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Research article

Blow-up for degenerate nonlinear parabolic problem

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Abstract: In this paper, we deal with the existence, uniqueness, and finite time blow-up of the solution to the degenerate nonlinear parabolic problem: $u_\tau = \left(\xi^r u^m u_\xi\right)_\xi / \xi^r + u^p$ for $0 < \xi < a$, $0 < \tau < \Gamma$, $u(\xi, 0) = u_0(\xi)$ for $0 \le \xi \le a$, and $u(0, \tau) = 0 = u(a, \tau)$ for $0 < \tau < \Gamma$, where $u_0(\xi)$ is a positive function and $u_0(0) = 0 = u_0(a)$. In addition, we prove that u exists globally if a is small through constructing a global-exist upper solution, and u_τ blows up in a finite time.

Keywords: blow-up; degenerate nonlinear parabolic problem; global existence
Mathematics Subject Classification: 35K55, 35K57, 35K60, 35K65

1. Introduction

Let $\Gamma \in (0, \infty]$, r be a nonnegative constant less than 1, a and m be positive constants, and p be a positive constant greater than 1. We study the following degenerate nonlinear parabolic first initial-boundary value problem:

$$u_\tau = \frac{1}{\xi^r}\left(\xi^r u^m u_\xi\right)_\xi + u^p \text{ in } (0, a) \times (0, \Gamma), \tag{1.1}$$

$$u(\xi, 0) = u_0(\xi) \text{ on } [0, a], \; u(0, \tau) = 0 = u(a, \tau) \text{ for } \tau \in (0, \Gamma), \tag{1.2}$$

where $u_0(\xi)$ is a positive function in $(0, a)$ such that $u_0^{m+1}(\xi) \in C^{2+\alpha}\left(\bar{D}\right)$ for some $\alpha \in (0, 1)$ and $u_0(0) = 0 = u_0(a)$.

Problems (1.1)–(1.2) describe the creeping gravity flow of a power-law liquid on a rigid horizontal surface. The solution u is the thickness of the current and r represents the Cartesian symmetry, see [5]. It also explains the radial spreading of an axisymmetric current with ξ and $u^{m+1}/(m+1)$ corresponding respectively to the radial coordinate and the integral of velocity profile of the current, see [7]. If u represents the temperature, then it can be interpreted as a nonlinear heat conduction problem with u^m being the thermal diffusivity, see [12, pp. 73–74]. When $m = 0$ and $r = 0.5$, it exemplifies heat transfer into one face of a flat cylinder with a small ratio of depth to diameter, see [2, 15]. Problems (1.1)–(1.2) can illustrate population dynamics when $r = 0$, see [6]. (1.1) is a degenerate equation because the thermal diffusivity $u^m \to 0$ when $\xi \to 0$ or $\xi \to a$.

Let $\xi = ax$, $\tau = a^2(m+1)t$, $\Gamma = a^2(m+1)T$, $D = (0,1)$, $\Omega = D\times(0,T)$, $\bar{D} = [0,1]$, $\bar{\Omega} = \bar{D}\times[0,T)$, $\partial D = \{0,1\}$, and $\partial\Omega = \left(\bar{D}\times\{0\}\right)\cup(\partial D\times(0,T))$. Then, the problems (1.1)–(1.2) are transformed into the degenerate nonlinear parabolic problem below,

$$u_t = (m+1)\frac{1}{x^r}\left(x^r u^m u_x\right)_x + a^2(m+1)u^p \text{ in } \Omega, \tag{1.3}$$

$$u(x,0) = u_0(x) \text{ on } \bar{D},\ u(0,t) = 0 = u(1,t) \text{ for } t \in (0,T). \tag{1.4}$$

When $r = 0$ and $u_0(x) \geq 0$ on $\bar{D}$, the multi-dimensional version of the problems (1.3)–(1.4) have been studied by [4,8,11,13,14]. Let μ_1 be the first eigenvalue of the following Sturm-Liouville problem,

$$\varphi'' + \mu\varphi = 0 \text{ in } D,\ \varphi(0) = 0 = \varphi(1).$$

When $p = m+1$, Sacks [13] proved that if $a^2(m+1) > \mu_1$, the solution blows up in a finite time. If $a^2(m+1) \leq \mu_1$ (that is, the domain size is sufficiently small), the problems (1.3)–(1.4) have a global solution (also see [14]). In the case of $p > m+1$, the solution may or may not exist for all time which depends on the initial condition u_0, see [8,11,13]. Galaktionov [4] proved that the problems (1.3)–(1.4) have a global solution if $p < m+1$.

