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Research article

Averaging methods for piecewise-smooth ordinary differential equations

Michal Fečkan[1,2,*], **Július Pačuta**[1], **Michal Pospíšil**[1,2] **and Pavol Vidlička**[1]

[1] Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

[2] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

* **Correspondence:** Email: michal.feckan@fmph.uniba.sk

Abstract: The averaging method is developed for periodic piecewise-smooth systems. We discuss the behavior of solutions intersecting the discontinuity boundary and the problems it introduces. We illustrate these difficulties on specific examples. In the case of transversal and sliding solutions, we introduce conditions that allow us to prove averaging theorems for piecewise-smooth periodic differential equations.

Keywords: piecewise-smooth differential equations; averaging method; Filippov regularization
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1. Introduction

Averaging method has many applications and it is well-developed for ordinary and partial differential equations [4, 8], for impulsive and differential inclusions [7], and for more general types of differential equations as well [5, 6]. Non-smooth dynamical systems also possess many applications [2]. The goal of this paper is to extend the tool set available for analyzing periodic piecewise-smooth differential equations by generalizing the averaging method. Several intrinsic problems surfaced while trying to generalize the results to the piecewise-smooth setting. We focus on the cases of transversal and sliding solutions of the piecewise-smooth systems. In these cases, we discuss the problems and try to shed some light on their nature and causes by considering concrete examples. Several conditions are introduced which allow us to prove results analogous to the averaging theorem for smooth systems.

Section 2 is a brief introduction to the theory of averaging for smooth systems and contains an averaging theorem whose statement serves as a prototype for statements of our generalizations for piecewise-smooth systems. Section 3 introduces piecewise-smooth systems and our main definitions.

Section 4 provides examples that motivated us for these definitions. Section 5 presents some auxiliary results concerning the discontinuity boundary and its intersection with a solution that are used in the next sections. Sections 6 and 7 deal with the proofs of our main results. The final Section 8 summarizes our results with a possible outline for further study.

Throughout the paper, we shall denote $u{\cdot}v$ the inner product of vectors $u, v \in \mathbb{R}^n$ and $|u|$ the Euclidean norm of $u \in \mathbb{R}^n$. Furthermore, the distance of sets $A, B \subset \mathbb{R}^n$ is defined as $\mathrm{dist}(A, B) = \inf\{|a - b| \mid a \in A,\ b \in B\}$.

2. Averaging for smooth systems

Let $\Omega \subset \mathbb{R}^n$ be a domain and $a \in \Omega$. We consider the initial value problem

$$\dot{x} = \epsilon f(x, t, \epsilon),\ x(0) = a \tag{2.1}$$

defined for $x \in \Omega$, $\epsilon \in [0, 1]$ and $t \in \mathbb{R}$ where $f(x, t, \epsilon)$ is a continuously differentiable periodic function in the time variable t with the period T.

Definition 2.1. Let f be the right-hand side of initial value problem (2.1). Denote

$$\bar{f}(z) = \frac{1}{T}\int_0^T f(z, s, 0)\,\mathrm{d}s.$$

The *averaged system* associated with (2.1) is the system

$$\dot{z} = \epsilon \bar{f}(z), \tag{2.2}$$

and the *guiding system* associated with (2.1) is the system

$$\dot{w} = \bar{f}(w). \tag{2.3}$$

Initial value problem (2.1) gives naturally a rise to the initial value problem for the averaged system (2.2) with $z(0) = a$, and for the guiding system (2.3) with $w(0) = a$. The guiding system is only a rescaled averaged system.

The following theorem from [8, Theorem 2.8.1] is the key result we need and it will be used throughout the whole paper.

Theorem 2.2. *Consider initial value problem* (2.1). *Suppose that f is Lipschitz continuous and $w : [0, L] \to \Omega$ is solution of the associated guiding system. Then there exist positive constants ϵ_0, C such that for all $\epsilon \in (0, \epsilon_0]$ there exists a unique solution $x_\epsilon : [0, \frac{L}{\epsilon}] \to \Omega$ such that*

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon$$

for $0 \leq t \leq \frac{L}{\epsilon}$.

To simplify some of the statements in the rest of the paper, we collect some of the common assumptions on the right-hand side f in the following definition.

Definition 2.3. We say, that a function $f : \Omega \times \mathbb{R} \times [0,1] \to \mathbb{R}^n$ is T-PCDLB if it is T-periodic in the second argument, continuously differentiable, Lipschitz continuous and bounded.

The following lemma is an extension of the averaging theorem for smooth systems which will be useful in the next sections.

Lemma 2.4. *Let f be a T-PCDLB function, $w : [L_1, L_2] \to \Omega$ be a solution of the guiding system $\dot{w} = \bar{f}(w)$ with $w(L_1) = a$ and $A \geq 0$ be a constant. Let $\epsilon \in (0,1]$. Consider the local solution of initial value problem $\dot{x} = \epsilon f(x,t,\epsilon)$, $x(\frac{L_1}{\epsilon} + A) = b_\epsilon$ for $b_\epsilon \in \Omega$ such that $|a - b_\epsilon| \leq D\epsilon$ for some constant $D > 0$.*

Then there exist $\epsilon_0, C > 0$ and a unique solution $x_\epsilon : [\frac{L_1}{\epsilon}+A, \frac{L_2}{\epsilon}] \to \Omega$ of $\dot{x} = \epsilon f(x,t,\epsilon)$, $x(\frac{L_1}{\epsilon}+A) = b_\epsilon$ for all $\epsilon \in (0,\epsilon_0]$ such that for all $t \in [\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}]$ it holds

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon. \tag{2.4}$$

Proof. Let us at first assume the existence of x_ϵ for sufficiently small $\epsilon > 0$ and prove estimate (2.4). Let A_ϵ be such that $A_\epsilon \geq A$, $|A_\epsilon - A| \leq T$, $\frac{L_1}{\epsilon} + A_\epsilon = k_\epsilon T$ for some $k_\epsilon \in \mathbb{Z}$. The existence of such A_ϵ for all ϵ is obvious. Let $\epsilon_0 > 0$ be so small that $\frac{L_1}{\epsilon} + A_\epsilon \leq \frac{L_2}{\epsilon}$ for all $\epsilon \in (0,\epsilon_0]$. Let M be a bound of $|f|$.

First, we prove inequality (2.4) for $t \in [\frac{L_1}{\epsilon} + A, \frac{L_1}{\epsilon} + A_\epsilon]$. Using the assumption $|a - b_\epsilon| \leq D\epsilon$ and the Lipschitz continuity of solutions x_ϵ and w, we obtain

$$\begin{aligned}|x_\epsilon(t) - w(\epsilon t)| &\leq \left|x_\epsilon(t) - x_\epsilon\left(\frac{L_1}{\epsilon} + A\right)\right| + \left|x_\epsilon\left(\frac{L_1}{\epsilon} + A\right) - w\left(\epsilon\frac{L_1}{\epsilon}\right)\right| + \left|w\left(\epsilon\frac{L_1}{\epsilon}\right) - w(\epsilon t)\right| \\ &\leq \epsilon M|A_\epsilon - A| + D\epsilon + \epsilon M A_\epsilon \\ &\leq \epsilon(D + MT + M(A+T)).\end{aligned}$$

Next, we prove inequality (2.4) for $t \in [\frac{L_1}{\epsilon} + A_\epsilon, \frac{L_2}{\epsilon}]$. Since the solution w is defined on interval $[L_1, L_2]$, we will use the averaging theorem for smooth systems for function $\widetilde{w}(\tau) = w(\tau + L_1)$. Hence there exists some $C > 0$ such that for all $\epsilon > 0$ sufficiently small, there is a solution $y_\epsilon : [0, \frac{L_2 - L_1}{\epsilon}] \to \Omega$ of the initial value problem $\dot{y} = \epsilon f(y,t,\epsilon)$, $y(0) = a$, and for all $t \in [0, \frac{L_2 - L_1}{\epsilon}]$ it holds

$$|y_\epsilon(t) - \widetilde{w}(\epsilon t)| \leq C\epsilon. \tag{2.5}$$

Using y_ϵ we can define a function $u_\epsilon : [\frac{L_1}{\epsilon} + A_\epsilon, \frac{L_2}{\epsilon} + A_\epsilon] \to \Omega$ by $u_\epsilon(t) = y_\epsilon(t - (\frac{L_1}{\epsilon} + A_\epsilon))$. The function u_ϵ satisfies the equation

$$\dot{u}_\epsilon(t) = \dot{y}_\epsilon\left(t - \left(\frac{L_1}{\epsilon} + A_\epsilon\right)\right) = \epsilon f\left(u_\epsilon(t), t - \left(\frac{L_1}{\epsilon} + A_\epsilon\right), \epsilon\right) = \epsilon f(u_\epsilon(t), t, \epsilon),$$

since $\frac{L_1}{\epsilon} + A_\epsilon = k_\epsilon T$ and f is T-periodic in the second argument.

