AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(5): 1508–1522.
DOI:10.3934/math.2019.5.1508
Received: 03 July 2019
Accepted: 20 August 2019
Published: 23 September 2019

Research article

Nonlinear boundary value problems for a parabolic equation with an unknown source function

Nataliya Gol'dman*

Science Research Computer Center, Moscow State University, Moscow 119992, Russia

* **Correspondence:** Email: nlgold40@yandex.ru; Tel: +7(495)9395424; Fax: +7(495)9392090.

Abstract: We study nonlinear problems for a parabolic equation with unknown source functions. One of the problems is a system which contains the boundary value problem of the first kind and the equation for a time dependence of the sought source function. In the other problem the corresponding system is distinguished by boundary conditions. For these nonlinear systems, conditions of unique solvability in a class of smooth functions are obtained on the basis of the Rothe method. The proposed approach involves the proof of *a priori* estimates in the difference-continuous analogs of Hölder spaces for the corresponding differential-difference nonlinear systems that approximate the original systems by the Rothe method. The considered nonlinear parabolic problems essentially differ from usual boundary value problems but have not only the theoretical interest. The present investigation is connected with the mathematical modeling of nonstationary filtration processes in porous media.

Keywords: parabolic equations; nonlinear boundary value problems; Rothe method; *a priori* estimates; unique solvability in Hölder spaces
Mathematics Subject Classification: 35K59, 35K61, 35Q35

1. Introduction

The goal of the work is to investigate nonlinear parabolic problems that arise in the mathematical modeling of some nonstationary filtration processes in underground hydrodynamics. These models are formulated as a system that involves a boundary value problem for a quasilinear parabolic equation with an unknown source function and, moreover, an additional relationship for a time dependence of this sought function. Justification of the corresponding mathematical statements is an important task since such statements essentially differ from usual boundary value problems (see the well known monographs [1, 2]). Our main aim is to obtain conditions for existence and uniqueness of their smooth solutions. Investigation of such conditions is carried out by using the Rothe method and *a priori* estimates in the difference-continuous analogs of Hölder spaces for the corresponding differential-difference nonlinear system that approximates the original system. The approach that is proposed in the present work allows one to avoid additional assumptions of the smoothness of the input data, which have usually been imposed by the Rothe method (see, e.g., [1]). Thus, the faithful character of differential relations between the input data and the solution in the chosen function spaces is determined for each of the considered nonlinear parabolic problems.

This article is organized as follows. Beforehand, in section 2 we present some definitions of the function spaces that are used in our analysis. In particular, the difference-continuous analogs of Hölder classes are determined for the grid functions. In section 3, we analyze the nonlinear parabolic problem with the boundary conditions of the first kind. In order to obtain the unique solvability result, the proof of the corresponding *a priori* estimates is split into several stages. In section 4 the similar analysis is carried out for the nonlinear parabolic problem with the boundary conditions of the second kind. Section 5 contains an example of some mathematical models of filtration processes in underground fluid mechanics. Such models arise in exploitation of oil-gas fields in the case of cracked porous media. Finally, a short conclusion in section 6 summarizes the results of this work.

2. Basic designations

In our work we use standard definitions for the function spaces from [1]. In particular, the Hölder class $H^{2+\lambda,1+\lambda/2}(\overline{Q})$ $(0<\lambda<1)$ is determined as the space of functions $u(x,t)$ continuous on the closed set $\overline{Q}=\{0\le x\le l, 0\le t\le T\}$ together with their derivatives u_{xx}, u_t which satisfy the Hölder condition as functions of x, t with the corresponding exponents λ and $\lambda/2$. The space $O^1[0,T]$ is determined as the set of continuous functions having the bounded derivative for $0\le t\le T$.

For a convenient presentation, the following designation is also used.

$H^{1,\lambda/2,1}(\overline{D})$ is the space of functions which are continuous for $(x,t,u)\in\overline{D}=\overline{Q}\times[-M_0,M_0]$ together with their derivatives with respect to x, u and, moreover, satisfy the Hölder condition as functions of t with the exponent $\lambda/2$.

Moreover, in connection with application of the Rothe method we use analogs of the Hölder classes in the case of the grid functions $\hat{u}=(u_0,\dots,u_n,\dots,u_N)$ defined on the grid $\overline{\omega}_\tau=\{t_n\}=\{n\tau, n=\overline{0,N}, \tau=TN^{-1}\}$ and in the case of the grid-continuous functions $\hat{u}(x)=(u_0(x),\dots,u_n(x),\dots,u_N(x))$ defined on the set $\overline{Q}_\tau=\{0\le x\le l, t_n\in\overline{\omega}_\tau\}$. Just as in [3] these analogs are determined in the following way.

$H_\tau^{1+\lambda/2}(\overline{\omega}_\tau)$ is the difference analog of the space $H^{1+\lambda/2}[0,T]$ (see [1]) for the functions $\hat{u}$ having a finite norm

$$|\hat{u}|^{1+\lambda/2}_{\overline{\omega}_\tau}=\max_{0\le n\le N}|u_n|+\max_{1\le n\le N}|u_{n\bar{t}}|+\langle\hat{u}_{\bar{t}}\rangle^{\lambda/2}_{\overline{\omega}_\tau},$$

$$u_{n\bar{t}}=(u_n-u_{n-1})\tau^{-1},\ n=\overline{1,N},\ \langle\hat{u}_{\bar{t}}\rangle^{\lambda/2}_{\overline{\omega}_\tau}=\max_{1\le n<n'\le N}\{|u_{n\bar{t}}-u_{n'\bar{t}}||t_n-t_{n'}|^{-\lambda/2}\}.$$

$H_\tau^{\lambda,\lambda/2}(\overline{Q}_\tau)$ is the difference-continuous analog of the space $H^{\lambda,\lambda/2}(\overline{Q})$ (see [1]) for the functions $\hat{u}(x)$ continuous in x for $(x,t_n)\in\overline{Q}_\tau$ and having a finite norm

$$\begin{aligned}
|\hat{u}(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} &= \max_{(x,t_n)\in\overline{Q}_\tau}|u_n(x)|+\langle\hat{u}(x)\rangle^{\lambda}_{x,\overline{Q}_\tau}+\langle\hat{u}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau},\\
\langle\hat{u}(x)\rangle^{\lambda}_{x,\overline{Q}_\tau} &= \sup_{(x,t_n),(x',t_n)\in\overline{Q}_\tau}\{|u_n(x)-u_n(x')||x-x'|^{-\lambda}\},\\
\langle\hat{u}(x)\rangle^{\lambda/2}_{t,\overline{Q}_\tau} &= \sup_{(x,t_n),(x,t'_n)\in\overline{Q}_\tau}\{|u_n(x)-u_{n'}(x)||t_n-t_{n'}|^{-\lambda/2}\}.
\end{aligned}$$

$H_\tau^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q}_\tau)$ is the difference-continuous analog of the space $H^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q})$ (see [1]) for the functions $\hat{u}(x)$ continuous in x together with their derivatives with respect to x for $(x, t_n) \in \overline{Q}_\tau$ and having a finite norm

$$|\hat{u}(x)|_{\overline{Q}_\tau}^{1+\lambda,\frac{1+\lambda}{2}} = \max_{(x,t_n)\in\overline{Q}_\tau} |u_n(x)| + |\hat{u}_x(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2} + \langle \hat{u}(x)\rangle_{t,\overline{Q}_\tau}^{\frac{1+\lambda}{2}},$$

where $\hat{u}_x(x) = (u_{0x}(x), \ldots, u_{nx}(x), \ldots, u_{Nx}(x))$.

