). Since $C_2C_1 2\epsilon < 1$, the Fixed point theorem ([21, Satz 1.24]) gives that there exists $\mathbf{v} \in X$ such that $\mathbf{u}^{\mathbf{v}} = \mathbf{v}$. So, $(\mathbf{u}^{\mathbf{v}}, p^{\mathbf{v}})$ is a solution of (3.18), (3.19) in $[C^{k,\beta}(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)] \times [C^{k-1,\beta}(\overline{\Omega}) \cap C^1(\Omega)]$ satisfying (3.20). □

In Theorem 3.4 we suppose that $a \equiv 0$ for $k \geq 3$. This assumption cannot be removed for $k > 3$ as Remark 3.3 shows. The following theorem is devoted to the case $k = 3$.

Theorem 3.5. *Let $0 < \beta \leq \alpha < 1$ and $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class $C^{3,\alpha}$. Let $a, b, \lambda \in C^{1,\beta}(\overline{\Omega})$ and $\lambda \geq 0$. Then there exist $\delta, \epsilon \in (0, \infty)$ such that the following holds: If $\mathbf{g} \in C^{3,\beta}(\partial\Omega; \mathbb{R}^m)$ satisfies* (2.3)*, $\mathbf{F} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)$ and*

$$\|\mathbf{g}\|_{C^{3,\beta}(\partial\Omega)} + \|\mathbf{F}\|_{C^{1,\beta}(\overline{\Omega})} < \delta, \tag{3.27}$$

then there exists a unique solution $(\mathbf{u}, p) \in C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{2,\beta}(\overline{\Omega})$ of the Dirichlet problem for the Darcy-Forchheimer-Brinkman system (3.18)*,* (3.19) *such that*

$$\|\mathbf{u}\|_{C^{3,\beta}(\overline{\Omega})} < \epsilon. \tag{3.28}$$

Proof. Let L_b be defined by (3.1), A_a be given by (3.4). We conclude from Lemma 3.1 and Lemma 3.2 that there exists a constant C_1 such that

$$\|L_b(\mathbf{v}, \mathbf{v})\|_{C^{1,\beta}(\overline{\Omega})} + \|A_a\mathbf{v}\|_{C^{1,\beta}(\overline{\Omega})} \leq C_1\|\mathbf{v}\|^2_{C^{3,\beta}(\overline{\Omega})}, \tag{3.29}$$

$$\|L_b(\mathbf{v}, \mathbf{v}) - L_b(\mathbf{u}, \mathbf{u})\|_{C^{1,\beta}(\overline{\Omega})} \leq C_1\|\mathbf{v} - \mathbf{u}\|_{C^{3,\beta}(\overline{\Omega})} \left[\|\mathbf{v}\|_{C^{3,\beta}(\overline{\Omega})} + \|\mathbf{u}\|_{C^{3,\beta}(\overline{\Omega})}\right]. \tag{3.30}$$

According to Theorem 2.3 there exists a constant C_2 such that for each $\mathbf{g} \in C^{3,\beta}(\partial\Omega; \mathbb{R}^m)$ satisfying (2.3) and $\mathbf{f} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)$ there exists a unique solution $(\mathbf{u}, p) \in C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{2,\beta}(\overline{\Omega})$ of the Dirichlet problem (2.2), (3.19). Moreover,

$$\|\mathbf{u}\|_{C^{3,\beta}(\overline{\Omega})} + \|p\|_{C^{2,\beta}(\overline{\Omega})} \leq C_2 \left(\|\mathbf{g}\|_{C^{3,\beta}(\partial\Omega)} + \|\mathbf{f}\|_{C^{1,\beta}(\overline{\Omega})}\right). \tag{3.31}$$

Suppose now that

$$0 < \epsilon < \frac{1}{4(C_1 + 1)(C_2 + 1)}, \quad 0 < \delta < \frac{\epsilon}{2(C_2 + 1)}. \tag{3.32}$$

Put $X_\epsilon := \{\mathbf{v} \in C^{3,\beta}(\overline{\Omega}, \mathbb{R}^m); \|\mathbf{v}\|_{C^{3,\beta}(\overline{\Omega})} \leq \epsilon\}$. Fix $\mathbf{g} \in C^{3,\beta}(\partial\Omega; \mathbb{R}^m)$ and $\mathbf{F} \in C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)$ satisfying (2.3) and (3.27). For $\mathbf{v} \in X_\epsilon$ there exists a unique solution $(\mathbf{u}^\mathbf{v}, p^\mathbf{v}) \in C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{2,\beta}(\overline{\Omega})$ of the Dirichlet problem (2.2), (3.19) with $\mathbf{f} = \mathbf{F} - L_b(\mathbf{v}, \mathbf{v})$. Moreover, there is a unique solution $(\tilde{\mathbf{u}}^\mathbf{v}, \tilde{p}^\mathbf{v}) \in C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{2,\beta}(\overline{\Omega})$ of