Now, let $t \in [\frac{L_1}{\epsilon} + A_\epsilon, \frac{L_2}{\epsilon}]$. Using (2.5) we estimate

$$\begin{aligned}|u_\epsilon(t) - w(\epsilon t)| &= \left|y_\epsilon\left(t - A_\epsilon - \frac{L_1}{\epsilon}\right) - w(\epsilon t)\right| \\ &\leq \left|y_\epsilon\left(t - A_\epsilon - \frac{L_1}{\epsilon}\right) - w(\epsilon(t - A_\epsilon))\right| + |w(\epsilon(t - A_\epsilon)) - w(\epsilon t)| \\ &\leq \epsilon(C + M(A+T)).\end{aligned} \tag{2.6}$$

The functions u_ϵ, x_ϵ satisfy the same equation (2.1) (however, with possibly different initial values), hence it holds

$$|u_\epsilon(t) - x_\epsilon(t)| \leq \left|u_\epsilon\left(\frac{L_1}{\epsilon} + A_\epsilon\right) - x_\epsilon\left(\frac{L_1}{\epsilon} + A_\epsilon\right)\right| + \epsilon \int_{\frac{L_1}{\epsilon}+A_\epsilon}^{t} |f(u_\epsilon(s), s, \epsilon) - f(x_\epsilon(s), s, \epsilon)| \, \mathrm{d}s.$$

Using the Lipschitz continuity of f and the Gronwall lemma, we obtain

$$|u_\epsilon(t) - x_\epsilon(t)| \leq C\left|u_\epsilon\left(\frac{L_1}{\epsilon} + A_\epsilon\right) - x_\epsilon\left(\frac{L_1}{\epsilon} + A_\epsilon\right)\right|$$

for some constant $C > 0$. The estimates (2.4), (2.6) for $t = \frac{L_1}{\epsilon} + A_\epsilon$ imply

$$\left|u_\epsilon\left(\frac{L_1}{\epsilon} + A_\epsilon\right) - x_\epsilon\left(\frac{L_1}{\epsilon} + A_\epsilon\right)\right| \leq C\epsilon.$$

Thus the proof of the estimate (2.4) for $t \in [\frac{L_1}{\epsilon} + A_\epsilon, \frac{L_2}{\epsilon}]$ is finished.

Now we prove the existence and uniqueness. The uniqueness and the local existence of x_ϵ are consequences of the Lipschitz continuity. The only issue for global existence of x_ϵ would be if x_ϵ approaches the boundary $\partial\Omega$.

Denote $d = \mathrm{dist}(\partial\Omega, w([L_1, L_2]))$. Let $\epsilon_0 > 0$ be such small that $C\epsilon_0 \leq \frac{d}{2}$. Then we have

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon \leq \frac{d}{2}$$

for any $\epsilon \in (0, \epsilon_0]$ and $t \in [\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}]$. Consequently,

$$\begin{aligned} d &\leq \mathrm{dist}(\partial\Omega, w([L_1 + A\epsilon, L_2])) \\ &\leq \mathrm{dist}\left(\partial\Omega, x_\epsilon\left(\left[\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}\right]\right)\right) + \mathrm{dist}\left(x_\epsilon\left(\left[\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}\right]\right), w([L_1 + A\epsilon, L_2])\right) \\ &\leq \mathrm{dist}\left(\partial\Omega, x_\epsilon\left(\left[\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}\right]\right)\right) + \frac{d}{2}, \end{aligned}$$

which means that $\mathrm{dist}(\partial\Omega, x_\epsilon(t)) \geq \frac{d}{2}$ for all $t \in [\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}]$. Hence, x_ϵ exists globally on interval $[\frac{L_1}{\epsilon} + A, \frac{L_2}{\epsilon}]$. □

3. Definition of a piecewise-smooth system

We start with a definition of a discontinuity boundary.

Definition 3.1. We say that a function $G : \Omega \to \mathbb{R}$ is a boundary function if G is continuously differentiable and $\nabla G(x) \neq 0$ for every $x \in \Omega$ such that $G(x) = 0$.

For a boundary function G, let us define

$$\begin{aligned} \Omega_0 &= \{x \in \Omega \mid G(x) = 0\}, \\ \Omega_+ &= \{x \in \Omega \mid G(x) > 0\}, \\ \Omega_- &= \{x \in \Omega \mid G(x) < 0\}. \end{aligned}$$

The set Ω_0 is a C^1-manifold and is called a *discontinuity boundary* or a *discontinuity surface*. The notation introduced above will be used throughout our paper.

Definition 3.2. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $f^-, f^+ : \Omega \times \mathbb{R} \times [0,1] \to \mathbb{R}^n$ and $G : \Omega \to \mathbb{R}$ be functions. We say that the tuple (Ω, G, f^-, f^+) is a piecewise-smooth system if f^- and f^+ are continuously differentiable and G is a boundary function. We say that the tuple (Ω, G, f^-, f^+) is a well-behaved piecewise-smooth system if f^- and f^+ are T-PCDLB and G is a boundary function.

A shorthand $\dot{x} = f^{\pm}(x, t, \epsilon)$ will be often used instead of the tuple (Ω, G, f^-, f^+) while talking about a piecewise-smooth system, implicitly including a set Ω and a boundary function G.

Definition 3.3. An absolutely continuous function $x : [a, b] \to \Omega$ is called a solution of the piecewise-smooth system (Ω, G, f^-, f^+) if for almost all $t \in [a, b]$,

$$\dot{x}(t) \in \begin{cases} \{f^-(x(t), t, \epsilon)\}, & \text{if } x(t) \in \Omega_-, \\ \{f^+(x(t), t, \epsilon)\}, & \text{if } x(t) \in \Omega_+, \\ \operatorname{conv}(\{f^-(x(t), t, \epsilon), f^+(x(t), t, \epsilon)\}), & \text{if } x(t) \in \Omega_0. \end{cases}$$

Furthermore, we say that a solution of the piecewise-smooth system is a solution of an initial value problem $\dot{x} = \epsilon f^{\pm}(x, t, \epsilon)$, $x(a) = x_a \in \Omega$ if $x(a) = x_a$.

The formalism that we just introduced is called the *Filippov regularization* of a piecewise-smooth system (see [1, 2]).

Next we define two properties for which we will prove the averaging theorem.

Definition 3.4. We say that a well-behaved piecewise-smooth system (Ω, G, f^-, f^+) is *uniformly transversal at the point* $x_0 \in \Omega_0$ if there exists a positive constant m such that

$$f^{\pm}(x_0, t, \epsilon) \cdot \nabla G(x_0) \geq m$$

for all $t \in \mathbb{R}$ and $\epsilon \in [0, 1]$.

The uniform transversality at x_0 also implies that $\bar{f}^{\pm}(x_0) \cdot \nabla G(x_0) \geq m$.

Definition 3.5. We say that a well-behaved piecewise-smooth system (Ω, G, f^-, f^+) is *uniformly sliding*, if there exists a positive constant m such that

$$f^-(x, t, \epsilon) \cdot \nabla G(x) \geq m \quad \text{and} \quad f^+(x, t, \epsilon) \cdot \nabla G(x) \leq -m$$

for all $x \in \Omega_0$, $t \in \mathbb{R}$ and $\epsilon \in [0, 1]$.

The condition of uniform sliding itself is not sufficient to prove an averaging theorem. We need to reformulate the guiding system in a following way. Suppose that the condition of uniform sliding is true for every $x \in \Omega_0$. This implies, that any solution sliding at t_0 has to stay on Ω_0 up to some $t_1 > t_0$. Hence the derivative of the solution is tangent to the manifold Ω_0 or, equivalently, it is orthogonal to $\nabla G(x)$. A short calculation shows that for a given (x, t, ϵ) such that the piecewise smooth system is sliding at (x, t, ϵ), there is a unique element of $\operatorname{conv}(\{f^-(x, t, \epsilon), f^+(x, t, \epsilon)\})$, which is orthogonal to $\nabla G(x)$. We define function $g : \Omega_0 \times \mathbb{R} \times [0, 1] \to \mathbb{R}^n$,

$$g = \alpha f^- + (1 - \alpha) f^+ \tag{3.1}$$

where

$$\alpha = \frac{f^+ \cdot \nabla G}{f^+ \cdot \nabla G - f^- \cdot \nabla G}. \tag{3.2}$$

The function g can be used to define an induced differential equation on some neighborhood of (x, t, ϵ) in the set $\Omega_0 \times \mathbb{R} \times [0, 1]$. The smoothness of g depends on the smoothness of f^-, f^+ and ∇G; see [1] for details. We can apply averaging to this function to get

$$\bar{g}(x) = \frac{1}{T} \int_0^T g(x, s, 0)\, \mathrm{d}s \tag{3.3}$$

which serves as a right-hand side of the guiding system $\dot{u} = \bar{g}(u)$ for the equation induced on the discontinuity surface.

Definition 3.6. Let (Ω, G, f^-, f^+) be a well-behaved piecewise-smooth system that is uniformly sliding. Let $\bar{g}$ be defined as in (3.3). The *sliding adjusted guiding system* associated with the initial value problem $\dot{x} = \epsilon f^\pm(x, t, \epsilon)$, $x(0) = a \in \Omega_-$ is given by

$$\dot{w} = \begin{cases} \bar{f}^-(w), & x \in \Omega_-, \\ \bar{g}(w), & x \in \Omega_0. \end{cases}$$

An absolutely continuous function $w : [0, L] \to \Omega$ is a solution of the sliding adjusted guiding system, if it satisfies the above equality almost everywhere.