$H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ is the difference-continuous analog of the space $H^{2+\lambda,1+\lambda/2}(\overline{Q})$ for the functions $\hat{u}(x)$ continuous in x together with their derivatives $\hat{u}_{xx}(x)$ and $\hat{u}_{\bar{t}}(x)$ for $(x, t_n) \in \overline{Q}_\tau$ and having a finite norm

$$|\hat{u}(x)|_{\overline{Q}_\tau}^{2+\lambda,1+\lambda/2} = \max_{(x,t_n)\in\overline{Q}_\tau} |u_n(x)| + \max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| + |\hat{u}_{xx}(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2} + |\hat{u}_{\bar{t}}(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2},$$

where

$$\begin{aligned} \hat{u}_{xx}(x) &= (u_{0xx}(x), \ldots, u_{nxx}(x), \ldots, u_{Nxx}(x)), \\ \hat{u}_{\bar{t}}(x) &= (u_{1\bar{t}}(x), \ldots, u_{n\bar{t}}(x), \ldots, u_{N\bar{t}}(x)), \\ u_{n\bar{t}}(x) &= (u_n(x) - u_{n-1}(x))\tau^{-1}, n = \overline{1, N}. \end{aligned}$$

3. Unique solvability of a nonlinear parabolic problem with the boundary conditions of the first kind

3.1. The statement for a quasilinear parabolic equation with an unknown source function

We formulate the present statement as a system for determination of the functions $\{u(x, t), p(x, t)\}$ in the domain $\overline{Q} = \{0 \le x \le l, 0 \le t \le T\}$ that satisfy the boundary value problem of the first kind

$$c(x, t, u)u_t - Lu = f(x, t)p(x, t), \quad (x, t) \in Q, \tag{1}$$

$$u(x, t)|_{x=0} = w(t), \quad u(x, t)|_{x=l} = v(t), \quad 0 < t \le T, \tag{2}$$

$$u(x, t)|_{t=0} = \varphi(x), \quad 0 \le x \le l, \tag{3}$$

and the additional relationship

$$p_t(x, t) = \chi(t)p(x, t) + \gamma(x, t, u), \quad (x, t) \in Q, \quad p(x, t)|_{t=0} = p^0(x), \quad 0 \le x \le l, \tag{4}$$

where a uniformly elliptic operator Lu has the form

$$Lu \equiv (a(x, t, u)u_x)_x - b(x, t, u)u_x - d(x, t, u)u.$$

All the input data in equation (1), boundary conditions (2), initial condition (3), and in relationship (4) are the known functions of their arguments; $a \ge a_{\min} > 0$, $c \ge c_{\min} > 0$, $a_{\min}$, $c_{\min} = \text{const} > 0$.

3.2. Conditions of unique solvability in the Hölder spaces

The result for unique definition of the smooth solution $\{u(x, t), p(x, t)\}$ of systems (1)–(4) is supplied by the following theorem.

Theorem 3.1. *Let the following conditions be satisfied.*

1. *For* $(x,t) \in \overline{Q}$ *and any* u, $|u| < \infty$, *the input data of the boundary value problems (1)–(3) are uniformly bounded functions of their arguments, where the coefficient* $a(x,t,u)$ *— together with the derivatives* $a_x(x,t,u)$ *and* $a_u(x,t,u)$*, moreover,* $0 < a_{\min} \le a(x,t,u) \le a_{\max}$, $0 < c_{\min} \le c(x,t,u) \le c_{\max}$.
2. *For* $(x,t,u) \in \overline{D} = \overline{Q} \times [-M_0, M_0]$ *(where* $M_0 \ge \max_{(x,t)\in\overline{Q}} |u|$, M_0 *is the constant from the maximum principle for boundary value problems (1)–(3)) the functions* $a(x,t,u)$, $a_x(x,t,u)$, $a_u(x,t,u)$, $b(x,t,u)$*, and* $d(x,t,u)$ *have the uniformly bounded derivatives with respect to u and Hölder continuous in x and t with the corresponding exponents* λ *and* $\lambda/2$*; moreover, the functions* $c(x,t,u)$ *and* $f(x,t)$ *are in* $H^{1,\lambda/2,1}(\overline{D})$ *and* $H^{\lambda,\lambda/2}(\overline{Q})$*, respectively.*
3. *The functions* $w(t)$ *and* $v(t)$ *are in* $H^{1+\lambda/2}[0,T]$*, the functions* $\varphi(x)$ *and* $p^0(x)$ *are in* $H^{2+\lambda}[0,l]$ *and* $C^1[0,l]$*, respectively,* $\max_{0\le x\le l} |p^0(x)| \le p^0_{\max}$, $\max_{0\le x\le l} |p^0_x(x)| \le p^0_{x\max}$, $p^0_{\max}, p^0_{x\max} = \text{const} > 0$*; there hold the matching conditions*

$$c(x,0,\varphi)w_t - L\varphi|_{x=0,t=0} = f(x,0)p^0(x)|_{x=0}, \tag{5}$$

$$c(x,0,\varphi)v_t - L\varphi|_{x=l,t=0} = f(x,0)p^0(x)|_{x=l}. \tag{6}$$

4. *The function* $\chi(t)$ *is in* $C[0,T]$, $\max_{0\le t\le T} |\chi(t)| \le \chi_{\max}$, $\chi_{\max} = \text{const} > 0$*; the function* $\gamma(x,t,u)$ *is uniformly bounded for* $(x,t) \in \overline{Q}$, $|u| < \infty$*, and is continuous for* $(x,t,u) \in \overline{D}$ *together with the derivatives with respect to x and u,*

$$|\gamma(x,t,u)| \le \gamma_{\max}, \quad \max_{(x,t,u)\in\overline{D}} |\gamma_x(x,t,u)| \le \gamma_{x\max}, \quad \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \le \gamma_{u\max},$$

$\gamma_{\max}, \gamma_{x\max}, \gamma_{u\max} = \text{const} > 0.$

Then there exists a unique solution $\{u(x,t), p(x,t)\}$ *of the nonlinear systems (1)–(4) which has properties*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad |u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}} \le M, \quad M = \text{const} > 0,$$

$$p(x,t) \in H^{\lambda,\lambda/2}(\overline{Q}), \quad |p(x,t)|^{\lambda,\lambda/2}_{\overline{Q}} \le \mathcal{M}, \quad \mathcal{M} = \text{const} > 0.$$