This paper is organized as follows. In section 2, we prove the existence and uniqueness of the classical solution of the problems (1.1)–(1.2). In section 3, we show that u blows up in a finite time when $p \geq m+1$. Then, we prove that there is a global solution when a is sufficiently small. Different from [13], our method does not require additional conditions on p and m. In section 4, we prove that u_t blows up in a finite time when u is unbounded.

2. Existence and uniqueness of the solution

We assume that the initial data $u_0(x)$ satisfies the condition below,

$$\frac{d^2(u_0)^{m+1}}{dx^2} + \frac{r}{x}\frac{d(u_0)^{m+1}}{dx} + a^2(m+1)(u_0)^p \geq 0 \text{ in } D. \tag{2.1}$$

We note that $u_0 = \left[Kx\sin\left(\pi(1-x)^2/2\right)\right]^{1/(m+1)}$, where K is a positive constant, satisfies (2.1) and $u_0(x) = 0$ on ∂D. Let $v = u^{m+1}$, the problems (1.3)–(1.4) become

$$v_t = (m+1)v^{m/(m+1)}\left[v_{xx} + \frac{r}{x}v_x + a^2(m+1)v^{p/(m+1)}\right] \text{ in } \Omega, \tag{2.2}$$

$$v(x,0) = v_0(x) \text{ on } \bar{D},\ v(0,t) = 0 = v(1,t) \text{ for } t \in (0,T), \tag{2.3}$$

where $v_0(x) = u_0^{m+1}(x)$. To prove the existence of a solution, Chan and Chan [1] consider the following nonlinear parabolic problem with ε being a small positive number less than 1,

$$v_{\varepsilon_t} = (m+1)v_\varepsilon^{m/(m+1)}\left[v_{\varepsilon_{xx}} + \frac{r}{x}v_{\varepsilon_x} + a^2(m+1)v_\varepsilon^{p/(m+1)}\right] \text{ in } \Omega,$$

$$v_\varepsilon(x,0) = v_0(x) + \varepsilon \text{ on } \bar{D},\ v_\varepsilon(0,t) = \varepsilon = v_\varepsilon(1,t) \text{ for } t \in (0,T).$$

They prove that $v_\varepsilon \in C\left(\bar{\Omega}\right) \cap C^{2+\alpha,1+\alpha/2}(D \times [0,T))$, and the sequence of solutions: $\{v_\varepsilon\}$ converges to $v \in C\left(\bar{\Omega}\right) \cap C^{2+\alpha,1+\alpha/2}(D \times [0,T))$ when $\varepsilon \to 0$. They also show that $v > 0$ in $D \times [0,T)$ and $v(x,t) \geq v_0(x)$ on $\bar{D} \times [0,T)$. Using these results, they prove that the problems (1.3)–(1.4) have a solution $u \in C\left(\bar{\Omega}\right) \cap C^{2+\alpha,1+\alpha/2}(D \times [0,T))$, $u > 0$ in $D \times [0,T)$, and $u(x,t) \geq u_0(x)$ on $\bar{D} \times [0,T)$. By (2.1), they show that $u_t \geq 0$ and $v_t \geq 0$ in $D \times [0,T)$. Further, they prove that u is unbounded in $D \times (0,T)$ if $T < \infty$. For ease of reference, let us state their Theorem 2.8 below.

Theorem 2.1. *Problems (1.3)–(1.4) have a solution* $u \in C\left(\bar{\Omega}\right) \cap C^{2+\alpha,1+\alpha/2}(D \times [0,T))$*. If* $T < \infty$*, then* u *is unbounded in* $D \times (0,T)$*.*

Let $Lv = v^{-m/(m+1)}v_t/(m+1) - v_{xx} - rv_x/x$ and $\beta(x,t)$ be a bounded function on $\bar{\Omega}$. Here is a comparison theorem.