Suppose that we are given an initial value problem for a piecewise-smooth system $(\Omega, G, \epsilon f^-, \epsilon f^+)$,

$$\dot{x} = \epsilon f^\pm(x, t, \epsilon), x(0) = a \tag{3.4}$$

such that both f^- and f^+ are T-periodic. Our goal is to generalize the averaging theorem, Theorem 2.2, to this setting. Akin to Definition 2.1, we can define

$$\bar{f}^\pm(w) = \frac{1}{T} \int_0^T f^\pm(w, s, 0)\, \mathrm{d}s$$

and also define a guiding system associated with (3.4) as

$$\dot{w} = \bar{f}^\pm(w), w(0) = a. \tag{3.5}$$

Given a solution $w : [0, T] \to \Omega$ of the guiding system (3.5), it is clear that if $(\operatorname{Im} w) \cap \Omega_0 = \emptyset$, we can use the result for smooth systems by restricting ourselves to a neighborhood of $\operatorname{Im} w$ that entirely lies inside Ω_- or inside Ω_+.

That is the reason why we will be concerned only with solutions of the guiding system (3.5), which intersect Ω_0. Furthermore, we will assume that $w(0) = a \in \Omega_-$. It is easy to see that if $a \in \Omega_+$, it suffices to exchange G for $-G$ and all the results hold. We are not going to explicitly consider $a \in \Omega_0$, but such an initial condition is a special case that can usually be easily inferred from presented results.

4. Some useful examples

Let us start with an example, which shows that we cannot simply adapt the averaging theorem by adding the word 'piecewise' to the statement. More specifically, we construct a well-behaved piecewise-smooth system and a transversal solution of the associated initial value problem and we show that the result of the averaging theorem does not hold.

Example 4.1. Let us define g as a periodic function with period 18, which is defined on the interval $[0, 18]$ by the following formula (see Figure 1):

$$g(t) = \begin{cases} -1, & t \in [0, 2], \\ 20(t-2) - 1, & t \in [2, 3], \\ -20(t-3) + 19, & t \in [3, 4], \\ -1, & t \in [4, 18]. \end{cases}$$

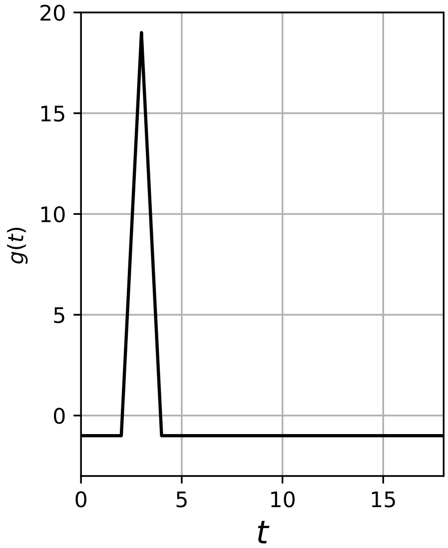


Figure 1. Graph of function g.

We can calculate that

$$\int_0^{18} g(t)\,\mathrm{d}t = -18 + 20 \cdot 2 \cdot \frac{1}{2} = 2.$$

We will use the standard mollifier as defined in [3, p.629],

$$\mu(t) = \begin{cases} \left(\int_{-1}^{1} \mathrm{e}^{\frac{1}{y^2-1}}\,\mathrm{d}y \right)^{-1} \mathrm{e}^{\frac{1}{t^2-1}}, & t \in (-1, 1), \\ 0, & t \notin (-1, 1). \end{cases}$$

For each $\delta > 0$, we set

$$\mu_\delta(t) = \frac{1}{\delta}\mu\left(\frac{t}{\delta}\right)$$

and we define

$$g^{\delta}(t) = (\mu_{\delta} * g)(t) = \int_{-\delta}^{\delta} \mu_{\delta}(y) g(t-y)\,\mathrm{d}y.$$

By [3, p.630, Theorem 6(i)], $g^{\delta} \in C^{\infty}(\mathbb{R})$. Let $0 < \delta < 2$ be fixed such that

$$\sup_{t\in\mathbb{R}} |g(t) - g^{\delta}(t)| = \sup_{t\in[0,18]} |g(t) - g^{\delta}(t)| < \frac{1}{2}$$

and $\int_0^{18} g^{\delta}(t)\,\mathrm{d}t > 0$, which can be done due to [3, p.630, Theorem 6(iii)]. We can immediately see that

$$g^{\delta}(t) \begin{cases} \leq -\frac{1}{2}, & t \in \bigcup_{k\in\mathbb{Z}} \left([0+18k, 2+18k] \cup [4+18k, 20+18k]\right), \\ \geq \frac{1}{2}, & t \in \bigcup_{k\in\mathbb{Z}} [2.1+18k, 3.9+18k]. \end{cases} \tag{4.1}$$

We will show that g^{δ} has exactly two zeros in $[0,18]$. The derivative of g^{δ} is

$$(g^{\delta})'(t) = (\mu_{\delta} * h)(t) = \int_{-\delta}^{\delta} \mu_{\delta}(y) h(t-y)\,\mathrm{d}y$$

where h is the periodic continuation of

$$h(t) = \begin{cases} 0, & t \in [0,2], \\ 20, & t \in [2,3], \\ -20, & t \in [3,4], \\ 0, & t \in [4,18]. \end{cases}$$

This implies that $(g^{\delta})'(t) > 0$ for $t \in (2-\delta, 3)$ and $(g^{\delta})'(t) < 0$ for $t \in (3, 4+\delta)$. Hence, by (4.1), g^{δ} has on $[0,18]$ just two zeros: $r_1 \in (2,3)$, $r_2 \in (3,4)$.

Consider $\dot{x} = \epsilon f^{\pm}(x,t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$ with a boundary function $G(x) = x$ and

$$f^{-}(x,t) = g^{\delta}(t),$$
$$f^{+}(x,t) = g^{\delta}(-t).$$

Such a system is well-behaved. The corresponding averaged version of these functions is

$$\bar{f}^{+} = \bar{f}^{+}(x) = \frac{1}{18}\int_0^{18} g^{\delta}(t)\,\mathrm{d}t = I > 0$$

and the guiding system is $\dot{w} = I$. Consider the initial value problem $w(0) = -C$ whose solution is $w : [0,L] \to \mathbb{R}$, $w(t) = -C + It$ where $C \in \mathbb{R}^{+}$ and $L > \frac{C}{I}$.

We will show that although the averaged system has a transversal solution for all choices of C and L satisfying the conditions above, the original system has no solution that begins in $\mathbb{R}^{-}$ and then crosses over to $\mathbb{R}^{+}$.

Suppose that $x_{\epsilon}(t)$ is such a solution, meaning that $x_{\epsilon}(0) < 0$ and there is a t^{*} such that $x_{\epsilon}(t^{*}) = 0$ (that is, it intersects the boundary $\Omega_0 = \{0\}$).

Let $t_0 = \sup\{t \mid \forall s \in [0,t) : x_{\epsilon}(s) < 0\}$. The definition of t_0 and existence of t^{*} imply that $x_{\epsilon}(t_0) = 0$. Since for all $s < t_0$ we have $x_{\epsilon}(s) < 0$, for each $m \in \mathbb{N}$ there is $t_m \in (t_0 - \frac{1}{m}, t_0)$ such that

$\epsilon g^{\delta}(t_m) = \dot{x}_\epsilon(t_m) > 0$. This implies that $t_0 \in [r_1 + 18l, r_2 + 18l]$ for some $l \in \mathbb{Z}$. Consequently, for all $t \in [t_0, r_2 + 18l]$ it holds that $\epsilon f^-(t) \geq 0$, $\epsilon f^+(t) < 0$ implying that $x_\epsilon(t) = 0$.

For all $t \in (r_2 + 18l, (18 - r_2) + 18l)$ it holds that $\epsilon f^-(t) < 0$, $\epsilon f^+(t) < 0$ which means that $\dot{x}_\epsilon(t) < 0$. Thus, $x_\epsilon(t) < 0$ and also $x_\epsilon((18 - r_2) + 18l) < 0$.

For all $t \in [(18 - r_2) + 18l, r_1 + 18(l + 1)]$, $\epsilon f^-(t) < 0$ which implies

$$x_\epsilon(r_1 + 18(l + 1)) < x((18 - r_2) + 18l) < 0.$$

Since

$$\int_{r_2+18l}^{r_2+18(l+1)} \epsilon f^-(t)\,\mathrm{d}t > 0,$$

there is $t_1 \in (r_1+18(l+1), r_2+18(l+1))$ such that $x_\epsilon(t_1) = 0$ and $t_1 = \sup\{t \mid \forall s \in (r_2+18l, t) : x_\epsilon(s) < 0\}$.

The same analysis as for t_0 can be repeated for t_1, etc. The solution can be extended up to infinity, but does not cross the boundary. However, the solution of the guiding system crosses the boundary without a problem. The problem is that the guiding system does not reflect the behavior of the original equation at the discontinuity boundary.

Now, we provide an example of an initial value problem for a well-behaved piecewise-smooth system. The solution of the associated guiding system is transversal, while there is a solution of the original system for which the estimate from the averaging theorem holds and this solution is not transversal.

Example 4.2. Let $G(x) = x$ be the boundary function and consider a system $\dot{x} = \epsilon f^{\pm}(x, t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$ with

$$f^-(x, t) = 1,$$
$$f^+(x, t) = 1 - 2\cos t$$

and an initial condition $x(0) = -2\pi$.