In order to prove Theorem 3.1 and to establish the existence of the smooth solution with the matching conditions (5) and (6) we approximate this system using the discretization procedure of the Rothe method on the uniform grid $\overline{\omega}_\tau = \{t_n\} \in [0,T]$ with time-step $\tau = TN^{-1}$:

$$c_n u_{n\bar{t}} - (a_n u_{nx})_x + b_n u_{nx} + d_n u_n = f_n p_n, \quad (x,t_n) \in Q_\tau = \{0 < x < l\} \times \omega_\tau, \tag{7}$$

$$u_n|_{x=0} = w_n, \quad u_n|_{x=l} = v_n, \quad 0 < t_n \le T, \tag{8}$$

$$u_0(x) = \varphi(x), \quad 0 \le x \le l, \tag{9}$$

$$p_{n\bar{t}} = \chi_{n-1} p_{n-1} + \gamma_{n-1}, \ (x,t_n) \in Q_\tau, \quad p_n(x)|_{n=0} = p^0(x), \quad 0 \le x \le l. \tag{10}$$

The approximating system can be formulated as follows: Find $\{u_n(x), p_n(x)\}$ — approximate values of the functions $u(x,t)$ and $p(x,t)$ for $t = t_n$ — satisfying conditions (7)–(10) in which a_n, b_n, c_n, and d_n are the values of the corresponding coefficients at the point (x,t_n,u_n); $f_n = f(x,t_n)$, $w_n = w(t_n)$, $v_n = v(t_n)$, $\chi_{n-1} = \chi(t_{n-1})$, and $\gamma_{n-1} = \gamma(x,t_{n-1},u_{n-1})$. In system (7)–(10) the following designations are also used: $u_{n\bar{t}} = (u_n(x) - u_{n-1}(x))\tau^{-1}$, $u_{nx} = du_n(x)/dx$, $p_{n\bar{t}} = (p_n(x) - p_{n-1}(x))\tau^{-1}$.

The proof of solvability of systems (1)–(4) by the Rothe method involves several stages.

Stage 1. Investigation of the differential-difference boundary value problems (7)–(9) in the difference-continuous Hölder space $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ under assumption that $p_n(x)$ is the known function. The aim of this stage is to prove unique solvability of problems (7)–(9) and to drive the corresponding *a priori* estimates for the solution $u_n(x)$ (independent of x, τ, n).

Stage 2. The proof of existence and uniqueness of the solution $\{u_n(x), p_n(x)\}$ to the differential-difference systems (7)–(10) in the corresponding function spaces by using the results of stage 1.

Stage 3. The passage to the limit as time-step τ goes to 0 (i.e., $n \to \infty$) in conditions (7)–(10) by using the compactness of the set $\{u_n(x), p_n(x)\}$ thanks to the estimates obtained at stage 2. The aim of this last stage is to show that original systems (1)–(4) have at least one solution in the corresponding Hölder spaces.

3.3. A priori estimates in the difference-continuous Hölder spaces

Passing to these stages we show the proof in details only if the justification of the Rothe method must take into account specific properties of systems (1)–(4). Otherwise, we only sketch the proof referring to the known results.

The conditions of unique solvability of problems (7)–(9) in $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ are formulated by the next lemma under assumption that $p_n(x)$ in the differential-difference equation (7) is the given source function in $H_\tau^{\lambda,\lambda/2}(\overline{Q}_\tau)$ with $|\hat{p}(x)|_{Q_\tau}^{\lambda,\lambda/2} \le \mathcal{M}$, $\mathcal{M} = \text{const} > 0$.

Lemma 3.1. *Assume that the conditions 1–3 of Theorem 3.1 hold and let $p_n(x)$ be a function with the above-mentioned properties. Then the differential-difference boundary value problems (7)–(9) has a unique solution $u_n(x)$ in the domain $\overline{Q}_\tau$ (for any sufficiently small time-step τ of the grid $\overline{\omega}_\tau$) and the following estimates are valid*

$$\max_{(x,t_n)\in\overline{Q}_\tau} |u_n(x)| \le M_0, \qquad \max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| \le M_1,$$

$$|\hat{u}(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2} \le M_2, \quad |\hat{u}_x(x)|_{\overline{Q}_\tau}^{\lambda,\lambda/2} \le M_3, \quad |\hat{u}(x)|_{\overline{Q}_\tau}^{2+\lambda,1+\lambda/2} \le M_4, \tag{11}$$

where $M_i > 0$ ($i = \overline{0,4}$) are positive constants independent of x, τ, and n.

The conclusion of Lemma 3.1 is based on results of Theorem 4.3.3 [3] about unique solvability of the differential-difference boundary value problems of the first kind in the Hölder class $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$. The proof of this theorem is supplied by the Leray-Schauder principle on the existence of the fixed points of the completely continuous transforms. The following remarks must be added.

Remark 3.1. For the present problems (7)–(9) the constant M_0 from the maximum principle has the form

$$M_0 = \left\{c_{\min}^{-1} f_{\max} p_{\max} T + \max(w_{\max}, v_{\max}, \varphi_{\max})\right\} \exp(K_1 T),$$

$$K_1 \ge (1+\varepsilon) d_{\max} c_{\min}^{-1}, \quad \varepsilon > 0 \text{ is arbitrary}, \quad \tau \le \tau_0 = \varepsilon K_1^{-1}. \tag{12}$$

In order to derive the estimate of the maximum principle auxiliary functions are used

$$\eta_n^{\pm}(x) = \pm u_n(x)(1 + K_1\tau)^{-n} + c_{\min}^{-1} f_{\max} p_{\max} t_n + \max(w_{\max}, v_{\max}, \varphi_{\max}).$$

Remark 3.2. In order to obtain the estimate $\max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| \le M_1$ for problems (7)–(9), we apply the discrete analog of the known technique [4]. This approach allows one to avoid differentiating equation (7) with respect to x and hence does not require additional smoothness of the input data. Namely, we apply the odd extension of the function $u_n(x)$ into domains $Q_\tau^- = \{-l < x < 0\} \times \omega_\tau$ and $Q_\tau^+ = \{l < x < 2l\} \times \omega_\tau$ with the next introduction of an additional space variable z and a function $W_n(x,z) = u_n(x) - u_n(z)$. For this function the estimate $|W_n(x,z)| \le M_1|x - z|$ is derived that leads to the desired estimate $\max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| \le M_1$ in (11). The constant M_1 depends on values M_0, $\max_{0\le x\le l} |\varphi_x(x)|$, $\max_{0\le t\le T} |w_t(t)|$, $\max_{0\le t\le T} |v_t(t)|$ (for details see Lemma 4.3.5 from [3]).

Passing to stage 2 we consider systems (7)–(10) in order to find $\{u_n(x), p_n(x)\}$. The values of $p_n(x)$ are beforehand unknown and simultaneously determined with $u_n(x)$. This requires additional reasonings for proving the solvability of systems (7)–(10).