Lemma 2.2. *Suppose that* y *and* $s \in C\left(\bar{\Omega}\right) \cap C^{2,1}(\Omega)$*, and*

$$Ly - \beta y \geq Ls - \beta s \text{ in } \Omega, \; y \geq s \text{ on } \partial\Omega. \tag{2.4}$$

Then, $y \geq s$ *on* $\bar{\Omega}$*.*

Proof. If not, let us assume that $s > y$ somewhere, say, $(\bar{x},\bar{t}) \in \Omega$. By the continuity of s and y over $\bar{\Omega}$, there exists an interval $(a_1,a_2) \subset D$ such that $\bar{x} \in (a_1,a_2)$, $s(a_1,\bar{t}) - y(a_1,\bar{t}) = 0$, $s(a_2,\bar{t}) - y(a_2,\bar{t}) = 0$, $s(x,\bar{t}) > y(x,\bar{t})$ for $x \in (a_1,a_2)$, and $s \leq y$ in $[a_1,a_2] \times [0,\bar{t})$. Then,

$$\int_{a_1}^{a_2} \left(s^{1/(m+1)}(x,\bar{t}) - y^{1/(m+1)}(x,\bar{t})\right) dx > 0. \tag{2.5}$$

Let $\tilde{\phi}(x)$ and $\tilde{\lambda}$ be the first eigenfunction and eigenvalue of the following Sturm-Liouville problem,

$$(x^r w')' + \lambda x^r w = 0 \text{ in } D,\; w(a_1) = 0 = w(a_2).$$

By Theorem 3.1.2 of Pao [9, p. 97], $\tilde{\phi}(x)$ exists and $\tilde{\lambda} > 0$. Further, $\tilde{\phi}(x) > 0$ in (a_1,a_2). Let γ be a positive real number to be determined. By the above equation, we have

$$\int_0^{\bar{t}} \int_{a_1}^{a_2} (s-y)\tilde{\lambda} x^r \tilde{\phi} e^{\gamma t} dx dt = -\int_0^{\bar{t}} \int_{a_1}^{a_2} (s-y)\left(x^r \tilde{\phi}'\right)' e^{\gamma t} dx dt. \tag{2.6}$$

Using integration by parts, $\tilde{\phi}'(a_1) \geq 0$, and $\tilde{\phi}'(a_2) \leq 0$, we have

$$\int_0^{\bar{t}} \int_{a_1}^{a_2} (s-y)\left(x^r \tilde{\phi}'\right)' e^{\gamma t} dx dt \geq \int_0^{\bar{t}} \int_{a_1}^{a_2} \left[(s-y)_x x^r\right]_x \tilde{\phi} e^{\gamma t} dx dt.$$

By (2.4), we get

$$x^r \left[y^{1/(m+1)} - s^{1/(m+1)}\right]_t - \beta x^r (y-s) \geq -x^r (s_{xx} - y_{xx}) - r x^{r-1}(s_x - y_x) = -\left[(s-y)_x x^r\right]_x.$$

From this, we have

$$\int_0^{\bar{t}} \int_{a_1}^{a_2} (s-y)\left(x^r \tilde{\phi}'\right)' e^{\gamma t} dx dt \geq -\int_0^{\bar{t}} \int_{a_1}^{a_2} \left[y^{1/(m+1)} - s^{1/(m+1)}\right]_t x^r \tilde{\phi} e^{\gamma t} dx dt + \int_0^{\bar{t}} \int_{a_1}^{a_2} \beta(y-s) x^r \tilde{\phi} e^{\gamma t} dx dt.$$

By (2.6), we obtain

$$\begin{aligned}
&-\int_0^{\bar{t}}\int_{a_1}^{a_2}(s-y)\,\tilde{\lambda}x^r\tilde{\phi}e^{\gamma t}dxdt\\
&\geq -\int_0^{\bar{t}}\int_{a_1}^{a_2}\left[y^{1/(m+1)}-s^{1/(m+1)}\right]_t x^r\tilde{\phi}e^{\gamma t}dxdt+\int_0^{\bar{t}}\int_{a_1}^{a_2}\beta\,(y-s)\,x^r\tilde{\phi}e^{\gamma t}dxdt\\
&=-\int_{a_1}^{a_2}\left[y^{1/(m+1)}(x,\bar{t})-s^{1/(m+1)}(x,\bar{t})\right]x^r\tilde{\phi}e^{\gamma\bar{t}}dx\\
&+\int_{a_1}^{a_2}\left[y^{1/(m+1)}(x,0)-s^{1/(m+1)}(x,0)\right]x^r\tilde{\phi}dx\\
&+\int_0^{\bar{t}}\int_{a_1}^{a_2}\left[y^{1/(m+1)}-s^{1/(m+1)}\right]\gamma x^r\tilde{\phi}e^{\gamma t}dxdt+\int_0^{\bar{t}}\int_{a_1}^{a_2}\beta\,(y-s)\,x^r\tilde{\phi}e^{\gamma t}dxdt.
\end{aligned}$$