The guiding system is $\dot{w} = 1$, which is transversal at $x = 0$ and we can immediately write the solution of the guiding system as

$$w(\tau) = -2\pi + \tau$$

which means that

$$w_\epsilon(t) = -2\pi + \epsilon t.$$

The solution of the original system up to the intersection with the discontinuity, that is for $t \in [0, \frac{2\pi}{\epsilon}]$, is given by

$$x_\epsilon(t) = -2\pi + \epsilon t = w(\epsilon t).$$

The solution x_ϵ can cross Ω_0 transversally at $t = \frac{2\pi}{\epsilon}$ only if $f^+(0, \frac{2\pi}{\epsilon}) > 0$. That implies

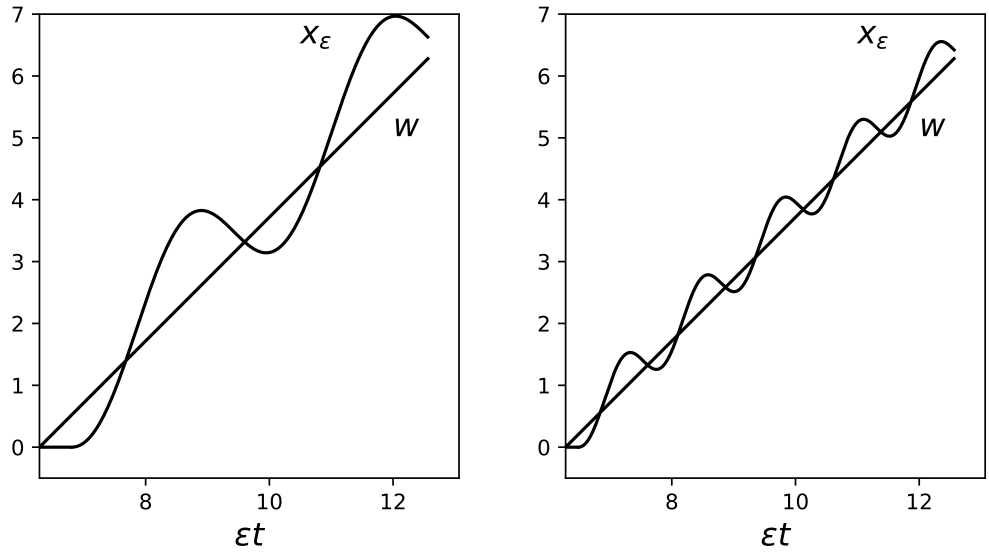
$$t \notin \bigcup_{k\in\mathbb{Z}} \left[-\frac{\pi}{3} + 2k\pi, \frac{\pi}{3} + 2k\pi\right].$$

Let $\epsilon_k > 0$ be such that $\frac{1}{\epsilon_k} \in (-\frac{1}{6} + k, \frac{1}{6} + k)$ for some $k \in \mathbb{Z}$ (see Figure 2). For such ϵ_k, the solution x_{ϵ_k} makes the first contact with the boundary at $t = \frac{2\pi}{\epsilon_k} \in (-\frac{\pi}{3} + 2k\pi, \frac{\pi}{3} + 2k\pi)$. However, since $f^+(t) \leq 0$ up to $\frac{\pi}{3} + 2k\pi$, we can say that the solution "sticks" to $x = 0$ up to $t = \frac{\pi}{3} + 2k\pi$.

By solving the system $\dot{x} = \epsilon_k f^+(x, t)$, $x(\frac{\pi}{3} + 2k\pi) = 0$ we obtain a continuation of the solution x_{ϵ_k}, namely

$$x_{\epsilon_k}(t) = \epsilon_k t - 2\epsilon_k \sin t + C_k$$

where $C_k = 2\epsilon_k \sin(\frac{\pi}{3} + 2k\pi) - \epsilon_k(\frac{\pi}{3} + 2k\pi)$.



(a) $\epsilon = 0.5$ **(b)** $\epsilon = 0.2$

Figure 2. Comparison of w and a "sticking" solution x_ϵ. Graph of the solution w of the guiding system and x_ϵ for $t \geq \frac{1}{\epsilon}$ and $\frac{1}{\epsilon} \in (-\frac{1}{6} + k, \frac{1}{6} + k)$ for some $k \in \mathbb{Z}$.

Such a solution does not make a contact with the boundary again. To see that, it only suffices to look at the values of $t - 2\sin t - \frac{\pi}{3} + 2\sin(\frac{\pi}{3})$ for $t > \frac{\pi}{3}$.

Now, we show that condition (2.4) is satisfied. For $t \in [\frac{2\pi}{\epsilon_k}, 2k\pi + \frac{\pi}{3}]$,

$$|w_{\epsilon_k}(t) - x_{\epsilon_k}(t)| = |w(\epsilon_k t) - 0| \leq |\epsilon_k| \left| t - \frac{2\pi}{\epsilon_k} \right| \leq \frac{2\pi}{3} |\epsilon_k|.$$

For $t > 2k\pi + \frac{\pi}{3}$,

$$|w_{\epsilon_k}(t) - x_{\epsilon_k}(t)| = | - 2\pi + \epsilon_k t - \epsilon_k t + 2\epsilon_k \sin t - C_k|$$
$$= |\epsilon_k| \left| 2\sin t - 2\sin\left(\frac{\pi}{3} + 2k\pi\right) + \frac{\pi}{3} + 2k\pi - \frac{2\pi}{\epsilon_k} \right| \leq |\epsilon_k| \left(4 + \frac{2\pi}{3}\right)$$

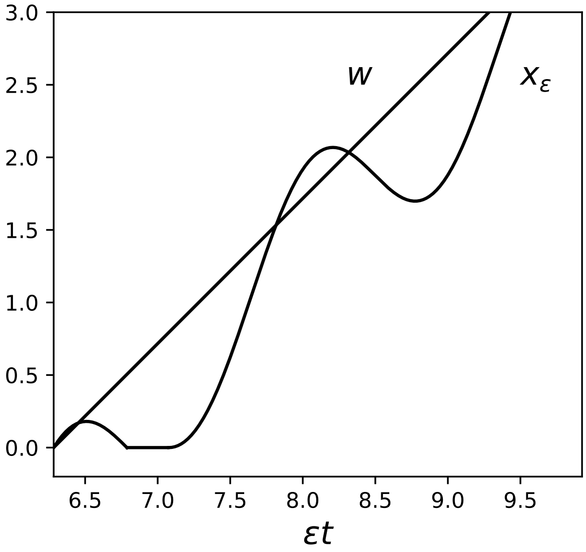
because

$$\left| \frac{\pi}{3} + 2k\pi - \frac{2\pi}{\epsilon_k} \right| \leq \frac{2\pi}{3}.$$

The analysis for other values of ϵ is a bit more tricky. The solution crosses Ω_0 transversally, but it can make a contact with the boundary again. If it makes the second contact with the boundary, it will continue with the same motion as we just analyzed (see Figure 3).

However, the difference in values of w_ϵ and x_ϵ will be less than $C|\epsilon|$ for some $C > 0$. That means that the averaging estimate still holds on an interval $[0, T]$ for any T.

(a) $\epsilon = 0.4$



(b) $\epsilon = 0.27$

Figure 3. Comparison of w and a "non-sticking" solution x_ϵ. Graph of the solution w of the guiding system and x_ϵ for $t \geq \frac{1}{\epsilon}$ and $\frac{1}{\epsilon} \notin [-\frac{1}{6}+k, \frac{1}{6}+k]$ for each $k \in \mathbb{Z}$. The solution for $\epsilon = 0.27$ intersects the discontinuity boundary again and sticks to it for a bit before continuing to Ω_+.

Note that the condition that the solution of the guiding system w is transversal translates to a condition that the system $\dot{w} = \bar{f}^{\pm}(w)$ is transversal at x_0. However, that does not imply that the original system $\dot{x} = \epsilon f(x, t, \epsilon)$ is transversal at x_0 for all $t \in \mathbb{R}, \epsilon \in [0, 1]$. This means that the behavior of the original system near $x_0 \in \Omega_0$ is not reflected by the guiding system and can be more complex. For this reason, we introduce the notion of uniform transversality of the system $\dot{x} = \epsilon f(x, t, \epsilon)$ at x_0 (uniform in t and ϵ).

Now consider the guiding system $\dot{w} = \bar{f}^{\pm}(w)$ associated with $\dot{x} = \epsilon f^{\pm}(x, t, \epsilon)$. The right-hand side of the differential equation induced by this guiding system on the discontinuity boundary is given by

$$h(w) = \beta(x)\bar{f}^{-}(w) + (1 - \beta(x))\bar{f}^{+}(w)$$

where β is given by

$$\beta(x) = \frac{\bar{f}^{+}(x) \cdot \nabla G(x)}{\bar{f}^{+}(x) \cdot \nabla G(x) - \bar{f}^{-}(x) \cdot \nabla G(x)}.$$

The sliding solution of the guiding system $\dot{w} = \bar{f}^{\pm}(w)$ is governed by h, but $\bar{g}$ of (3.1) is the true right-hand side of the guiding system for the sliding motion on Ω_0.