Lemma 3.2. *Assume that the input data of systems (1)–(4) satisfy the hypotheses of Theorem 3.1. Then in the domain $\overline{Q}_\tau$ for any time-step $\tau \le \tau_0$ ($\tau_0 > 0$ is the constant defined by estimate (12)) there exists a unique solution $\{u_n(x), p_n(x)\}$ of the differential-difference systems* (7)–(10) *having the properties*

$$u_n(x) \in H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau), \quad p_n(x) \in H_\tau^{\lambda,\lambda/2}(\overline{Q}_\tau),$$

$$\max_{(x,t_n)\in\overline{Q}_\tau} |p_n(x)| \le p_{\max}, \quad \max_{(x,t_n)\in\overline{Q}_\tau} |p_{nx}(x)| \le p_{x\max}, \quad \max_{(x,t_n)\in\overline{Q}_\tau} |p_{n\bar{t}}(x)| \le p_{t\max}, \tag{13}$$

where

$$p_{\max} = (p^0_{\max} + T\gamma_{\max})\exp(T\chi_{\max}),$$
$$p_{x\max} = \{p^0_{x\max} + T(\gamma_{x\max} + \gamma_{u\max}M_1)\}\exp(T\chi_{\max}),$$
$$p_{t\max} = \chi_{\max}(p^0_{\max} + T\gamma_{\max})\exp(T\chi_{\max}) + \gamma_{\max}.$$

Proof of Lemma 3.2. Starting with the initial conditions for $t_0 = 0$, we assume that for each of time layers t_j ($j = \overline{1, n-1}$) the solutions $\{u_j(x), p_j(x)\}$ are found and the corresponding estimates are established. The conditions of Theorem 3.1 concerning the functions $p^0(x)$, $\gamma(x,t,u)$ and $\chi(t)$ allow one to conclude that for $0 \le x \le l$, $t = t_n$ there hold from (10)

$$|p_n(x)| \le (1 + \tau\chi_{\max})|p_{n-1}(x)| + \tau\gamma_{\max}$$

$$\le (1 + \tau\chi_{\max})^n p^0_{\max} + \sum_{j=0}^{n-1}(1 + \tau\chi_{\max})^j \tau\gamma_{\max},$$

$$\max_{(x,t_n)\in\overline{Q}_\tau} |p_n(x)| \le (p^0_{\max} + T\gamma_{\max})\exp(T\chi_{\max}).$$

Moreover, from (10) it is not difficult to obtain

$$p_{nx}(x) = (1 + \tau\chi_{n-1})p_{n-1x}(x) + \tau(\gamma_{n-1x} + \gamma_{n-1u}u_{n-1x}(x)),$$

$$|p_{nx}(x)| \le (1 + \tau\chi_{\max})^n p^0_{x\max} + \sum_{j=0}^{n-1}(1 + \tau\chi_{\max})^j \tau(\gamma_{x\max} + \gamma_{u\max}M_1),$$

$$\max_{(x,t_n)\in\overline{Q}_\tau} |p_{nx}(x)| \le \{p^0_{x\max} + T(\gamma_{x\max} + \gamma_{u\max}M_1)\}\exp(T\chi_{\max}).$$

Next we note from (10) that

$$\max_{(x,t_n)\in\overline{Q}_\tau} |p_{n\bar{t}}(x)| \le \chi_{\max} p_{\max} + \gamma_{\max}$$
$$\le \chi_{\max}(p^0_{\max} + T\gamma_{\max})\exp(T\chi_{\max}) + \gamma_{\max}.$$

Thus, for $t = t_n$ estimates (13) are received since we assume that the corresponding estimates for t_j ($j = \overline{1, n-1}$) are already known.

As a result of (13) the grid-continuous source function $p_n(x)$, which is determined from (10) by using the given values of $p_{n-1}(x)$ and $u_{n-1}(x)$, belongs to $H^{\lambda,\lambda/2}_\tau(\overline{Q}_\tau)$ with the norm $|\hat{p}(x)|^{\lambda,\lambda/2}_{Q_\tau} \le \mathcal{M}$, where $\mathcal{M} \le p_{\max} + p_{x\max} + p_{t\max}$. This claim easily follows from the definition of the norm in the Hölder class $H^{\lambda,\lambda/2}_\tau(\overline{Q}_\tau)$.

By Lemma 3.1 this means that the differential-difference boundary value problem of the first kind (7)–(9) with such a source function $p_n(x)$ has a unique solution $u_n(x)$ in $H^{2+\lambda,1+\lambda/2}_\tau(\overline{Q}_\tau)$ for which bounds (11) hold. Thus Lemma 3.2 is proved.

Remark 3.3. We have already indicated in Remarks 3.1 and 3.2 that the constants M_0 and M_1 in estimates (11) for $|u_n(x)|$ and $|u_{nx}(x)|$ depend of the value of $p_{\max}$. This means that the present estimates can be derived as soon as the estimate for $|p_n(x)|$ is established.

Passing to stage 3 we note that the uniform estimates (11), (13) (independent of x, τ, and n) mean the compactness of the set $\{u_n(x), p_n(x)\}$ in the corresponding spaces. By taking the limit as τ goes to 0 (i.e., as $n \to \infty$) in conditions (7)–(10), we can show in a standard way that the original problem (1)–(4) has at least one solution $\{u(x,t), p(x,t)\}$ such that $u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q})$, $p(x,t) \in H^{\lambda,\lambda/2}(\overline{Q})$.

Thus the proof of the solvability in the Hölder spaces of nonlinear boundary value problem (1)–(4) by the Rothe method is completed.

3.4. *Proof of uniqueness of the solution* $\{u(x,t), p(x,t)\}$

In order to finish the proof of Theorem 3.1, it remains to show that the solution of problems (1)–(4) is unique in the class of smooth functions

$$\sup_{(x,t)\in\overline{Q}} |u, u_x, u_{xx}, u_t| < \infty, \qquad \sup_{(x,t)\in\overline{Q}} |p, p_x, p_t| < \infty.$$

Assume that for $t \in [0, t^0]$, $0 \le t^0 < T$, the uniqueness is already proved. Let us show the uniqueness result for $t \in [t^0, t^0 + \Delta t]$, where $\Delta t > 0$ is a sufficiently small but bounded time interval that allows us exhaust all the segment $[0, T]$ by a fixed number of steps. We will use a contradiction argument. Assume that for $t \in [t^0, t^0 + \Delta t]$ there exist two solutions of systems (1)–(4) $\{u(x,t), p(x,t)\}$ and $\{\overline{u}(x,t), \overline{p}(x,t)\}$. It is easily seen from (4) that in the domain $\overline{Q}_{t^0} = \{0 \le x \le l, t^0 \le t \le t^0 + \Delta t\}$ the differences

$$\eta(x,t) = u(x,t) - \overline{u}(x,t), \qquad \zeta(x,t) = p(x,t) - \overline{p}(x,t)$$

satisfy the following relationship

$$\zeta_t(x,t) = \chi(t)\zeta(x,t) + \gamma_u(x,t,\overline{u})\eta(x,t).$$

By taking into account that $p(x,t^0) = \overline{p}(x,t^0)$, i.e., $\zeta(x,t^0) = 0$ for $0 \le x \le l$, we obtain

$$\zeta(x,t) = \int_{t^0}^{t} \chi(\tau)\zeta(x,\tau)\,d\tau + \int_{t^0}^{t} \gamma_u(x,\tau,\overline{u}(x,\tau))\eta(x,\tau)\,d\tau.$$

Hence in $\overline{Q}_{t^0} = \{0 \le x \le l, t^0 \le t \le t^0 + \Delta t\}$ there holds

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| \le \Delta t\, \chi_{\max} \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| + \Delta t \gamma_{u\max} \max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)|. \tag{14}$$