$$\nabla \tilde{p}^\mathbf{v} - \Delta \tilde{\mathbf{u}}^\mathbf{v} + \lambda \tilde{\mathbf{u}}^\mathbf{v} = -a|\mathbf{v}|\mathbf{v}, \quad \nabla \cdot \tilde{\mathbf{u}}^\mathbf{v} = 0 \quad \text{in } \Omega,$$

$$\tilde{\mathbf{u}}^\mathbf{v} = 0 \qquad \text{on } \partial\Omega, \qquad \int_\Omega \tilde{p}^\mathbf{v} \, \mathrm{d}x = 0.$$

If $\mathbf{u}^\mathbf{v} + \tilde{\mathbf{u}}^\mathbf{v} = \mathbf{v}$ then $(\mathbf{v}, p^\mathbf{v} + \tilde{p}^\mathbf{v})$ is a solution of the problem (3.18), (3.19). If $\mathbf{w} \in X_\epsilon$ then

$$\|\mathbf{u}^\mathbf{v}\|_{C^{3,\beta}(\overline{\Omega})} + \|\tilde{\mathbf{u}}^\mathbf{w}\|_{C^{3,\beta}(\overline{\Omega})} \leq C_2\big(\|\mathbf{g}\|_{C^{3,\beta}(\partial\Omega)} + \|\mathbf{F}\|_{C^{1,\beta}(\overline{\Omega})} + \|L_b(\mathbf{v}, \mathbf{v})\|_{C^{1,\beta}(\overline{\Omega})}$$

$$+\|A_a \mathbf{w}\|_{C^{1,\beta}(\overline{\Omega})}\big) \leq C_2(\delta + C_1 \|\mathbf{v}\|^2_{C^{3,\beta}(\overline{\Omega})} + C_1 \|\mathbf{w}\|^2_{C^{3,\beta}(\overline{\Omega})}) < \epsilon$$

by (3.31), (3.27), (3.29) and (3.32). So, $\mathbf{u}^\mathbf{v} + \tilde{\mathbf{u}}^\mathbf{w} \in X_\epsilon$. According to (3.31), (3.30) and (3.32)

$$\|\mathbf{u}^\mathbf{v} - \mathbf{u}^\mathbf{w}\|_{C^{3,\beta}(\overline{\Omega})} \leq C_2 \|L_b(\mathbf{v}, \mathbf{v}) - L_b(\mathbf{w}, \mathbf{w})\|_{C^{1,\beta}(\overline{\Omega})}$$

$$\leq C_1 C_2 \|\mathbf{v} - \mathbf{w}\|_{C^{3,\beta}(\overline{\Omega})} [\|\mathbf{v}\|_{C^{3,\beta}(\overline{\Omega})} + \|\mathbf{w}\|_{C^{3,\beta}(\overline{\Omega})}] < \frac{1}{2} \|\mathbf{v} - \mathbf{w}\|_{C^{3,\beta}(\overline{\Omega})}.$$

So $\mathbf{v} \mapsto \mathbf{u}^\mathbf{v}$ is a contractive mapping on X_ϵ. Since $A_a : C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \to C^{1,\beta}(\overline{\Omega}; \mathbb{R}^m)$ is a compact continuous mapping by Lemma 3.2, the mapping $\mathbf{v} \mapsto \tilde{\mathbf{u}}^\mathbf{v}$ is a compact continuous mapping on X_ϵ. Lemma 3.8 forces that there exists $\mathbf{v} \in X_\epsilon$ such that $\mathbf{u}^\mathbf{v} + \tilde{\mathbf{u}}^\mathbf{v} = \mathbf{v}$. Hence $(\mathbf{v}, p^\mathbf{v} + \tilde{p}^\mathbf{v})$ is a solution of the problem (3.18), (3.19), (3.28) in $C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{1,\beta}(\overline{\Omega})$. By Theorem 3.4, for sufficiently small ϵ and δ there exists at most one solution of the problem (3.18), (3.19), (3.28) in $C^{3,\beta}(\overline{\Omega}; \mathbb{R}^m) \times C^{1,\beta}(\overline{\Omega})$. □

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Conflict of interest

The author declares no conflict of interest in this paper.