A question that naturally arise is: What is the relationship between $\bar{g}$ and h? We could hope that these two functions are equal. However, this is not the case in general. The function $\bar{g}$ is a result of first inducing an equation on Ω_0 and then averaging. For h, those operations are reversed. That is, we first average and induce an equation on Ω_0 afterwards. Their relationship is not that simple.

The following example shows a specific system, where $\bar{g}$ and h are not equal.

Example 4.3. Let the discontinuity surface be defined by $G : \mathbb{R}^2 \to \mathbb{R}$, $G(x) = x_1$ and consider the system $\dot{x} = \epsilon f^{\pm}(x, t)$, $x \in \mathbb{R}^2$, $t \in \mathbb{R}$ with

$$f^-(x,t) = (1,1),$$
$$f^+(x,t) = \left(-1 + \frac{1}{2}\sin t, 3\right)$$

along with the initial condition $x(0) = (-1, -1)$.

The differential equation induced on $\Omega_0 = \{0\} \times \mathbb{R}$ is $\dot{x} = \epsilon g(x, t)$, where g depends only on t, and is given by

$$g(x,t) = (0, \alpha(t) \cdot 1 + (1 - \alpha(t)) \cdot 3).$$

The function α is defined by the equality

$$0 = \alpha(t) \cdot 1 + (1 - \alpha(t))\left(-1 + \frac{1}{2}\sin t\right).$$

By solving for α and substituting into the previous equality, we get

$$g(x,t) = \left(0, \frac{4 - \frac{1}{2}\sin t}{2 - \frac{1}{2}\sin t}\right).$$

By averaging over the period, we obtain

$$\bar{g}(x) = \left(0, 1 + \frac{4}{\sqrt{15}}\right) \doteq (0, 2.0328).$$

The guiding system for this example is given by the right-hand side

$$\bar{f}^-(x) = (1,1),$$
$$\bar{f}^+(x) = (-1,3)$$

which induces a differential equation on Ω_0, $\dot{x} = \epsilon h(x)$, where

$$h(x) = \left(0, \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3\right) = (0,2) \neq \bar{g}(x).$$

Let us compare the solution of the original initial value problem x_ϵ and w_ϵ obtained from the guiding system for $\tau \in [0, 2]$ (that is, $t \in [0, \frac{2}{\epsilon}]$).

For $t \in [0, \frac{1}{\epsilon}]$ we have

$$x_\epsilon(t) = w_\epsilon(t) = (-1 + \epsilon t, -1 + \epsilon t)$$

and the solutions make contact with the boundary at $t = \frac{1}{\epsilon}$ in $x = (0, 0)$. After that time, both solutions slide along the discontinuity boundary Ω_0. The motion of x_ϵ is given by the differential equation $\dot{x}_\epsilon = \epsilon g(x_\epsilon, t)$, while the motion of w_ϵ is given by $\dot{w}_\epsilon = \epsilon h(w_\epsilon) = \epsilon(0, 2)$ (see Figure 4).

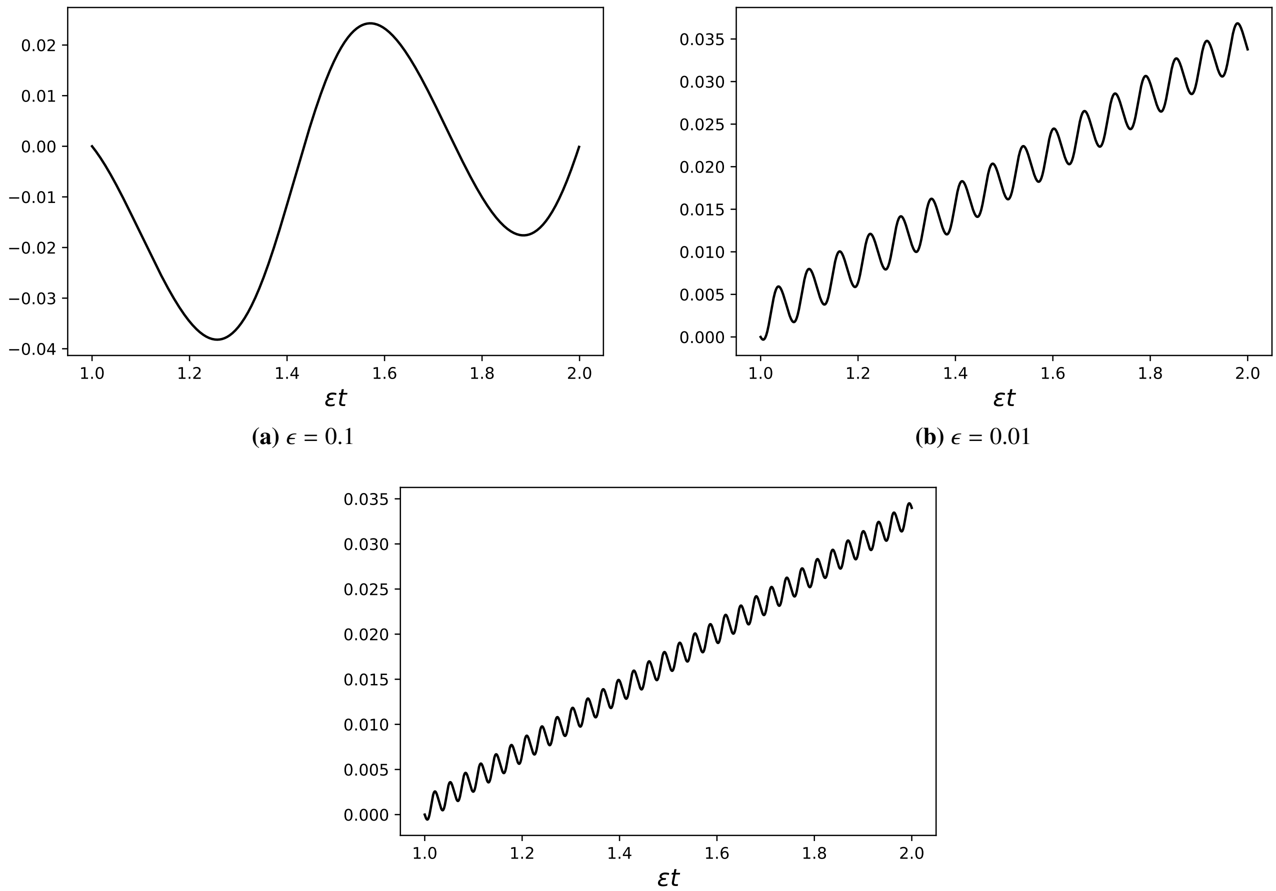
To analyze the behavior of x_ϵ on the boundary for sufficiently small ϵ, we can employ the averaging theorem for smooth systems. In the following, let us drop the first coordinate (which is constant and equal to 0).

The guiding system for the equation on the boundary has the right-hand side equal to $\bar{g}_0$. Using Lemma 2.4 (an extension of the averaging theorem), there is $\epsilon_0 > 0$, $C > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and $t \in [\frac{1}{\epsilon}, \frac{2}{\epsilon}]$,

$$\left| x_\epsilon(t) - \epsilon \left(1 + \frac{4}{\sqrt{15}}\right)\left(t - \frac{1}{\epsilon}\right)\right| \leq C\epsilon.$$

In particular, at $t = \frac{2}{\epsilon}$ we have

$$\left| x_\epsilon\left(\frac{2}{\epsilon}\right) - \left(1 + \frac{4}{\sqrt{15}}\right)\right| \leq C\epsilon.$$



(a) $\epsilon = 0.1$ **(b)** $\epsilon = 0.01$

(c) $\epsilon = 0.005$

Figure 4. Difference between x_ϵ and w_ϵ on the discontinuity boundary.

On the other side,

$$w_\epsilon\left(\frac{2}{\epsilon}\right) = 2\epsilon\left(t - \frac{1}{\epsilon}\right)\bigg|_{t=\frac{2}{\epsilon}} = 2 \neq 1 + \frac{4}{\sqrt{15}},$$

which means that it cannot be true that there exists a constant $\widetilde{C} > 0$ such that for all $t \in [\frac{1}{\epsilon}, \frac{2}{\epsilon}]$ and all $\epsilon > 0$ sufficiently small,

$$|x_\epsilon(t) - w_\epsilon(t)| \leq \widetilde{C}\epsilon.$$

Indeed, otherwise one would get a contradiction from

$$\left|2-\left(1+\frac{4}{\sqrt{15}}\right)\right| \leq \left|x_\epsilon\left(\frac{2}{\epsilon}\right)-2\right| + \left|x_\epsilon\left(\frac{2}{\epsilon}\right)-\left(1+\frac{4}{\sqrt{15}}\right)\right| \leq \epsilon(\widetilde{C}+C)$$

(see Figure 5).

