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Research article

Convexity and inequalities related to extended beta and confluent hypergeometric functions

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Abstract: In the paper, the authors establish the logarithmic convexity and some inequalities for the extended beta function and, by using these inequalities for the extended beta function, find the logarithmic convexity and the monotonicity for the extended confluent hypergeometric function.

Keywords: extended beta function; extended confluent hypergeometric function; inequality; logarithmic convexity; monotonicity
Mathematics Subject Classification: Primary 33B15; Secondary 26D15, 33B99

1. Preliminaries

In [6], the extended beta function was defined by

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-p/t(1-t)}\,\mathrm{d}\,t, \tag{1.1}$$

where $\Re(p), \Re(x), \Re(y) > 0$. It is clear that, if $p = 0$, then $B(x, y; 0) = B(x, y)$ is just the classical beta function [28].

In [7], the extended confluent hypergeometric function was defined as

$$\Phi_p(\beta,\gamma;z)=\sum_{n=0}^{\infty}\frac{B(\beta+n,\gamma-\beta;p)}{B(\beta,\gamma-\beta)}\frac{z^n}{n!},$$

where $\Re(p) > 0$ and $\beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0, -1, -2, \ldots$. An integral representation of $\Phi_p(\beta, \gamma; z)$ was given in [7, Eq. (3.7)] by

$$\Phi_p(\beta, \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \exp\left(zt - \frac{p}{t(1-t)}\right) \mathrm{d}\, t \tag{1.2}$$

for $p \geq 0$ and $\Re(\gamma) > \Re(\beta) > 0$.

In [30], the extended beta function $B(x, y; p)$ defined by (1.1) was generalized as

$$B_\lambda^p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_\lambda\left(-\frac{p}{t(1-t)}\right) \mathrm{d}\, t, \tag{1.3}$$

where $\Re(p), \Re(x), \Re(y), \lambda > 0$, $E_\lambda(x) = E_{\lambda,1}(x)$ denotes the Mittag–Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and $\Gamma(z)$ is the classical Euler gamma function which can be defined [16, 23, 36] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$$

or by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0.$$

It is clear that

$$B_1^p(x, y) = B(x, y; p) \quad \text{and} \quad B_1^1(x, y) = B(x, y).$$

It is well known [11, 15] that, when $\lambda \in [0, 1]$, the Mittag-Leffler function $E_\lambda(-w)$ is completely monotonic on $(0, \infty)$. Hence, when $\lambda \in [0, 1]$, the Mittag-Leffler function $E_\lambda(-w)$ is positive on $(0, \infty)$. For detailed information on complete monotonicity, please refer to [17, 24] and the closely related references therein.

In [30], the extended confluent hypergeometric function $\Phi_p(\beta, \gamma; z)$ defined by (1.2) was generalized as

$$\Phi_p^\lambda(\beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{B_\lambda^p(\beta + n, \gamma - \beta)}{B(\beta, \gamma - \beta)} \frac{z^n}{n!}. \tag{1.4}$$

In [30, Eq. (31)], an integral representation of $\Phi_p^\lambda(\beta, \gamma; z)$ was given by

$$\Phi_p^\lambda(\beta, \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \exp(zt) E_\lambda\left(-\frac{p}{t(1-t)}\right) \mathrm{d}\, t \tag{1.5}$$

for $p \geq 0$, $\lambda > 0$, and $\Re(\gamma) > \Re(\beta) > 0$.

It is obvious that $\Phi_p^1(\beta, \gamma, z) = \Phi_p(\beta, \gamma; z)$ and $\Phi_0^1(\beta, \gamma, z) = \Phi(\beta, \gamma; z)$ which is the classical confluent hypergeometric series [28] and is the limit case of the Gauss hypergeometric function [27, 32–35].

This paper is organized as follows. In Section 2, we recall definitions of some convex functions and recite several lemmas needed in this paper. In Section 3, we present some inequalities for extended beta functions $B_\lambda^p(x, y)$ defined in (1.3). In Section 4, we find the monotonicity and the logarithmic convexity for functions related to extended confluent hypergeometric functions $\Phi_p(\beta, \gamma; z)$ defined in (1.4).

2. Definitions and lemmas

Now we recall definitions of some convex functions and recite several lemmas.

Definition 2.1 ([5, 22])**.** Let X be a convex set in a real vector space and let $f : X \to \mathbb{R}$ be a function. Then f is said to be convex on X if the inequality

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2)$$

is valid for any $x_1, x_2 \in X$ and $\alpha \in [0, 1]$.