Moreover, thanks to (1)–(3) $\eta(x,t)$ and $\zeta(x,t)$ satisfy the relationships

$$c(x,t,u)\eta_t - (a(x,t,u)\eta_x)_x + \mathcal{A}_0\eta_x + \mathcal{A}_1\eta = f(x,t)\zeta(x,t), \quad (x,t) \in Q_{t^0},$$

$$\eta|_{x=0} = 0, \quad \eta|_{x=l} = 0, \quad t^0 < t \le t^0 + \Delta t,$$
$$\eta(x,t^0) = 0, \quad 0 \le x \le l,$$

where the coefficients $\mathcal{A}_0$ and $\mathcal{A}_1$ depend in the corresponding way on the derivatives a_u, a_{xu}, a_{uu}, b_u, c_u, and d_u at the point $(x,t,\sigma u + (1-\sigma)\overline{u})$ $(0 < \sigma < 1)$. Moreover, $\mathcal{A}_0$ and $\mathcal{A}_1$ depend on $u(x,t)$ and the derivatives $u_x(x,t)$, $u_{xx}(x,t)$, and $u_t(x,t)$.

All the input data of this linear boundary value problem of the first kind are uniformly bounded in the domain $\overline{Q}_{t^0}$ as functions of (x,t). This allows one to apply the maximum principle that leads to the following estimate

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| \le K_2 \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)|, \quad K_2 = \text{const} > 0. \tag{15}$$

From (14) by taking into account this estimate we obtain

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| \le \Delta t\, (\chi_{\max} + K_2\gamma_{u\max}) \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)|.$$

Choosing then $\Delta t > 0$ such that

$$\Delta t\, (\chi_{\max} + K_2\gamma_{u\max}) \le 1 - \mu, \quad 0 < \mu < 1,$$

we output the following relationship

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| \le (1-\mu) \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)|,$$

i.e., $\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| = 0$. Thanks to (15) we can conclude from here that $\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| = 0$. Thus, the uniqueness result is completely proved for $t \in [t^0, t^0 + \Delta t]$.

By repeating the analogous arguments for $t \in [t^1, t^2]$ $(t^1 = t^0 + \Delta t,\ t^2 = t^1 + \Delta t)$, $t \in [t^2, t^3]$, etc., up to the final time T, we drive the uniqueness result for problems (1)–(4) on all the segment $[0, T]$.

Thus, there exists a unique solution $\{u(x,t), p(x,t)\}$ of the nonlinear systems (1)–(4) in the class of smooth functions. Theorem 3.1 is completely proved.

3.5. Error estimates of the Rothe method for the boundary conditions of the first kind

Our next aim is to show that the Rothe method is applicable for construction of approximate solutions of the present nonlinear system. It is necessary to estimate the differences

$$\omega_n(x) = u_n(x) - u(x,t_n), \quad \xi_n(x) = p_n(x) - p(x,t_n),$$

where $\{u(x,t_n), p(x,t_n)\}$ solves the original problems (1)–(4) for $t = t_n$, $\{u_n(x), p_n(x)\}$ solves the approximating systems (7)–(10).

Theorem 3.2. *Assume that the input data satisfy the conditions of Theorem 3.1. Then for any sufficiently small time-step τ of the grid $\overline{\omega}_\tau$ there hold the error estimates for the Rothe method*

$$\max_{(x,t_n)\in\overline{Q}_\tau} |\omega_n(x)| \le K_3(\Psi + \psi), \qquad \max_{(x,t_n)\in\overline{Q}_\tau} |\xi_n(x)| \le K_4(\Psi + \psi), \tag{16}$$

where $\Psi = \max_{(x,t_n)\in\overline{Q}_\tau} \Psi_n(x)$, $\psi = \max_{(x,t_n)\in\overline{Q}_\tau} \psi_n(x)$, $\Psi_n(x)$ is the discretization error for the differential-difference boundary value problems (7)–(9) and $\psi_n(x)$ is the discretization error for equation (10), K_3 and K_4 are positive constants independent of x, t, τ, and n.

The proof repeats—with the corresponding modification—the above proof of the uniqueness result in Theorem 3.1. We only note that estimates (16) are shown step by step for the bounded time intervals $[0, t_{n_0}]$, $[t_{n_0}, t_{n_1}]$, $[t_{n_1}, t_{n_2}]$, etc., up to the final time $t_N = T$. Existence of such estimates allows one to apply the Rothe method for approximate solving the nonlinear problems (1)–(4) with the unknown source function. The solution $\{u(x,t), p(x,t)\}$ can be obtained as the limit of the solution $\{u_n(x), p_n(x)\}$ of the approximating systems (7)–(10) as the time-step τ of the grid $\overline{\omega}_\tau$ goes to 0.

4. The nonlinear parabolic problem with the boundary conditions of the second kind

4.1. The statement and conditions of unique solvability in the Hölder spaces

Now we consider the nonlinear parabolic problem which is distinguished by the boundary conditions from system (1)–(4)

$$a(x,t)u_x|_{x=0} = g(t), \quad a(x,t)u_x|_{x=l} = q(t), \quad 0 < t \le T. \tag{17}$$

Here and in what follows we assume that the coefficient a has the form $a = a(x,t)$. Conditions of unique solvability of such a nonlinear parabolic problem with an unknown source function are established by the following theorem.

Theorem 4.1. *Let the following conditions be satisfied.*

1. *For $(x,t) \in \overline{Q}$ and any u, $|u| < \infty$, the input data of the corresponding problem for equation (1) with the boundary conditions (17) are uniformly bounded functions of their arguments, where the coefficient $a(x,t)$ — together with the derivatives $a_x(x,t)$ and $a_t(x,t)$; moreover, $a_x(x,t)$ and $f(x,t)$ are in $H^{\lambda,\lambda/2}(\overline{Q})$, there hold*

$$0 < a_{\min} \le a(x,t) \le a_{\max}, \; 0 < c_{\min} \le c(x,t,u) \le c_{\max},$$

2. *For $(x,t,u) \in \overline{D} = \overline{Q} \times [-\overline{M}_0, \overline{M}_0]$ ($\overline{M}_0 > 0$ is the constant from the maximum principle for boundary value problem (1), (17),(3)) the functions $b(x,t,u)$ and $d(x,t,u)$ are Hölder continuous in x, t with the corresponding exponents λ and $\lambda/2$ and have the uniformly bounded derivatives with respect to u; the function $c(x,t,u)$ is in $H^{1,\lambda/2,1}(\overline{D})$.*
3. *The functions $\varphi(x)$, $g(t)$, and $q(t)$ are in $H^{2+\lambda}[0,l]$ and $O^1[0,T]$, respectively; there hold the matching conditions $a(x,0)\varphi_x|_{x=0} = g(0)$, $a(x,0)\varphi_x|_{x=l} = q(0)$.*
4. *The functions $p^0(x)$ and $\chi(t)$ are in $C^1[0,l]$ and $C[0,T]$, respectively,*

$$\max_{0\le x\le l} |p^0(x)| \le p^0_{\max}, \; \max_{0\le x\le l} |p^0_x(x)| \le p^0_{x\max}, \; \max_{0\le t\le T} |\chi(t)| \le \chi_{\max},$$

where $p^0_{\max}$, $p^0_{x\max}$, $\chi_{\max} = \text{const} > 0$*; the function* $\gamma(x,t,u)$ *is uniformly bounded for* $(x,t) \in \overline{Q}$, $|u| < \infty$, *and is continuous for* $(x,t,u) \in \overline{D}$ *together with the derivatives with respect to x and u,*

$$|\gamma(x,t,u)| \le \gamma_{\max}, \quad \max_{(x,t,u)\in\overline{D}} |\gamma_x(x,t,u)| \le \overline{\gamma}_{x\max}, \quad \max_{(x,t,u)\in\overline{D}} |\gamma_u(x,t,u)| \le \overline{\gamma}_{u\max},$$

where $\gamma_{\max}$, $\overline{\gamma}_{x\max}$, $\overline{\gamma}_{u\max} = \text{const} > 0$.