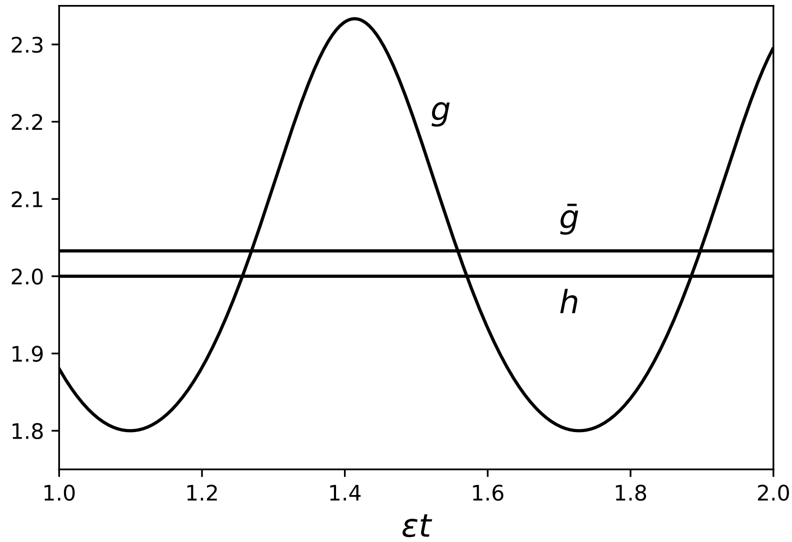


Figure 5. Comparison of g, $\bar{g}$ and h.

5. Analysis of the discontinuity boundary Ω_0

For the purpose of this section, we denote $B_r(a) = \{x \in \mathbb{R}^n \mid |x-a| < r\}$. The next theorem shows that the function G can be thought of as a distance function on some neighborhood of a point $x_0 \in \Omega_0$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $G : \Omega \to \mathbb{R}$ a continuously differentiable function. Suppose that $x_0 \in \Omega_0 = \{x \in \Omega \mid G(x) = 0\}$ and that $\nabla G(x_0) \neq 0$. Then there exist positive numbers $d, \alpha, \beta \in \mathbb{R}$, such that $\overline{B_d(x_0)} \subset \Omega$, and for each $x \in \overline{B_d(x_0)}$,*

$$\alpha|G(x)| \leq \operatorname{dist}(x, \Omega_0) \leq \beta|G(x)|.$$

Proof. Since the functions $(x,y) \mapsto \nabla G(x)\cdot\nabla G(y)$ defined on $\Omega\times\Omega$ and $x \mapsto |\nabla G(x)|$ defined on Ω are continuous and $\nabla G(x_0)\cdot\nabla G(x_0) > 0$, $|\nabla G(x_0)| > 0$, there are positive constants d, a, A, b, B such that $\overline{B_{2d}(x_0)} \subset \Omega$ and for all $x, y \in \overline{B_{2d}(x_0)}$,

$$a \leq \nabla G(x)\cdot\nabla G(y) \leq A, \quad b \leq |\nabla G(x)| \leq B. \tag{5.1}$$

Suppose that $x \in \overline{B_d(x_0)}$ and $z \in \Omega_0$ is such that $|x-z| = \operatorname{dist}(x,\Omega_0)$. Since $x_0 \in \Omega_0$ and $|x-x_0| \leq d$, it follows that $|x-z| \leq d$. Hence $|x_0 - z| \leq 2d$, i.e., $z \in \overline{B_{2d}(x_0)}$. Consequently, $\nabla G(z) \neq 0$. It follows from the geometry that $x - z = k\nabla G(z)$ where $k = \operatorname{sgn} G(x)\,\frac{|x-z|}{|\nabla G(z)|} = \operatorname{sgn} G(x)\,\frac{\operatorname{dist}(x,\Omega_0)}{|\nabla G(z)|}$. Hence, it holds

$$G(x) = G(z) + \int_0^1 \nabla G(z + t(x-z))\cdot(x-z)\,\mathrm{d}t$$

$$= \operatorname{sgn} G(x) \, \frac{\operatorname{dist}(x, \Omega_0)}{|\nabla G(z)|} \int_0^1 \nabla G(z + t(x - z)) \cdot \nabla G(z) \, \mathrm{d}t.$$

Using the inequalities (5.1), we arrive at

$$\frac{a}{B} \operatorname{dist}(x, \Omega_0) \leq |G(x)| \leq \frac{A}{b} \operatorname{dist}(x, \Omega_0)$$

and the statement easily follows. □

In the next lemma, we consider a system with C^1-smooth right-hand side but included in $\Omega = \Omega_+ \cup \Omega_0 \cup \Omega_-$. So, we can estimate the time and position of the contact with Ω_0.

Lemma 5.2. *Let f be T-PCDLB and G be a boundary function. Let $w : [0, L] \to \Omega$ be a solution of the initial value problem $\dot{w} = \bar{f}(w)$, $w(0) = a \in \Omega_-$ and $x_\epsilon : [0, \frac{L}{\epsilon}] \to \Omega$ be a solution to $\dot{x} = \epsilon f(x, t, \epsilon)$, $x(0) = a$. Assume that w intersects Ω_0 in exactly one point x_0 at $\tau_0 \in (0, L)$ and $\bar{f}(x_0) \cdot \nabla G(x_0) > 0$.*

Then there exist positive constants ϵ_0, A, B such that for all $\epsilon \in (0, \epsilon_0]$, the solution x_ϵ intersects Ω_0 in some point and if we define t_ϵ to be the smallest t such that $x_\epsilon(t) \in \Omega_0$, then

$$\left| t_\epsilon - \frac{\tau_0}{\epsilon} \right| \leq A \quad \textit{and} \quad |x_\epsilon(t_\epsilon) - x_0| \leq B\epsilon.$$

Proof. Let M be a bound of $|f|$. Due to the smooth averaging, Lemma 2.4 can be used to show that there exist $C > 0$ and $\epsilon_0 > 0$ sufficiently small such that for all $\epsilon \in (0, \epsilon_0]$ and $t \in [0, \frac{L}{\epsilon}]$, it holds

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon. \tag{5.2}$$

In the rest of the proof, ϵ_0 may decrease.

Since $w(L) \in \Omega_+$, we have $G(w(L)) > 0$. Then the estimate (5.2) and the continuity of G imply that $G(x_\epsilon(\frac{L}{\epsilon})) > 0$ for all $\epsilon \in (0, \epsilon_0]$. Since $G(x_\epsilon(0)) = G(a) < 0$, the solution x_ϵ crosses Ω_0 in some point.

Due to the continuity of mapping $x \mapsto \bar{f}(x) \cdot \nabla G(x)$ and Theorem 5.1, there exist positive constants d, m, α, β such that $\overline{B_d(x_0)} \subset \Omega$, and for all $x \in \overline{B_d(x_0)}$, it holds

$$\bar{f}(x) \cdot \nabla G(x) \geq m \quad \text{and} \quad \alpha |G(x)| \leq \operatorname{dist}(x, \Omega_0) \leq \beta |G(x)|. \tag{5.3}$$

Let A be a constant such that $A > \frac{C}{m\alpha}$ where C is from (5.2). Let ϵ_0 be such that $\tau_0 - \epsilon_0 A > 0$, $\tau_0 + \epsilon_0 A < L$ and $MA\epsilon_0 \leq d$. If we choose $\epsilon \in (0, \epsilon_0]$, then for all $\tau \in [\tau_0 - A\epsilon, \tau_0 + A\epsilon] \subset [0, L]$, it holds

$$|w(\tau) - x_0| \leq \int_{\tau_0}^{\tau} |\bar{f}(w(s))| \, \mathrm{d}s \leq M|\tau - \tau_0| \leq MA\epsilon \leq d,$$

i.e., $w(\tau) \in \overline{B_d(x_0)}$. Since $w(\tau_0 + A\epsilon) \in \Omega_+$, we have $G(w(\tau_0 + A\epsilon)) > 0$ and we obtain

$$G(w(\tau_0 + A\epsilon)) = G(w(\tau_0)) + \int_{\tau_0}^{\tau_0 + A\epsilon} \bar{f}(w(s)) \cdot \nabla G(w(s)) \, \mathrm{d}s \geq A\epsilon m$$

due to the first inequality of (5.3). Now, we can use the other inequality of (5.3) to get

$$\operatorname{dist}(w(\tau_0 + A\epsilon), \Omega_0) \geq \alpha |G(w(\tau_0 + A\epsilon))| \geq \alpha A\epsilon m.$$

Observe that the set $w([\tau_0 + A\epsilon_0, L])$ is compact and lies in Ω_+. Therefore, $\mathrm{dist}(w([\tau_0 + A\epsilon_0, L], \Omega_0) > 0$. Consequently, for some $\epsilon_0 > 0$ and all $\epsilon \in (0, \epsilon_0]$, we obtain

$$\begin{aligned}&\mathrm{dist}(w([\tau_0 + A\epsilon, L]), \Omega_0)\\ &\geq \min\{\mathrm{dist}(w([\tau_0 + A\epsilon, \tau_0 + A\epsilon_0]), \Omega_0), \mathrm{dist}(w([\tau_0 + A\epsilon_0, L]), \Omega_0)\} \geq \alpha A\epsilon m.\end{aligned}$$

The estimate (5.2) implies that for any $\epsilon \in (0, \epsilon_0]$,

$$\mathrm{dist}\left(x_\epsilon\left(\left[\frac{\tau_0 + A\epsilon}{\epsilon}, \frac{L}{\epsilon}\right]\right), \Omega_0\right) \geq \mathrm{dist}(w([\tau_0 + A\epsilon, L]), \Omega_0) - \sup_{t\in\left[\frac{\tau_0+A\epsilon}{\epsilon}, \frac{L}{\epsilon}\right]} |x_\epsilon(t) - w(\epsilon t)| \geq (\alpha Am - C)\epsilon.$$

Due to our choice of A, we deduce that $x_\epsilon(t) \in \Omega_+$ for all $t \in [\frac{\tau_0+A\epsilon}{\epsilon}, \frac{L}{\epsilon}]$, $\epsilon \in (0, \epsilon_0]$.