A function f is said to be concave if $-f$ is convex.

A function f is said to be logarithmically convex (or logarithmically concave respectively) on X if $f > 0$ and $\ln f$ (or $-\ln f$ respectively) is convex (or concave respectively) on X.

Lemma 2.1 ([8, 10, 21])**.** *Let $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be integrable and satisfy*

$$[f(x) - f(y)][g(x) - g(y)] \gtreqless 0$$

for all $x, y \in [a, b]$ and let $p(x) : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a positive integrable function. Then

$$\int_a^b p(x) f(x) \,\mathrm{d}\, x \int_a^b p(x) g(x) \,\mathrm{d}\, x \gtreqless \int_a^b p(x) \,\mathrm{d}\, x \int_a^b p(x) f(x) g(x) \,\mathrm{d}\, x. \tag{2.1}$$

Lemma 2.2 ([29, 31])**.** *Let θ_1 and θ_2 be positive numbers such that $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$ and let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then*

$$\left|\int_a^b f(x) g(x) \,\mathrm{d}\, x\right| \le \left(\int_a^b |f(x)|^{\theta_1} \,\mathrm{d}\, x\right)^{1/\theta_1} \left(\int_a^b |g(x)|^{\theta_2} \,\mathrm{d}\, x\right)^{1/\theta_2}. \tag{2.2}$$

Lemma 2.3 ([4])**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, with $a_n \in \mathbb{R}$ and $b_n > 0$ for all n, converge on $(-\alpha, \alpha)$. If the sequence $\{\frac{a_n}{b_n}\}_{n \ge 0}$ is increasing (or decreasing respectively), then $x \mapsto \frac{f(x)}{g(x)}$ is also increasing (or decreasing respectively) on $(0, \alpha)$.*

3. Inequalities for extended beta functions

Now we start off to establish inequalities for functions involving extended beta functions.

Theorem 3.1. *If x, y, x_1, y_1 are positive numbers such that $(x - x_1)(y - y_1) \ge 0$, then, when $\lambda \in [0, 1]$,*

$$B_p^\lambda(x, y_1) B_p^\lambda(x_1, y) \le B_p^\lambda(x_1, y_1) B_p^\lambda(x, y). \tag{3.1}$$

Proof. Consider the mappings $f, g, h : [0, 1] \to [0, \infty)$ given by $f(t) = t^{x - x_1}$, $g(t) = (1-t)^{y - y_1}$, and

$$h(t) = t^{x_1 - 1}(1-t)^{y_1 - 1} E_\lambda\left(-\frac{p}{t(1-t)}\right).$$

Since $f'(t) = (x - x_1) t^{x - x_1 - 1}$ and $g'(t) = (y_1 - y)(1-t)^{y - y_1 - 1}$, the functions f and g have the same monotonicity on $[0, 1]$. Applying Chebyshev's integral inequality (2.1) to f, g, and h, we have

$$\int_a^b t^{x-1}(1-t)^{y_1-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\int_a^b t^{x_1-1}(1-t)^{y-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t$$
$$\le\int_a^b t^{x_1-1}(1-t)^{y_1-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\int_a^b t^{x-1}(1-t)^{y-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t.$$

This can be rearranged as (3.1). The proof of Theorem 3.1 is complete. □

Corollary 3.1. *For $x, y > 0$ and $\lambda \in [0, 1]$, we have*

$$B_\lambda^p(x,y) \ge [B_\lambda^p(x,x)B_\lambda^p(y,y)]^{1/2}.$$

Proof. This follows from Theorem 3.1 directly. □

Theorem 3.2. *The function $(x, y) \mapsto B_\lambda^p(x, y)$ is logarithmically convex on $(0,\infty)\times(0,\infty)$ for all $p \ge 0$ and $0 \le \lambda \le 1$. Consequently,*

$$\Big[B_\lambda^p\Big(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\Big)\Big]^2 \le B_\lambda^p(x_1,y_1)B_\lambda^p(x_2,y_2). \tag{3.2}$$

Proof. Let $(x_1, y_1), (x_2, y_2) \in (0,\infty)^2$ and let $c, d \ge 0$ with $c + d = 1$. Then

$$B_\lambda^p(c(x_1,y_1)+d(x_2,y_2)) = B_\lambda^p(cx_1+\mathrm{d}\,x_2, cy_1+dy_2).$$