Then the present nonlinear parabolic problem has a unique solution $\{u(x,t), p(x,t)\}$ *which satisfies the conditions*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad p(x,t) \in H^{\lambda,\lambda/2}(\overline{Q}),$$
$$|u(x,t)|^{2+\lambda,1+\lambda/2}_{\overline{Q}} \le \overline{M}, \quad |p(x,t)|^{\lambda,\lambda/2}_{\overline{Q}} \le \overline{\mathcal{M}}, \quad \overline{M}, \overline{\mathcal{M}} = \text{const} > 0.$$

This solution is the limit of the solution $\{u_n(x), p_n(x)\}$ *of the corresponding differential-difference nonlinear system that approximates the original system by the Rothe method.*

4.2. The scheme of proof of Theorem 4.1

Claims of Theorem 4.1 are proved by analogy with the proof of the corresponding claims of Theorem 3.1. Namely, solvability of the original nonlinear system with the boundary conditions (17) is established with the help of the differential-difference approximation of this system, which is is distinguished from (7)–(10) by the boundary conditions

$$a_n u_{nx}|_{x=0} = g_n, \quad a_n u_{nx}|_{x=l} = q_n, \quad 0 < t_n \le T, \tag{18}$$

where $a_n = a(x,t_n)$, $g_n = g(t_n)$, and $q_n = q(t_n)$.

Investigation of the present approximation involves several stages similar to ones in section 3. In the corresponding stage 1 we take into account specific properties of conditions (18) and establish unique solvability in $H^{2+\lambda,1+\lambda/2}_\tau(\overline{Q}_\tau)$ of the differential-difference boundary value problem of the second kind under assumption that the source function $p_n(x)$ in equation (7) is given. Moreover, $p_n(x)$ is in $H^{\lambda,\lambda/2}_\tau(\overline{Q}_\tau)$ with $|\hat{p}(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} \le \overline{\mathcal{M}}$, $\overline{\mathcal{M}} = \text{const} > 0$, $\max_{(x,t_n)\in\overline{Q}_\tau} |p_n(x)| \le p_{\max}$, $p_{\max} = \text{const} > 0$.

Lemma 4.1. *Assume that the conditions 1–3 of Theorem 4.1 hold and let* $p_n(x)$ *be a functon with the above-mentioned properties. Then for any sufficiently small time-step* τ *of the grid* $\overline{\omega}_\tau$ *there exists one and only one solution of the differential-difference boundary value problem with conditions (18) which belongs to the class* $H^{2+\lambda,1+\lambda/2}_\tau(\overline{Q}_\tau)$ *and satisfies the estimates*

$$\max_{(x,t_n)\in\overline{Q}_\tau} |u_n(x)| \le \overline{M}_0, \quad \max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| \le \overline{M}_1,$$

$$|\hat{u}(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} \le \overline{M}_2, \quad |\hat{u}_x(x)|^{\lambda,\lambda/2}_{\overline{Q}_\tau} \le \overline{M}_3, \quad |\hat{u}(x)|^{2+\lambda,1+\lambda/2}_{\overline{Q}_\tau} \le \overline{M}_4, \tag{19}$$

where $\overline{M}_i > 0$ $(i = \overline{0,4})$ *are positive constants independent of x,* τ*, and n.*

The conclusion of Lemma 4.1 is based on results of Theorem 1 from [5] which we apply by the corresponding way to problem (7), (18), (9). Note that this theorem about unique solvability of the differential-difference boundary value problem of the second kind in the Hölder class $H^{2+\lambda,1+\lambda/2}_\tau(\overline{Q}_\tau)$ is proved by the Leray-Schauder principle on the existence of the fixed points of the completely continuous transforms. In order to apply this principle, *a priori* estimates in $H_\tau^{1+\lambda,\frac{1+\lambda}{2}}(\overline{Q}_\tau)$ for $u_n(x)$ must be derived. For the present nonlinear problem we make the following remarks.

Remark 4.1. For problem (7), (18), (9) the constant $\overline{M}_0$ from the maximum principle has the form

$$\overline{M}_0 = K_6 T \exp(K_5 T) + K_7 l\left(1 + \frac{l}{4}\right), \tag{20}$$

in which K_5, K_6, and K_7 are positive constants,

$$\begin{aligned} K_5 &\geq (1+\varepsilon) d_{\max} c_{\min}^{-1},\ \tau \leq \tau_0 = \varepsilon K_5^{-1}, \varepsilon > 0 \textit{ is arbitrary},\\ K_6 &\geq c_{\min}^{-1}\Big\{f_{\max} p_{\max} + 2K_7 a_{\max} + K_7 l(a_{x\max} + b_{\max} + (1+\tfrac{l}{4})d_{\max})\Big\},\\ K_7 &\geq \max(l^{-1}\varphi_{\max},\ l^{-1}a_{\min}^{-1} g_{\max},\ l^{-1}a_{\min}^{-1} q_{\max}). \end{aligned}$$

In order to derive the estimate of the maximum principle auxiliary functions are used (for details see Lemma 1 from [5])

$$\eta_n^{\pm}(x) = (1 + K_5\tau)^{-n}\Big\{u_n(x) \pm K_7(x - \frac{l}{2})^2 \pm K_7 l\Big\} \pm K_6 t_n.$$

Remark 4.2. In order to obtain the estimate $\max_{(x,t_n)\in\overline{Q}_\tau} |u_{nx}(x)| \leq \overline{M}_1$ for problem (7), (18), (9), we apply the approach proposed in [5]. This approach allows one to avoid differentiating equation (7) with respect to x and hence does not require additional smoothness of the input data. Here we only sketch the proof, for details see Lemma 2 from [5]. A substitution is carried out

$$\vartheta_n(x) = u_n(x) - x^2\psi_n^l + (x-l)^2\psi_n^0, \quad (x, t_n) \in \overline{Q}_\tau,$$

$$\psi_n^0 = g_n\left(2la_n|_{x=0}\right)^{-1}, \quad \psi_n^l = q_n\left(2la_n|_{x=l}\right)^{-1}, \quad n = \overline{1, N},$$

that reduces the boundary conditions at $x = 0$ and $x = l$ to homogeneous ones

$$\vartheta_{nx}(x) = u_{nx}(x) - 2x\psi_n^l + 2(x-l)\psi_n^0, \quad \vartheta_{nx}(x)|_{x=0} = 0, \quad v_{nx}(x)|_{x=l} = 0.$$

Next we use even extension of the function $\vartheta_n(x)$ into domains $Q_\tau^- = \{-l < x < 0\} \times \omega_\tau$ and $Q_\tau^+ = \{l < x < 2l\} \times \omega_\tau$ with the next introduction of an additional space variable z and a function $W_n(x, z) = \vartheta_n(x) - \vartheta_n(z)$. The main aim is to obtain the estimate $|W_n(x, z)| \leq \overline{M}_1|x - z|$ that leads to the estimate of the derivative $\vartheta_{nx}(x)$, i.e., to the desired estimate for $u_{nx}(x)$ in (19).