The above expression is equivalent to

$$\begin{aligned}
&\int_0^{\bar{t}}\int_{a_1}^{a_2}\left(\beta-\tilde{\lambda}\right)(s-y)\,x^r\tilde{\phi}e^{\gamma t}dxdt+\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,0)-y^{1/(m+1)}(x,0)\right]x^r\tilde{\phi}dx\\
&+\int_0^{\bar{t}}\int_{a_1}^{a_2}\left[s^{1/(m+1)}-y^{1/(m+1)}\right]\gamma x^r\tilde{\phi}e^{\gamma t}dxdt\\
&\geq\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,\bar{t})-y^{1/(m+1)}(x,\bar{t})\right]x^r\tilde{\phi}e^{\gamma\bar{t}}dx.
\end{aligned}$$

By the mean value theorem, there exists an ζ between $s^{1/(m+1)}$ and $y^{1/(m+1)}$ such that

$$\begin{aligned}
&\int_0^{\bar{t}}\int_{a_1}^{a_2}\left[s^{1/(m+1)}-y^{1/(m+1)}\right]\left[(m+1)\left(\beta-\tilde{\lambda}\right)\zeta^m+\gamma\right]x^r\tilde{\phi}e^{\gamma t}dxdt\\
&+\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,0)-y^{1/(m+1)}(x,0)\right]x^r\tilde{\phi}dx\\
&\geq\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,\bar{t})-y^{1/(m+1)}(x,\bar{t})\right]x^r\tilde{\phi}e^{\gamma\bar{t}}dx.
\end{aligned}$$

By the Gronwall inequality (cf. Walter [16, pp. 14–15]),

$$\begin{aligned}
&\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,\bar{t})-y^{1/(m+1)}(x,\bar{t})\right]x^r\tilde{\phi}e^{\gamma\bar{t}}dx\\
&\leq\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,0)-y^{1/(m+1)}(x,0)\right]x^r\tilde{\phi}dx\left[1+\int_0^{\bar{t}}\left[(m+1)\left(\beta-\tilde{\lambda}\right)\zeta^m+\gamma\right]e^{\int_t^{\bar{t}}\left[(m+1)\left(\beta-\tilde{\lambda}\right)\zeta^m+\gamma\right]dt}dt\right].
\end{aligned}$$

As β is bounded, we choose γ such that $\gamma\geq(m+1)\left(\tilde{\lambda}-\beta\right)\zeta^m$. By $y\geq s$ in $[a_1,a_2]\times[0,\bar{t})$, we have

$$\int_{a_1}^{a_2}\left[s^{1/(m+1)}(x,\bar{t})-y^{1/(m+1)}(x,\bar{t})\right]x^r\tilde{\phi}e^{\gamma\bar{t}}dx\leq 0.$$

Since $x^r\tilde{\phi}e^{\gamma\bar{t}}>0$ in (a_1,a_2), the above inequality contradicts (2.5). Therefore, $y\geq s$ in Ω. As $y\geq s$ on $\partial\Omega$, $y\geq s$ on $\bar{\Omega}$. The proof is complete. □

Let $\mathcal{L}y = v^{-m/(m+1)}y_t/(m+1) - y_{xx} - ry_x/x$. Based on a similar computation of Lemma 2.2, we have the following result.

Lemma 2.3. *Suppose that y and $s \in C\left(\bar{\Omega}\right) \cap C^{2,1}(\Omega)$, and*

$$\mathcal{L}y - \beta y \geq \mathcal{L}s - \beta s \text{ in } \Omega,\ y \geq s \text{ on } \partial\Omega.$$

Then, $y \geq s$ on $\bar{\Omega}$.

By Theorem 2.1 and Lemma 2.2, we obtain the result of the existence and uniqueness of solution.

Theorem 2.4. *Problems (1.3)–(1.4) and (2.2)–(2.3) have the unique classical solution.*

3. Blow-up of the solution and global existence

Instead of using condition (2.1), let us assume that u_0 satisfies the inequality below in the following two sections:

$$\frac{d^2(u_0)^{m+1}}{dx^2} + \frac{r}{x}\frac{d(u_0)^{m+1}}{dx} + a^2(m+1)(u_0)^p > 0 \text{ in } D. \tag{3.1}$$

Then, by (1.3) and $u \in C\left(\bar{\Omega}\right) \cap C^{2+\alpha,1+\alpha/2}(D \times [0,T))$, we have $u_t(x,0) > 0$ ($v_t(x,0) > 0$) in D. We want to prove that $v_t(x,t) > 0$ in D for $t > 0$. To achieve it, we have the following two results.