Using similar arguments, we can show that

$$\mathrm{dist}(w(\tau_0 - A\epsilon), \Omega_0) \geq \alpha A\epsilon m \quad \text{and} \quad x_\epsilon\left(\left[0, \frac{\tau_0 - A\epsilon}{\epsilon}\right]\right) \in \Omega_-$$

for all $\epsilon \in (0, \epsilon_0]$ for some (possibly smaller than before) $\epsilon_0 > 0$. Hence, there exists $t_\epsilon \in (\frac{\tau_0-A\epsilon}{\epsilon}, \frac{\tau_0+A\epsilon}{\epsilon})$ such that $x_\epsilon(t_\epsilon) \in \Omega_0$. Furthermore, the following estimate is valid:

$$|x_\epsilon(t_\epsilon) - w(\tau_0)| \leq |x_\epsilon(t_\epsilon) - w(\epsilon t_\epsilon)| + |w(\epsilon t_\epsilon) - w(\tau_0)| \leq C\epsilon + M|\epsilon t_\epsilon - \tau_0| \leq \epsilon(C + MA). \qquad \square$$

6. Uniformly transversal systems

In this section, we prove an averaging theorem for uniformly transversal systems. First we prove an auxiliary result.

Lemma 6.1. *Let (Ω, G, f^-, f^+) be a well-behaved piecewise-smooth system that is uniformly transversal at $x_0 \in \Omega_0$. Let $w : [0, L] \to \Omega$ be a solution of the guiding system $\dot{w} = \bar{f}^\pm(w), w(0) = a \in \Omega_-$ which intersects the boundary Ω_0 in exactly one point $w(\tau_0) = x_0$ at $\tau_0 \in (0, L)$.*

Suppose that there is $\epsilon_1 \in (0, 1]$ such that for all $\epsilon \in (0, \epsilon_1]$ there is a unique solution $x_\epsilon : [0, \frac{L}{\epsilon}] \to \Omega$ of the problem $\dot{x}_\epsilon = \epsilon f^\pm(x_\epsilon, t, \epsilon)$, $x_\epsilon(0) = a$ intersecting the boundary Ω_0 at exactly one point in time.

Then there are $\epsilon_0 \in (0, \epsilon_1]$ and $C > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and all $t \in [0, \frac{L}{\epsilon}]$,

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon. \tag{6.1}$$

Proof. We know that w is a solution of $\dot{w}(\tau) = \bar{f}^-(w(\tau))$ for $\tau \in [0, \tau_0]$, satisfying $w(0) = a$. Since the right-hand side $\bar{f}^-$ is defined on the whole Ω, there is a solution $w^- : [0, L^-] \to \Omega$ of the same problem for some $L^- > \tau_0$. This solution is equal to w on $[0, \tau_0]$. The uniform transversality at x_0 implies that $\dot{w}^-(\tau_0) \neq 0$. Hence the solution w^- intersects Ω_0 exactly once if L^- is sufficiently close to τ_0.

In what follows, the constants ϵ_0 and C may vary from step to step. By Theorem 2.2, for every $\epsilon \in (0, \epsilon_0]$, there is a unique solution $x^-_\epsilon : [0, \frac{L^-}{\epsilon}] \to \Omega$ of the initial value problem $\dot{x} = \epsilon f^-(x, t, \epsilon)$, $x(0) = a$ such that

$$|x^-_\epsilon(t) - w^-(\epsilon t)| \leq C\epsilon \tag{6.2}$$

for all $t \in [0, \frac{L^-}{\epsilon}]$. The assumption of uniform transversality at x_0 and Lemma 5.2 imply that there are positive constants A, B such that for all $\epsilon \in (0, \epsilon_1]$, the solution x_ϵ^- intersects Ω_0 and it holds

$$\left| t_\epsilon - \frac{\tau_0}{\epsilon} \right| \leq A \quad \text{and} \quad |x_\epsilon^-(t_\epsilon) - x_0| \leq B\epsilon \tag{6.3}$$

where t_ϵ is the earliest time of intersection. Our assumption on f^- implies that t_ϵ is the time of the only intersection. Clearly, the solution x_ϵ^- is equal to x_ϵ on $[0, t_\epsilon]$ and $x_\epsilon^-(t) \in \Omega_+$ for $t \in (t_\epsilon, L]$.

Since the functions $f^\pm$ are bounded, the solutions x_ϵ, w are Lipschitz continuous. For $t \in [\frac{\tau_0}{\epsilon} - A, \frac{\tau_0}{\epsilon} + A]$, we estimate

$$|x_\epsilon(t) - w(\epsilon t)| \leq \left| x_\epsilon(t) - x_\epsilon^-\left(\frac{\tau_0}{\epsilon} - A\right)\right| + \left| x_\epsilon^-\left(\frac{\tau_0}{\epsilon} - A\right) - w^-(\tau_0 - \epsilon A)\right| + |w^-(\tau_0 - \epsilon A) - w(\epsilon t)|.$$

Since $x_\epsilon^-(\frac{\tau_0}{\epsilon} - A) = x_\epsilon(\frac{\tau_0}{\epsilon} - A)$ and $w^-(\tau_0 - \epsilon A) = w(\tau_0 - \epsilon A)$, using Lipschitz continuity of x_ϵ, w and inequality (6.2), we arrive at (6.1) for $t \in [\frac{\tau_0}{\epsilon} - A, \frac{\tau_0}{\epsilon} + A]$.

Now, let $t \in [\frac{\tau_0}{\epsilon} + A, \frac{L}{\epsilon}]$. For all such t we have that $w(t), x_\epsilon(t) \in \Omega_+$. Hence x_ϵ is a solution of equation $\dot{x} = \epsilon f^+(x, t, \epsilon)$ and w is a solution of the corresponding guiding system. Lemma 2.4 with $L_1 = \tau_0$, $a = w(\tau_0)$ and $b_\epsilon = x_\epsilon(\frac{\tau_0}{\epsilon} + A)$ implies the existence of $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, it holds

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon$$

for all $t \in [\frac{\tau_0}{\epsilon} + A, \frac{L}{\epsilon}]$. This estimate concludes the proof. □

Theorem 6.2. *Let (Ω, G, f^-, f^+) be a well-behaved piecewise-smooth system that is uniformly transversal at $x_0 \in \Omega_0$. Let $w : [0, L] \to \Omega$ be a solution of the guiding system $\dot{w} = \bar{f}^\pm(w)$, $w(0) = a \in \Omega_-$ which intersects the boundary Ω_0 in exactly one point $w(\tau_0) = x_0$ at time $\tau_0 \in (0, L)$.*

Then there are $\epsilon_0 > 0$ and $C > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ there exists a unique solution x_ϵ of the problem $\dot{x}_\epsilon = \epsilon f^\pm(x, t, \epsilon)$, $x_\epsilon(0) = a$, intersecting Ω_0 exactly once, the intersection is transversal, and for all $t \in [0, \frac{L}{\epsilon}]$, it holds

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon. \tag{6.4}$$

Proof. The main idea of the proof is to use the averaging theorem, Theorem 2.2, for smooth systems in Ω_- and Ω_+ separately and to construct a solution of the piecewise-smooth system $\dot{x} = \epsilon f^\pm(x, t, \epsilon)$ that crosses Ω_0 only once.

The proof is divided into two parts. Lemma 6.1 shows that, given a transversal solution of $\dot{x} = \epsilon f^\pm(x, t, \epsilon)$, the averaging estimate (6.4) holds for all sufficiently small $\epsilon > 0$.

Now, it only remains to show that there exists a unique transversal solution for all sufficiently small ϵ. The uniform transversality at x_0 implies the existence of $d, m > 0$ such that $\overline{B_{2d}(x_0)} \subset \Omega$ and

$$f^\pm(x, t, \epsilon) \cdot \nabla G(x) \geq \frac{m}{2}$$

for all $x \in \overline{B_{2d}(x_0)}$, $t \in \mathbb{R}$, $\epsilon \in [0, 1]$.

Let M be a bound of both $|f^+|$ and $|f^-|$. Analogously to the proof of Lemma 6.1, there is $L^- > \tau_0$ and a solution $w^- : [0, L^-] \to \Omega$ of $\dot{w} = \bar{f}^-(w)$, $w(0) = a$ that intersects Ω_0 exactly once and $w^-(\tau) = w(\tau)$ for $\tau \in [0, \tau_0]$. Due to the averaging theorem for smooth systems, there is $\epsilon_1 \in (0, 1]$ and $C > 0$

such that for any $\epsilon \in (0, \epsilon_0]$, there is a unique solution $x_\epsilon^- : [0, \frac{L^-}{\epsilon}] \to \Omega$ of the initial value problem $\dot{x} = \epsilon f^-(x, t, \epsilon)$, $x(0) = a$ such that it holds

$$|x_\epsilon^-(t) - w^-(\epsilon t)| \leq C\epsilon$$

for all $t \in [0, \frac{L^-}{\epsilon}]$. In the following again, the constants ϵ_0 and C may vary from step to step. By Lemma 5.2, there are constants A, B such that for all $\epsilon \in (0, \epsilon_0]$ the solution x_ϵ^- intersects Ω_0 and it holds

$$\left| t_\epsilon - \frac{\tau_0}{\epsilon} \right| \leq A \quad \text{and} \quad |x_\epsilon^-(t_\epsilon) - x_0| \leq B\epsilon$$

where t_ϵ is the time of the first contact. Let ϵ_0 be so small that the following inequalities are true

$$B\epsilon_0 \leq d, \quad \epsilon_0 AM \leq \frac{d}{2}, \quad \epsilon_0 A \leq \min\left\{\frac{d}{2M}, L - \tau_0\right\}. \tag{6.5}$$

The first inequality of (6.5) implies

$$|x_\epsilon^-(t_\epsilon) - x_0| \leq d$$

for all $\epsilon \in (0, \epsilon_0]$.