The constant $\overline{M}_1$ depends on values $\overline{M}_0$, $p_{\max}$, $\varphi_{x\max}$, $g_{\max}$, $q_{\max}$. We especially note that $\overline{M}_1$ is independent of the derivative $p_{nx}(x)$ in $\overline{Q}_\tau$ thanks to the proposed approach.

The next stage in proof of Theorem 4.1 is to consider the corresponding approximate system with boundary conditions (18) for determination of $\{u_n(x), p_n(x)\}$.

Lemma 4.2. *Assume that the input data satisfy the hypotheses of Theorem 4.1. Then in the domain $\overline{Q}_\tau$ for any time-step $\tau \leq \tau_0$ ($\tau_0 > 0$ is the constant defined by estimate (20)) there exists a unique solution $\{u_n(x), p_n(x)\}$ of the present differential-difference system having the properties*

$$u_n(x) \in H_\tau^{2+\lambda, 1+\lambda/2}(\overline{Q}_\tau), \quad p_n(x) \in H_\tau^{\lambda,\lambda/2}(\overline{Q}_\tau),$$

$$\max_{(x,t_n)\in\overline{Q}_\tau} |p_n(x)| \leq p_{\max}, \quad \max_{(x,t_n)\in\overline{Q}_\tau} |p_{nx}(x)| \leq \overline{p}_{x\max}, \quad \max_{(x,t_n)\in\overline{Q}_\tau} |p_{n\bar{t}}(x)| \leq p_{t\max}, \tag{21}$$

where $p_{\max}$, $\overline{p}_{x\max}$, $p_{t\max}$ are positive constants independent of x, τ, and n.

The proof of Lemma 4.2 repeats the above proof of Lemma 3.2 with the corresponding use of *a priori* estimates of Lemma 4.1. The values of constants in estimates (21) are determined by analogy with ones in Lemma 3.2. These estimates mean that the grid-continuous source function $p_n(x)$, which is determined from (10) by using the given values of $p_{n-1}(x)$ and $u_{n-1}(x)$, belongs to $H_\tau^{\lambda,\lambda/2}(\overline{Q}_\tau)$ with the norm $|\hat{p}(x)|_{Q_\tau}^{\lambda,\lambda/2} \le \overline{\mathcal{M}}$, where $\overline{\mathcal{M}} \le p_{\max} + \overline{p}_{x\max} + p_{t\max}$. Hence by Lemma 4.1 the differential-difference boundary value problem of the second kind corresponding to this source function $p_n(x)$ has a unique solution $u_n(x)$ in $H_\tau^{2+\lambda,1+\lambda/2}(\overline{Q}_\tau)$ for which bounds (19) hold.

Remark 4.3. In Remarks 4.1 and 4.2 it is indicated that the constants $\overline{M}_0$ and $\overline{M}_1$ in estimates (19) for $|u_n(x)|$ and $|u_{nx}(x)|$ depend of the value $p_{\max}$. This means that the present estimates can be obtained as soon as the estimate for $|p_n(x)| \le p_{\max}$ is established.

As a result of the uniform estimates (19), (21) (independent of x, τ, and n) the set $\{u_n(x), p_n(x)\}$ is compact in the corresponding spaces. Passing to the last stage 3 we take the limit as τ goes to 0 (i.e., as $n \to \infty$) in conditions (7), (18), (9), and (10). This allows one to show in a standard way that the original problem with the boundary conditions of the second kind (17) has at least one solution $\{u(x,t), p(x,t)\}$ such that $u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q})$, $p(x,t) \in H^{\lambda,\lambda/2}(\overline{Q})$. Thus the solvability in the Hölder spaces of the present nonlinear boundary value problem is proved.

In order to complete the proof of Theorem 4.1, it remains to show the uniqueness of the solution $\{u(x,t), p(x,t)\}$ in the class of smooth functions. The present result is established by analogy with the corresponding result in Theorem 3.1 step by step for the bounded time intervals that allows us exhaust all the segment $[0,T]$ by a fixed number of steps. By a contradiction argument we assume that for $t \in [t^0, t^0+\Delta t]$ there exist two solutions $\{u(x,t), p(x,t)\}$ and $\{\overline{u}(x,t), \overline{p}(x,t)\}$. The corresponding linear boundary value problem for the differences $\eta(x,t) = u(x,t) - \overline{u}(x,t)$, $\zeta(x,t) = p(x,t) - \overline{p}(x,t)$ has the form in the domain $\overline{Q}_{t^0} = \{0 \le x \le l, t^0 \le t \le t^0+\Delta t\}$

$$c(x,t,u)\eta_t - (a(x,t)\eta_x)_x + \mathcal{A}_0\eta_x + \mathcal{A}_1\eta = f(x,t)\zeta(x,t), \quad (x,t) \in Q_{t^0},$$

$$a(x,t)\eta_x|_{x=0} = 0, \quad a(x,t)\eta_x|_{x=l} = 0, \quad t^0 < t \le t^0+\Delta t,$$

$$\eta(x,t^0) = 0, \quad 0 \le x \le l,$$

where all the input data are uniformly bounded in the domain $\overline{Q}_{t^0}$ as functions of (x,t) thanks to the established estimates. This allows one to apply the maximum principle that leads to the following estimate [1]

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| \le K_8\Delta t \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)|, \quad K_8 = \text{const} > 0.$$

Moreover,

$$\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| \le \Delta t\,\chi_{\max} \max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| + \Delta t\,\overline{\gamma}_{u\max} \max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)|.$$

By repeating the corresponding reasonings in the proof of Theorem 3.1 we obtain choosing Δt from the condition

$$\Delta t\,(\chi_{\max} + K_8\overline{\gamma}_{u\max}) \le 1-\mu, \quad 0<\mu<1,$$

that $\max_{(x,t)\in\overline{Q}_{t^0}} |\zeta(x,t)| = 0$, $\max_{(x,t)\in\overline{Q}_{t^0}} |\eta(x,t)| = 0$. Thus, the uniqueness result is proved for $t \in [t^0, t^0+\Delta t]$. By repeating the analogous arguments step by step for the next bounded time intervals we drive the uniqueness result on all the segment $[0,T]$.

Thus, Theorem 4.1 on unique solvability of the nonlinear parabolic problem with the boundary conditions of the second kind is completely proved.