Lemma 3.1. *$v(x,t) > v_0(x)$ in Ω.*

Proof. From (3.1), we obtain

$$\frac{d^2 v_0}{dx^2} + \frac{r}{x}\frac{dv_0}{dx} + a^2(m+1)v_0^{p/(m+1)} > 0 \text{ in } D.$$

As stated in section 2, we have $v(x,t) \geq v_0(x)$ on $\bar{D} \times [0,T)$. Subtract the above inequality from (2.2), it gives

$$\begin{aligned} v^{-m/(m+1)}v_t &> (m+1)\left[(v-v_0)_{xx} + \frac{r}{x}(v-v_0)_x + a^2(m+1)\left(v^{p/(m+1)} - v_0^{p/(m+1)}\right)\right] \\ &\geq (m+1)\left[(v-v_0)_{xx} + \frac{r}{x}(v-v_0)_x\right]. \end{aligned}$$

Further, we know that $v(x,t) = v_0(x) = 0$ on $\partial D \times (0,T)$ and $v(x,0) = v_0(x)$ on $\bar{D}$. Suppose that $v(\tilde{x},t) = v_0(\tilde{x})$ for some $\tilde{x} \in D$ and $t > 0$. Then, the set

$$\{t : v(x,t) = v_0(x) \text{ for some } x \in D \text{ and } t > 0\}$$

is non-empty. Let $\tilde{t}$ denote its infimum. Suppose that $\tilde{t} > 0$. Then, $v(\tilde{x},\tilde{t}) = v_0(\tilde{x})$ and $v(x,t) > v_0$ in $D \times (0,\tilde{t})$. Therefore, $\left(v(\tilde{x},\tilde{t}) - v_0(\tilde{x})\right)_t \leq 0$. From section 2, we have $v_t(\tilde{x},\tilde{t}) \geq 0$. Thus, $\left(v(\tilde{x},\tilde{t}) - v_0(\tilde{x})\right)_t = 0$. Further, $v(x,t) - v_0(x)$ attains its local minimum at $(\tilde{x},\tilde{t})$. This implies that $\left(v(\tilde{x},\tilde{t}) - v_0(\tilde{x})\right)_x = 0$ and $\left(v(\tilde{x},\tilde{t}) - v_0(\tilde{x})\right)_{xx} > 0$. Since $v(\tilde{x},\tilde{t}) > 0$, we have

$$0 = v^{-m/(m+1)}(\tilde{x},\tilde{t})\,v_t(\tilde{x},\tilde{t}) > (m+1)\left[\left(v(\tilde{x},\tilde{t}) - v_0(\tilde{x})\right)_{xx} + \frac{r}{\tilde{x}}\left(v(\tilde{x},\tilde{t}) - v_0(\tilde{x})\right)_x\right] > 0.$$

It leads to a contradiction. If $\tilde{t} = 0$, we have $v(x,0) = v_0(x)$ on $\bar{D}$ and $v(x,t) > v_0(x)$ for $t > 0$ in D. Hence, $v(x,t) > v_0(x)$ in Ω. □

Let h be a small positive real number and $q(x,t) = v(x,t+h)$. Further, q is the solution of the following problem:

$$q^{-m/(m+1)}q_t = (m+1)\left[q_{xx} + \frac{r}{x}q_x + a^2(m+1)q^{p/(m+1)}\right] \text{ in } \Omega, \tag{3.2}$$

$$q(x,0) = v(x,h) \text{ on } \bar{D},\ q(0,t) = 0 = q(1,t) \text{ for } t \in (0,T). \tag{3.3}$$

We follow a similar calculation of Lemma 3.1 to obtain the corollary below.

Corollary 3.2. $q(x,t) > v(x,t)$ *in* Ω.

Having these two results, we prove v_t being positive in the domain.

Lemma 3.3. $v_t > 0$ *in* Ω.