By definition, we have

$$B_\lambda^p(c(x_1,y_1)+d(x_2,y_2)) = \int_0^1 t^{cx_1+\mathrm{d}\,x_2-1}(1-t)^{cy_1+dy_2-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t$$
$$=\int_0^1 t^{cx_1+\mathrm{d}\,x_2-(c+d)}(1-t)^{cy_1+dy_2-(c+d)}\Big[E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big]^{c+d}\,\mathrm{d}\,t$$
$$=\int_0^1 t^{c(x_1-1)}t^{d(x_2-1)}(1-t)^{c(y_1-1)}(1-t)^{d(y_2-1)}\Big[E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big]^c\Big[E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big]^d\,\mathrm{d}\,t$$
$$=\int_0^1\Big[t^{x_1-1}(1-t)^{y_1-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big]^c\Big[t^{x_2-1}t^{y_2-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big]^d\,\mathrm{d}\,t.$$

Setting $\theta_1 = \frac{1}{c}$ and $\theta_2 = \frac{1}{d}$ and using the Hölder inequality (2.2) give

$$B_\lambda^p(c(x_1,y_1)+d(x_2,y_2)) \le \Big[\int_0^1 t^{x_1-1}(1-t)^{y_1-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\Big]^c\Big[\int_0^1 t^{x_2-1}t^{y_2-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\Big]^d$$
$$=\Big[B_\lambda^p(x_1,y_1)\Big]^c[B_\lambda^p(x_2,y_2)]^d.$$

Accordingly, the function $B_\lambda^p(x, y)$ is logarithmically convex on $(0,\infty)^2$.

When $c = d = \frac{1}{2}$, the above inequality reduces to (3.2). The proof of Theorem 3.2 is complete. □

Remark 3.1. Letting $x, y > 0$ such that $\min_{a\in\mathbb{R}}(x + a, x - a) > 0$ and taking $x_1 = x + a$, $x_2 = x - a$, $y_1 = y + b$, and $y_2 = y - b$ in (3.2) result in

$$[B_\lambda^p(x,y)]^2 \le B_\lambda^p(x+a,y+b)B_\lambda^p(x-a,y-b)$$

for all $p \ge 0$ and $\lambda > 0$.

4. Inequalities for extended confluent hypergeometric functions

Now we find the logarithmic convexity and the monotonicity related to the extended confluent hypergeometric function $\Phi_p^\lambda(\beta, \gamma; z)$ defined in (1.4).

Theorem 4.1. *Let $\beta \geq 0$ and $\gamma, \delta > 0$.*

1. *For $\gamma \geq \delta$, the function $x \mapsto \frac{\Phi_p^\lambda(\beta,\gamma;x)}{\Phi_p^\lambda(\beta,\delta;x)}$ is increasing on $(0, \infty)$.*
2. *For $\gamma \geq \delta$,*
$$\delta\Phi_p^\lambda(\beta + 1, \gamma + 1; x)\Phi_p^\lambda(\beta, \delta; x) \geq \gamma\Phi_p^\lambda(\beta, \gamma; x)\Phi_p^\lambda(\beta + 1, \delta + 1; x). \tag{4.1}$$
3. *The function $x \mapsto \Phi_p^\lambda(\beta, \gamma; x)$ is logarithmically convex on $\mathbb{R}$.*
4. *For $\sigma, \gamma, x > 0$, the function*
$$\beta \mapsto \frac{B(\beta, \gamma)\Phi_p^\lambda(\beta + \sigma, \gamma; x)}{B(\beta + \sigma, \gamma)\Phi_p^\lambda(\beta, \gamma; x)}$$
is decreasing on $(0, \infty)$.