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Appendix

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary of class C^1. Let $\lambda \in L^\infty(\Omega)$ be non-negative and $1 < q < \infty$. If $\mathbf{f} \in W^{-1,q}(\Omega, \mathbb{R}^m)$ and $\mathbf{g} \in W^{1-1/q,q}(\partial\Omega, \mathbb{R}^m)$, then there exists a solution $(\mathbf{v}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ of (2.2) if and only if (2.3) holds. A velocity $\mathbf{v}$ is unique and a pressure p is unique up to an additive constant. Moreover,*

$$\|\mathbf{v}\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \leq C\left(\|\mathbf{f}\|_{W^{-1,q}(\Omega)} + \|\mathbf{g}\|_{W^{1-1/q,q}(\partial\Omega)} + \left|\int_\Omega p \, \mathrm{d}x\right|\right).$$

Proof. If $(\mathbf{v}, p) \in W^{1,q}(\Omega, \mathbb{R}^m) \times L^q(\Omega)$ is a solution of (2.2), then (2.3) holds by Green's formula.

Denote by Y_q the set of all $\mathbf{g} \in W^{1-1/q,q}(\partial\Omega; \mathbb{R}^m)$ satisfying (2.3). Define $X_q := \{\mathbf{v} \in W^{1,q}(\Omega; \mathbb{R}^m); \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v}|_{\partial\Omega} \in Y_q\}$,

$$U_\lambda(\mathbf{v}, p) := \left[-\Delta\mathbf{v} + \lambda\mathbf{v} + \nabla p, \mathbf{v}, \int_\Omega p \, \mathrm{d}x\right].$$

Then $U_0 : X_q \times L^q(\Omega) \to W^{-1,q}(\Omega; \mathbb{R}^m) \times Y_q \times \mathbb{R}^1$ is an isomorphism by [22, Theorem 2.1]. Since $U_\lambda - U_0 : X_q \times L^q(\Omega) \to W^{-1,q}(\Omega; \mathbb{R}^m) \times Y_q \times \mathbb{R}^1$ is compact, $U_\lambda : X_q \times L^q(\Omega) \to W^{-1,q}(\Omega; \mathbb{R}^m) \times Y_q \times \mathbb{R}^1$ is a Fredholm operator with index 0. Let now $U_\lambda(\mathbf{v}, p) = 0$. Since $U_\lambda : X_2 \times L^2(\Omega) \to W^{-1,2}(\Omega; \mathbb{R}^m) \times Y_2 \times \mathbb{R}^1$ is a Fredholm operator with index 0, [23, Lemma 11.9.21] gives $(\mathbf{v}, p) \in X_2 \times L^2(\Omega)$. Since $-\Delta\mathbf{v} + \nabla p = -\lambda\mathbf{v}$, Green's formula forces

$$\int_\Omega \nabla\mathbf{v} \cdot \nabla\mathbf{\Phi} \, \mathrm{d}x = -\int_\Omega \lambda\mathbf{v} \cdot \mathbf{\Phi} \, \mathrm{d}x$$

for all $\mathbf{\Phi} \in C_c^\infty(\Omega; \mathbb{R}^m)$ with $\nabla \cdot \mathbf{\Phi} = 0$. As $\mathbf{v}$ is in the closure of the space of such $\mathbf{\Phi}$ by [24, Theorem 2.9], we infer

$$\int_\Omega [\nabla\mathbf{v} \cdot \nabla\mathbf{v} + \lambda\mathbf{v} \cdot \mathbf{v}] \, \mathrm{d}x = 0.$$

$\nabla v \equiv 0$ and therefore the velocity $\mathbf{v}$ is constant. We have $\mathbf{v} \equiv 0$, because $\mathbf{v} = 0$ on $\partial\Omega$. Moreover, $\nabla p = \Delta\mathbf{v} - \lambda\mathbf{v} \equiv 0$ forces that p is constant. The equality $\int_\Omega p \, \mathrm{d}x = 0$ gives that $p \equiv 0$. Therefore $U_\lambda : X_q \times L^q(\Omega) \to W^{-1,q}(\Omega; \mathbb{R}^m) \times Y_q \times \mathbb{R}^1$ is an isomorphism. □

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^m$ be open and $0 < \alpha \leq 1$. If $f, g \in C^{0,\alpha}(\overline{\Omega})$ then $fg \in C^{0,\alpha}(\overline{\Omega})$ and*

$$\|fg\|_{C^{0,\alpha}(\overline{\Omega})} \leq 2\|f\|_{C^{0,\alpha}(\overline{\Omega})}\|g\|_{C^{0,\alpha}(\overline{\Omega})}.$$

(See [14, Lemma 1.16.8].)

Lemma 3.8. *Let Z be a closed convex non-empty subset of a Banach space X. Suppose that A and B map Z into X and that*

- *$Ax + By \in Z$ for all $x, y \in Z$,*
- *A is a contraction mapping,*
- *B is compact and continuous.*

Then there exists $z \in Z$ such that $Az + Bz = z$.

(See [25, Theorem 4.4.1].)