Proof. From the result of section 2, $v_t \geq 0$ in $D \times [0,T)$. Let us assume that $v_t(\rho,\omega) = 0$ for some $(\rho,\omega) \in \Omega$. Then, there exists a neighborhood $(a_3,a_4) \times (t_1,t_2) \subset \Omega$ such that $(\rho,\omega) \in (a_3,a_4) \times (t_1,t_2)$. We differentiate (2.2) with respect to t to obtain

$$(v_t)_t = \frac{m}{(m+1)}v^{-1}(v_t)^2 + (m+1)v^{m/(m+1)}\left[(v_t)_{xx} + \frac{r}{x}(v_t)_x + a^2pv^{(p-m-1)/(m+1)}v_t\right]. \tag{3.4}$$

Since $v > 0$ in $(a_3,a_4) \times (t_1,t_2)$, it gives

$$(v_t)_t \geq (m+1)v^{m/(m+1)}\left[(v_t)_{xx} + \frac{r}{x}(v_t)_x + a^2pv^{(p-m-1)/(m+1)}v_t\right] \text{ in } (a_3,a_4) \times (t_1,t_2).$$

By the strong maximum principle (cf. Protter and Weinberger [10, pp. 168–169]), $v_t \equiv 0$ in $(a_3,a_4) \times (t_1,t_2)$. This contradicts Corollary 3.2 that v is strictly increasing in t in Ω. Therefore, $v_t > 0$ in $(a_3,a_4) \times (t_1,t_2)$. Since (ρ,ω) is arbitrary in Ω, $v_t > 0$ in Ω. □

To study the blow-up of the solution u, we let $z^{1/(1-r)} = x$. By a direct computation,

$$v_x = v_z\frac{1-r}{z^{r/(1-r)}},$$

$$v_{xx} = (1-r)^2 z^{-2r/(1-r)}v_{zz} - r(1-r)\frac{v_z}{z^{(1+r)/(1-r)}}.$$

Then, the problems (2.2)–(2.3) are transformed into

$$v_t = (m+1)v^{m/(m+1)}\left[(1-r)^2 z^{-2r/(1-r)}v_{zz} + a^2(m+1)v^{p/(m+1)}\right] \text{ in } \Omega, \tag{3.5}$$

$$v(z,0) = v_0(z) \text{ on } \bar{D},\ v(0,t) = 0 = v(1,t) \text{ for } t \in (0,T). \tag{3.6}$$

Let

$$F(t) = \frac{(m+1)^2}{p+1}\int_0^1 z^{2r/(1-r)}v^{(p+1)/(m+1)}dz. \tag{3.7}$$

Since $v > 0$ in $D \times [0,T)$, $F(t) > 0$ over $[0,T)$. We modify Lemma 4.3 of Deng, Duan and Xie [3] to obtain the result below.

Lemma 3.4. *If* $p \geq m+1$, *then*

$$(F'(t))^2 \leq \frac{p+1}{2p}F(t)F''(t).$$

Proof. By a direct computation, the derivative of $F(t)$ is given by

$$F'(t) = (m+1)\int_0^1 z^{2r/(1-r)} v^{(p-m)/(m+1)} v_t dz. \tag{3.8}$$

By $v_t(x,0) > 0$ in D and Lemma 3.3 $v_t > 0$ in Ω, we have $F'(t) > 0$ over $[0,T)$. By (3.5), (3.8) is rewritten as

$$F'(t) = (m+1)^2 \int_0^1 \left[(1-r)^2 v_{zz} + a^2 (m+1) z^{2r/(1-r)} v^{p/(m+1)}\right] v^{p/(m+1)} dz.$$

Differentiating $F'(t)$ with respect to t and by (3.5), we have

$$\begin{aligned} F''(t) = {} & p\int_0^1 v^{(p-2m-1)/(m+1)} z^{2r/(1-r)} (v_t)^2 dz + (m+1)^2 (1-r)^2 \int_0^1 v^{p/(m+1)} v_{zzt} dz \\ & + a^2 (m+1)^2 p \int_0^1 z^{2r/(1-r)} v^{[2p-(m+1)]/(m+1)} v_t dz. \end{aligned}$$

Using integration by parts and $p \geq m+1$, we obtain

$$\begin{aligned} F''(t) = {} & p\int_0^1 v^{(p-2m-1)/(m+1)} z^{2r/(1-r)} (v_t)^2 dz + a^2 (m+1)^2 p \int_0^1 z^{2r/(1-r)} v^{[2p-(m+1)]/(m+1)} v_t dz \\ & + (1-r)^2 p (m+1)\left(\frac{p}{m+1} - 1\right)\int_0^1 v^{p/(m+1)-2} v_t (v_z)^2 dz + p(m+1)(1-r)^2 \int_0^1 v^{p/(m+1)-1} v_t v_{zz} dz. \end{aligned}$$