Proof. By the definition in (1.4), we have
$$\frac{\Phi_p^\lambda(\beta, \gamma; x)}{\Phi_p^\lambda(\beta, \delta; x)} = \frac{\sum_{n=0}^\infty a_n(c)x^n}{\sum_{n=0}^\infty a_n(d)x^n},$$
where
$$a_n(z) = \frac{B_\lambda^p(\beta + n, z - \beta)}{B_\lambda^p(\beta, z - \beta)}.$$
If denoting $f_n = \frac{a_n(c)}{a_n(d)}$, then
$$f_n - f_{n+1} = \frac{a_n(c)}{a_n(d)} - \frac{a_{n+1}(c)}{a_{n+1}(d)} = \frac{B(\beta, \delta - \beta)}{B(\beta, \gamma - \beta)}\left[\frac{B_\lambda^p(\beta + n, \gamma - \beta)}{B_\lambda^p(\beta + n, \delta - \beta)} - \frac{B_\lambda^p(\beta + n + 1, \gamma - \beta)}{B_\lambda^p(\beta + n + 1, \delta - \beta)}\right].$$
When taking $x = \beta + n$, $y = \delta - \beta$, $x_1 = \beta + n + 1$, and $y_1 = \gamma - \beta$ in (3.1), since $(x - x_1)(y - y_1) = \gamma - \delta \geq 0$, it follows from Theorem 3.1 that
$$\frac{B_\lambda^p(\beta + n, \gamma - \beta)}{B_\lambda^p(\beta + n, \delta - \beta)} \leq \frac{B_\lambda^p(\beta + n + 1, \gamma - \beta)}{B_\lambda^p(\beta + n + 1, \delta - \beta)}$$
which is equivalent to say that $\{f_n\}_{n\geq 0}$ is an increasing sequence. Hence, with the aid of Lemma 2.3, we conclude that $x \mapsto \frac{\Phi_p^\lambda(\beta,\gamma;x)}{\Phi_p^\lambda(\beta,\delta;x)}$ is increasing on $(0, \infty)$.

Recall from [30] that
$$\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\Phi_p^\lambda(\beta, \gamma; x) = \frac{(\beta)_n}{(\gamma)_n}\Phi_p^\lambda(\beta + n, \gamma + n; x). \tag{4.2}$$
Since the increasing property of $x \mapsto \frac{\Phi_p^\lambda(\beta,\gamma;x)}{\Phi_p^\lambda(\beta,\delta;x)}$ is equivalent to
$$\frac{\mathrm{d}}{\mathrm{d}\, x}\left[\frac{\Phi_p^\lambda(\beta, \gamma; x)}{\Phi_p^\lambda(\beta, \delta; x)}\right] \geq 0,$$

together with (4.2), we further obtain

$$\Phi_p^\lambda(\beta,\gamma;x)\Phi_p^\lambda(\beta,\delta;x)-\Phi_p^\lambda(\beta,\gamma;x)\Phi_p^\lambda(\beta,\delta;x)$$
$$=\frac{\beta}{\gamma}\Phi_p^\lambda(\beta+1,\gamma+1;x)\Phi_p^\lambda(\beta,\delta;x)-\frac{\beta}{\delta}\Phi_p^\lambda(\beta,\gamma;x)\Phi_p^\lambda(\beta+1,\delta+1;x)\geq 0.$$

This implies the inequality (4.1).

The logarithmic convexity of $x\mapsto\Phi_p^\lambda(\beta,\gamma;x)$ can be proved by using the integral representation (1.5) and by applying the Hölder inequality (2.2) as follows:

$$\begin{aligned}\Phi_p^\lambda(\beta,\gamma;\alpha x+(1-\alpha)y)&=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\exp(\alpha xt+(1-\alpha)yt)E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\\&=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1\bigg[\Big(t^{\beta-1}(1-t)^{\gamma-\beta-1}\exp(xt)E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big)^\alpha\\&\quad\times\Big(t^{\beta-1}(1-t)^{\gamma-\beta-1}\exp(yt)E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\Big)^{1-\alpha}\bigg]\,\mathrm{d}\,t\\&\leq\Big[\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\exp(xt)E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\Big]^\alpha\\&\quad\times\Big[\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\exp(yt)E_\lambda\Big(-\frac{p}{t(1-t)}\Big)\,\mathrm{d}\,t\Big]^{1-\alpha}\\&=[\Phi_p^\lambda(\beta,\gamma;x)]^\alpha[\Phi_p^\lambda(\beta,\gamma;y)]^{1-\alpha}\end{aligned}$$

for $x,y>0$ and $\alpha\in[0,1]$. For the case $x<0$, the assertion follows immediately from the identity

$$\Phi_p^\lambda(\beta,\gamma;x)=e^x\Phi_p^\lambda(\gamma-\beta,\gamma;-z)$$

in [30].