4.3. Error estimates of the Rothe method for the boundary conditions of the second kind

In order to show that the Rothe method allows one to obtain approximate solutions for the considered nonlinear system, it is necessary to estimate the differences

$$\omega_n(x) = u_n(x) - u(x, t_n), \quad \xi_n(x) = p_n(x) - p(x, t_n),$$

where $\{u(x, t_n), p(x, t_n)\}$ solves the original problem with the boundary conditions (17) for $t = t_n$, $\{u_n(x), p_n(x)\}$ solves the approximating system with conditions (18).

Theorem 4.2. *Let the conditions of Theorem 4.1 be satisfied. Then for any sufficiently small time-step* τ *of the grid* $\overline{\omega}_\tau$ *there hold the error estimates for the Rothe method*

$$\max_{(x,t_n)\in\overline{Q}_\tau} |\omega_n(x)| \le K_9(\Psi + \psi), \qquad \max_{(x,t_n)\in\overline{Q}_\tau} |\xi_n(x)| \le K_{10}(\Psi + \psi), \tag{22}$$

where $\Psi = \max_{(x,t_n)\in\overline{Q}_\tau} \Psi_n(x)$, $\psi = \max_{(x,t_n)\in\overline{Q}_\tau} \psi_n(x)$, $\Psi_n(x)$ *is the discretization error for the differential-difference boundary value problem with conditions (18) and* $\psi_n(x)$ *is the discretization error for equation (10),* K_9 *and* K_{10} *are positive constants independent of* x*,* t*,* τ*, and* n*.*

This theorem is analogous to Theorem 3.2. Estimates (22) are derived step by step for the bounded time intervals up to the final time $t_N = T$. From (22) it follows that the solution $\{u(x, t), p(x, t)\}$ can be obtained as the limit of the solution $\{u_n(x), p_n(x)\}$ of the corresponding approximate system as the time-step τ of the grid $\overline{\omega}_\tau$ goes to 0.

5. Mathematical models of some filtration processes

The nonlinear parabolic problems that are investigated in sections 3, 4 have the wide applications. In particular, such statements are motivated by the needs of the modeling and control of nonstationary filtration processes in underground hydrodynamics. Below as an example we show a mathematical model that arises in exploitation of oil-gas fields in the case of cracked porous media (see, e.g., [6, 7]).

The present statement is connected with nonstationary filtration of liquid to vertical bore-hole in a circular stratum—to find the pressure distribution in cracked blocks that satisfies the relationships in the cylindrical coordinate system (r, t):

$$\beta_{cr} u_t = \mu^{-1} r^{-1} \big(k(u) r u_r\big)_r + \mu^{-1}\alpha(p - u), \quad (r, t) \in Q = \{r_{bh} < r < r_{fc}, 0 < t \le T\}, \tag{23}$$

$$u(r, t)|_{r=r_{bh}} = u_{bh}, \quad u(r, t)|_{r=r_{fc}} = u_{fc}, \quad 0 < t \le T, \tag{24}$$

$$\beta_{pb} p_t = -\mu^{-1}\alpha(p - u), \quad (r, t) \in Q, \tag{25}$$

$$u(r, t)|_{t=0} = \varphi(r), \quad p(r, t)|_{t=0} = \varphi(r), \quad r_{bh} \le r \le r_{fc}. \tag{26}$$

Here $u(r, t)$ is the pressure in the cracks, $p(r, t)$ is the pressure in the porous blocks, β_{cr} and β_{pb} are the corresponding coefficients of elastic capacity, r_{bh} is the radius of the bore-hole, r_{fc} is the radius of the feed contour, u_{bh} and u_{fc} are the pressure corresponding to these boundaries, $\varphi(r)$ is the initial pressure distribution in the stratum, μ is the liquid viscosity, α is the parameter of liquid shift between the blocks and the cracks, k represents the permeability of the stratum.

It is known that filtration properties of cracked porous stratums depend of changes of the pressure. Mathematical statements that are considered in section 3 for a quasilinear parabolic equation, allow one to take into account this dependence. In particular, the coefficient k in (23) has the form $k(u)$.

Besides (24) the boundary conditions in systems (23)–(26) can be given in the other form, in particular, of the second kind

$$2\pi H\mu^{-1}(k(u)ru_r)|_{r=r_{bh}} = q(t), \quad 2\pi H\mu^{-1}(k(u)ru_r)|_{r=r_{fc}} = 0, \quad 0 < t \le T,$$

or of the mixed kind

$$2\pi H\mu^{-1}(k(u)ru_r)|_{r=r_{bh}} = q(t), \quad u(r,t)|_{r=r_{fc}} = u_{fc}, \quad 0 < t \le T,$$

where H is the tickness of the stratum, $q(t)$ is the debit.

All these models can be considered as concrete examples of the nonlinear parabolic problems that are investigated in sections 3, 4. The pressure $p(r,t)$ in the porous blocks plays the role of sought source function in equation (23). The corresponding equation for a time dependence of $p(r,t)$ has the form (25).

6. Conclusion

In this work the nonlinear parabolic problems with an unknown source function are investigated. They are formulated as a system that involves a boundary value problem for a quasilinear parabolic equation and, moreover, an additional relationship for a time dependence of this sought function. Our main aim is to justify such problems in a class of smooth functions taking into account their essential distinction from usual boundary value problems. The following results of our analysis can be formulated—conditions of unique solvability in the Hölder spaces are proved for the corresponding nonlinear system with the boundary conditions both of the first and second kind.

To this end, *a priori* estimates in the corresponding spaces are established for the nonlinear differential-difference system that approximates the original system by the Rothe method. Thanks to these estimates in the differential-continuous analogs of Hölder classes we avoid additional assumptions of the smoothness of the input data (which have usually been imposed by the Rothe method). Thus, the proposed approach allows one to determine the faithful character of differential relations for the nonlinear parabolic problems of the considered type. These results are similar to the ones obtained in [1] for the boundary value problems in the case of quasilinear parabolic equations with the given right-hand side.

The error estimates established in the work for the Rothe method show that this method provides the approximate solutions for the present nonlinear parabolic problems. As an example of important applications, a model of nonstationary filtration in the cracked porous stratum is represented.

Acknowledgments

This work was supported by M.V. Lomonosov Moscow State University (the program Exposure and Support of New Perspective Scientific Directions).

Conflict of interest

The author declares no conflict of interest.

References

1. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Providence: American Mathematical Soc., 1968.
2. A. Friedman, *Partial Differential Equations of Parabolic Type*, Englewood Cliffs: Prentice Hall, 1964.
3. N. L. Gol'dman, *Inverse Stefan Problems*, Dordrecht: Kluwer Academic, 1997.
4. S. N. Kruzhkov, *A priori estimates of the derivative of a solution to a parabolic equation*, Vestnik Mosk. Univ. Ser. I Mat. Mekh., **2** (1967), 41–48.
5. N. L. Gol'dman, *Boundary value problems for a quasilinear parabolic equation with an unknown coefficient*, J. Differ. Eq., **266** (2019), 4925–4952.
6. G. I. Barenblatt, V. M. Entov, V. M. Ryzhik, *Theory of Fluid Flows Through Natural Rocks*, Netherlands: Springer, 1989.
7. G. G. Cherny, *Selected Works*, Moscow: Nauka, 2009.