By (3.5), the above expression becomes

$$\begin{aligned} F''(t) = {} & p\int_0^1 v^{(p-2m-1)/(m+1)} z^{2r/(1-r)} (v_t)^2 dz + a^2 (m+1)^2 p \int_0^1 z^{2r/(1-r)} v^{[2p-(m+1)]/(m+1)} v_t dz \\ & + (1-r)^2 p (m+1)\left(\frac{p}{m+1} - 1\right)\int_0^1 v^{p/(m+1)-2} v_t (v_z)^2 dz \\ & + p\int_0^1 v^{p/(m+1)-1} v_t (m+1)\left[\frac{v^{-m/(m+1)} z^{2r/(1-r)} v_t}{(m+1)} - a^2 (m+1) z^{2r/(1-r)} v^{p/(m+1)}\right] dz \\ = {} & 2p\int_0^1 v^{(p-2m-1)/(m+1)} z^{2r/(1-r)} (v_t)^2 dz + (1-r)^2 p (m+1)\left(\frac{p}{m+1} - 1\right)\int_0^1 v^{p/(m+1)-2} v_t (v_z)^2 dz. \end{aligned}$$

By assumption $p \geq m+1$, it yields

$$F''(t) \geq 2p\int_0^1 z^{2r/(1-r)} v^{(p-2m-1)/(m+1)} (v_t)^2 dz. \tag{3.9}$$

By (3.8) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} (F'(t))^2 & = (m+1)^2 \left[\int_0^1 z^{2r/(1-r)} v^{(p-m)/(m+1)} v_t dz\right]^2 \\ & \leq (m+1)^2 \int_0^1 z^{2r/(1-r)} v^{(p+1)/(m+1)} dz \int_0^1 z^{2r/(1-r)} v^{(p-2m-1)/(m+1)} (v_t)^2 dz. \end{aligned}$$

Then, by (3.7) and (3.9), we have

$$(F'(t))^2 \le \frac{p+1}{2p} F(t) F''(t). \tag{3.10}$$

This completes the proof. □

Lemma 3.5. *If $p \ge m+1$, then the solution u blows up somewhere on $\bar{D}$ in a finite time T.*

Proof. By a direct computation,

$$\begin{aligned}\frac{d^2}{dt^2} F^{-(p-1)/(p+1)}(t) &= -\frac{p-1}{p+1}\left[\frac{-2p}{p+1} F^{-(3p+1)/(p+1)} (F')^2 + F^{-2p/(p+1)} F''\right] \\ &= \frac{2p(p-1)}{(p+1)^2} F^{-(3p+1)/(p+1)} \left[(F')^2 - \frac{p+1}{2p} F F''\right].\end{aligned}$$

By (3.10), $p > 1$, and $F > 0$ over $[0, T)$, we have

$$\frac{d^2}{dt^2} F^{-(p-1)/(p+1)}(t) \le 0.$$

We integrate the above inequality over $(0, t)$ to get

$$\left(F^{-(p-1)/(p+1)}(t)\right)' - \left(F^{-(p-1)/(p+1)}(0)\right)' \le 0.$$

Equivalently,

$$\left(F^{-(p-1)/(p+1)}(t)\right)' \le -\frac{p-1}{p+1} F^{-2p/(p+1)}(0) F'(0).$$

Then, we integrate this inequality over $(0, t)$ to obtain

$$F^{-(p-1)/(p+1)}(t) \le -\frac{p-1}{p+1} F^{-2p/(p+1)}(0) F'(0) t + F^{-(p-1)/(p+1)}(0).$$

Since $F(0) > 0$, $F'(0) > 0$, and $p > 1$, the right side of the above inequality is a decreasing function in t and is equal to zero in a finite time. Therefore, there exists some finite T such that $F^{-(p-1)/(p+1)}(T) = 0$. Hence, $F(T) = \infty$. It implies that $v(z, t) \to \infty$ when $t \to T$ for some $z \in \bar{D}$. Thus, $u(x, t)$ blows up somewhere on $\bar{D}$ in a finite time T. □

Now, we prove that u exists globally if a is sufficiently small. This can be achieved through constructing a global-exist upper solution of the problems (2.2)–(2.3). In this proof, we do not have additional conditions on p and m.