Let $\beta'\geq\beta$ and

$$h(t)=t^{\beta'-1}(1-t)^{\gamma-\beta'-1}\exp(xt)E_\lambda\Big(-\frac{p}{t(1-t)}\Big),\quad f(t)=\Big(\frac{t}{1-t}\Big)^{\beta-\beta'},\quad g(t)=\Big(\frac{t}{1-t}\Big)^{\sigma}.$$

Using the integral representation (1.5), we have

$$\frac{B(\beta,\gamma)\Phi_p^\lambda(\beta+\sigma,\gamma;x)}{B(\beta+\sigma,c)\Phi_p^\lambda(\beta,\gamma;x)}-\frac{B(\beta',\gamma)\Phi_p^\lambda(\beta'+\sigma,\gamma;x)}{B(\beta'+\sigma,\gamma)\Phi_p^\lambda(\beta',\gamma;x)}=\frac{\int_0^1 f(t)g(t)h(t)\,\mathrm{d}\,t}{\int_0^1 f(t)h(t)\,\mathrm{d}\,t}-\frac{\int_0^1 g(t)h(t)\,\mathrm{d}\,t}{\int_0^1 h(t)\,\mathrm{d}\,t}.\tag{4.3}$$

One can easily determine that, when $\sigma\geq 0$ and $\beta'\geq\beta$, the function f is decreasing and the function g is increasing. Since h is a nonnegative function for $t\in[0,1]$, by Chebyshev's inequality (2.1), it follows that

$$\int_0^1 f(t)h(t)\,\mathrm{d}\,t\int_0^1 g(t)h(t)\,\mathrm{d}\,t\leq\int_0^1 h(t)\,\mathrm{d}\,t\int_0^1 f(t)g(t)h(t)\,\mathrm{d}\,t.$$

Combining this with (4.3) yields

$$\frac{B(\beta,\gamma)\Phi_p^\lambda(\beta+\sigma,\gamma;x)}{B(\beta+\delta,\gamma)\Phi_p^\lambda(\beta,\gamma;x)}-\frac{B(\beta',\gamma)\Phi_p^\lambda(\beta'+\sigma,\gamma;x)}{B(\beta'+\sigma,\gamma)\Phi_p^\lambda(\beta',\gamma;x)}\geq 0$$

which is equivalent to say that the function

$$\beta \mapsto \frac{B(\beta,\gamma)\Phi_p^\lambda(\beta+\sigma,\gamma;x)}{B(\beta+\sigma,\gamma)\Phi_p^\lambda(\beta,\gamma;x)}$$

is decreasing on $(0,\infty)$. The proof of Theorem 4.1 is complete. □

Remark 4.1. The decreasing property of the function

$$\beta \mapsto \frac{B(\beta,\gamma)\Phi_p^\lambda(\beta+\sigma,\gamma;x)}{B(\beta+\sigma,\gamma)\Phi_p^\lambda(\beta,\gamma;x)}$$

is equivalent to the inequality

$$[\Phi_p^\lambda(\beta+\sigma,\gamma;x)]^2 \ge \frac{B^2(\beta+\sigma,\gamma)}{B(\beta+2\sigma,\gamma)B(\beta,\gamma)}\Phi_p^\lambda(\beta+2\sigma,\gamma;x)\Phi_p^\lambda(\beta,\gamma;x). \tag{4.4}$$

When $\lambda = 1$, the inequality (4.4) becomes

$$\Phi_p^2(\beta+\sigma,\gamma;x) \ge \frac{B^2(\beta+\sigma,\gamma)}{B(\beta+2\sigma,\gamma)B(\beta,\gamma)}\Phi^p(\beta+2\sigma,\gamma;x)\Phi^p(\beta,\gamma;x)$$

which was established in [12]. When $\lambda = 1$ and $p = 0$, the inequality (4.4) reduces to

$$\Phi^2(\beta+\sigma,\gamma;x) \ge \frac{B^2(\beta+\sigma,\gamma)}{B(\beta+2\sigma,\gamma)B(\beta,\gamma)}\Phi(\beta+2\sigma,\gamma;x)\Phi(\beta,\gamma;x)$$

which recovers Theorem 4(b) in [9] and Eq. (24) in [12].

Remark 4.2. In recent years, some new results about the topic in this paper have been obtained in the papers [2, 3, 13, 14, 18–20, 25] and closely related references therein.

Remark 4.3. In this paper, we established some inequalities involving $B_\lambda^p(x,y)$ and $\Phi_p^\lambda(\beta,\gamma;z)$. Throughout this paper, if we take $\lambda = 1$, all results in this paper reduce to those in [12]; if we take $\lambda = 1$ and $p = 0$, all results in this paper reduce to corresponding ones in [1, 8].

Remark 4.4. When $\lambda > 1$, is the Mittag-Leffler function $E_\lambda(-w)$ still positive on $(0,\infty)$? When $\lambda > 1$, what about the validity of Theorems 3.1 and 3.2 and Corollary 3.1?

Remark 4.5. This paper is a slightly revised version of the preprint [26].

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Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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