Theorem 3.6. *If a is small enough, then u exists globally.*

Proof. It suffices to prove that $v(x, t)$ exists globally. Let $V(x) = kx^{1-r}(1-x)$ where k is a positive constant. Then, $V(x) \in C\left(\bar{D}\right) \cap C^2(D)$. We choose k such that $V(x) \ge v_0(x)$. Clearly, $V(x) = 0$ at $x = 0$ and $x = 1$. The expression of V_x and V_{xx} is below

$$V_x = k\left[(1-r) x^{-r} - (2-r) x^{1-r}\right],$$

$$V_{xx} = k\left[-r(1-r) x^{-r-1} - (2-r)(1-r) x^{-r}\right].$$

By a direct computation,

$$\begin{aligned}
&V_{xx} + \frac{r}{x} V_x + a^2 (m+1) V^{p/(m+1)} \\
&= k\left[-r(1-r)x^{-r-1} - (2-r)(1-r)x^{-r} + r(1-r)x^{-r-1} - r(2-r)x^{-r}\right] \\
&+ a^2 (m+1) k^{p/(m+1)} \left[x^{1-r}(1-x)\right]^{p/(m+1)} \\
&= -k(2-r)x^{-r} + a^2 (m+1) k^{p/(m+1)} \left[x^{1-r}(1-x)\right]^{p/(m+1)}.
\end{aligned}$$

If a is sufficiently small, then $V_{xx}+rV_x/x+a^2(m+1)V^{p/(m+1)} \leq 0\,(=V_t)$. By Lemma 2.2, $V(x) \geq v(x,t)$ on $\bar{D} \times [0,\infty)$. Therefore, v exists globally which implies u exists globally. □

4. Blow-up of u_t

In this section, we want to prove that u_t tends to infinity if u blows up. From Lemma 3.3, $v_t > 0$ in Ω. Let $J(x,t) = v_t(x,t) - \varepsilon v(x,t)$ where ε is a small positive number. Then, $J = 0$ on $\partial D \times [0,T)$. Let $t_3 \in (0,T)$. We choose ε such that $J(x,t_3) \geq 0$ on $\bar{D}$.

Lemma 4.1. *If $p \geq m+1$, then $J \geq 0$ on $\bar{D} \times [t_3, T)$.*

Proof. By a direct computation, $J_t = v_{tt} - \varepsilon v_t$, $J_x = v_{tx} - \varepsilon v_x$, and $J_{xx} = v_{txx} - \varepsilon v_{xx}$. From (3.4), we have

$$v_{tt} = \frac{m}{m+1} v^{-1} (v_t)^2 + (m+1) v^{m/(m+1)} \left[J_{xx} + \varepsilon v_{xx} + \frac{r}{x}(J_x + \varepsilon v_x) + a^2 p v^{(p-m-1)/(m+1)} v_t\right].$$

By Lemma 3.3, $J_t + \varepsilon v_t = v_{tt}$, and (2.2), we have

$$\begin{aligned}
J_t + \varepsilon v_t > {} & (m+1) v^{m/(m+1)} \left(J_{xx} + \frac{r}{x} J_x\right) + (m+1) v^{m/(m+1)} \varepsilon \left[\frac{v^{-m/(m+1)}}{m+1} v_t - a^2 (m+1) v^{p/(m+1)}\right] \\
& + a^2 (m+1) p v^{(p-1)/(m+1)} (J + \varepsilon v).
\end{aligned}$$

Simplifying the above inequality and by $p \geq m+1$, it gives

$$\begin{aligned}
J_t &> (m+1) v^{m/(m+1)} \left(J_{xx} + \frac{r}{x} J_x\right) + a^2 (m+1) p v^{(p-1)/(m+1)} J + \varepsilon a^2 (m+1) \left[p - (m+1)\right] v^{(p+m)/(m+1)} \\
&\geq (m+1) v^{m/(m+1)} \left(J_{xx} + \frac{r}{x} J_x\right) + a^2 (m+1) p v^{(p-1)/(m+1)} J.
\end{aligned}$$

By Lemma 2.3, we have $J \geq 0$ on $\bar{D} \times [t_3, T)$. □

Our main result below is immediately followed by Lemma 3.5 and Lemma 4.1.

Theorem 4.2. *If $p \geq m+1$ and u is unbounded somewhere on $\bar{D}$ in a finite time T, then u_t blows up at T.*

5. Conclusion

In this paper, we prove the existence and uniqueness of the solution of a degenerate nonlinear parabolic problem. This solution blows up in a finite time if $p \geq m+1$. Then, we show that u_t blows up somewhere in the domain in a finite time.

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Conflict of interest

The author declares that there are no conflicts of interest in this paper.

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