Now, let $\epsilon \in (0, \epsilon_0]$ be fixed. Denote $x_\epsilon^+(t) : [t_\epsilon, s] \to \Omega$ the unique local solution of the initial value problem $\dot{x} = \epsilon f^+(x, t, \epsilon)$, $x(t_\epsilon) = x^-(t_\epsilon)$ for some $s > t_\epsilon$. This solution can be extended until it approaches $\partial\Omega$. For $t \in [t_\epsilon, s]$, we estimate

$$|x_\epsilon^+(t) - x_0| \leq |x_\epsilon^+(t) - x_\epsilon^+(t_\epsilon)| + |x_\epsilon^+(t_\epsilon) - x_0| \leq \epsilon M|s - t_\epsilon| + d. \tag{6.6}$$

Denote $\Delta = \min\{\frac{d}{2M}, L - \tau_0\}$. From the third inequality of (6.5) we deduce that

$$\frac{\tau_0 + \Delta}{\epsilon} \geq \frac{\tau_0 + \epsilon A}{\epsilon} \geq t_\epsilon.$$

Assume that $s \leq \frac{\tau_0 + \Delta}{\epsilon}$. The inequality $t_\epsilon \geq \frac{\tau_0 - \epsilon A}{\epsilon}$ together with (6.5) imply

$$\epsilon M|s - t_\epsilon| \leq \epsilon M \left| \frac{\tau_0 + \Delta}{\epsilon} - \frac{\tau_0 - \epsilon A}{\epsilon} \right| = M(\Delta + \epsilon A) \leq \frac{d}{2} + \frac{d}{2} = d.$$

Combining this estimate with (6.6), we get

$$x_\epsilon^+(t) \in \overline{B_{2d}(x_0)}$$

for any $t \in [t_\epsilon, s]$. This means that the solution x_ϵ^+ is separated from the boundary $\partial\Omega$ on $[t_\epsilon, s]$ for arbitrary $s \leq \frac{\tau_0 + \Delta}{\epsilon}$. Hence, it exists on $[t_\epsilon, \frac{\tau_0 + \Delta}{\epsilon}]$. Moreover, since $x_\epsilon^+(t) \in \overline{B_{2d}(x_0)}$ for all $t \in [t_\epsilon, \frac{\tau_0 + \Delta}{\epsilon}]$, it holds

$$G(x_\epsilon^+(t)) = G(x_\epsilon^+(t_\epsilon)) + \epsilon \int_{t_\epsilon}^{t} f^+(x_\epsilon^+(s), s, \epsilon) \cdot \nabla G(x_\epsilon^+(s)) \, \mathrm{d}s \geq \frac{\epsilon m(t - t_\epsilon)}{2} > 0$$

for all $t \in [t_\epsilon, \frac{\tau_0 + \Delta}{\epsilon}]$, and so $x_\epsilon^+(t) \in \Omega_+$.

For $\epsilon \in [0, \epsilon_0]$, define $x_\epsilon : [0, \frac{\tau_0 + \Delta}{\epsilon}] \to \Omega$ as follows:

$$x_\epsilon(t) = \begin{cases} x_\epsilon^-(t), & t \in [0, t_\epsilon], \\ x_\epsilon^+(t), & t \in (t_\epsilon, \frac{\tau_0 + \Delta}{\epsilon}]. \end{cases}$$

This is a solution of the initial value problem $\dot{x} = f^{\pm}(x, t, \epsilon)$, $x(0) = a$, which intersects Ω_0 exactly once and the intersection is transversal. Lemma 6.1 implies validity of estimate (6.4) for $\epsilon \in (0, \epsilon_0]$ and $t \in [0, \frac{\tau_0+\Delta}{\epsilon}]$.

If $\tau_0 + \Delta = L$ the proof is done. Thus assume that $\tau_0 + \Delta < L$. Let

$$h = \operatorname{dist}(w([\tau_0 + \Delta, L]), \Omega_0) > 0.$$

Since $\dot{w}(\tau) = \bar{f}^{+}(w(\tau))$ for $\tau \in (\tau_0, L)$, we can use Lemma 2.4 with $L_1 = \tau_0 + \Delta$, $a = w(L_1)$, $b_\epsilon = x_\epsilon(\frac{\tau_0+\Delta}{\epsilon})$ and $A = 0$. Hence, there is a unique solution $x_\epsilon^{++} : [\frac{\tau_0+\Delta}{\epsilon}, \frac{L}{\epsilon}] \to \Omega$ such that

$$|x_\epsilon^{++}(t) - w(\epsilon t)| \leq C\epsilon$$

for all $t \in [\frac{\tau_0+\Delta}{\epsilon}, \frac{L}{\epsilon}]$ and $\epsilon \in (0, \epsilon_0]$. Observe that $x_\epsilon^{++}(t) \in \Omega_+$ for all $t \in [\frac{\tau_0+\Delta}{\epsilon}, \frac{L}{\epsilon}]$ and $\epsilon \in (0, \frac{h}{C}]$.

Finally for $\epsilon \in [0, \epsilon_0]$, we define the desired solution $x_\epsilon : [0, \frac{L}{\epsilon}] \to \Omega$ as

$$x_\epsilon(t) = \begin{cases} x_\epsilon^{-}(t), & t \in [0, t_\epsilon], \\ x_\epsilon^{+}(t), & t \in (t_\epsilon, \frac{\tau_0+\Delta}{\epsilon}], \\ x_\epsilon^{++}(t), & t \in (\frac{\tau_0+\Delta}{\epsilon}, \frac{L}{\epsilon}] \end{cases}$$

and estimate (6.4) is valid.

The uniqueness of the solution x_ϵ follows from the smoothness of f^-, f^+ and the transversality. □

7. Uniformly sliding systems

Notice that ∇G is used in the definition of the function g. To ensure that the function g is Lipschitz continuous, which is needed to use the averaging theorem for smooth systems, we will need to impose some further conditions on ∇G. One can see that we deal with a differential equation defined on $(n-1)$-dimensional surface. We will show that it can be analyzed using the techniques available for differential equations defined on open subsets of $\mathbb{R}^n$ by extending the equation to an open set in Ω.

Lemma 7.1. *Let* (Ω, G, f^-, f^+) *be a well-behaved piecewise-smooth system that is uniformly sliding and let* g *be defined by* (3.1)*. Let* $\mathcal{M}$ *be a compact subset of* Ω_0 *and let* ∇G *be continuously differentiable and Lipschitz continuous on some open neighborhood* V *of* $\mathcal{M}$*. Then there exists an open set* $U \subset V$ *such that* $\mathcal{M} \subset U$*,* g *can be defined by the formula* (3.1) *on* $U \times \mathbb{R} \times [0, 1]$*, and the function* $g : U \times \mathbb{R} \times [0, 1] \to \mathbb{R}^n$ *is T-PCDLB.*

Proof. We define a function $\phi : \Omega \to \mathbb{R}$ as

$$\phi(x) = \min_{t \in [0,T],\, \epsilon \in [0,1]} \{f^-(x, t, \epsilon) \cdot \nabla G(x) - f^+(x, t, \epsilon) \cdot \nabla G(x)\}.$$

The uniform sliding implies that $\phi(x) \geq 2m$ for all $x \in \Omega_0$, where m is a positive constant from the definition of uniformly sliding systems. For every $x \in \mathcal{M}$, there is a radius $d > 0$ such that $\overline{B_d(x)} \subset V$ and $\phi(y) \geq m$ for all $y \in B_d(x)$. Since the compact set $\mathcal{M}$ is covered by these balls, there exists a finite open subcover of $\mathcal{M}$. Let U be the union of the balls from the finite open subcover of $\mathcal{M}$. It is clear that $\overline{U} \subset V$ and $\phi(x) \geq m$ for all $x \in U$.

This implies that for all $x \in U$, $t \in \mathbb{R}$ and $\epsilon \in [0, 1]$,

$$f^-(x, t, \epsilon) \cdot \nabla G(x) - f^+(x, t, \epsilon) \cdot \nabla G(x) \geq m.$$

Hence, the function g can be extended to $\overline{U}$ and one can prove that it has the desired properties. □

We are now ready to formulate and prove an averaging theorem for sliding solutions of piecewise-smooth systems.

Theorem 7.2. *Let (Ω, G, f^-, f^+) be a well-behaved piecewise-smooth system that is uniformly sliding. Let $w : [0, L] \to \Omega$ be a solution of the sliding adjusted guiding system*

$$\dot{w} = \begin{cases} \bar{f}^-(w), & x \in \Omega_-, \\ \bar{g}(w), & x \in \Omega_0 \end{cases}$$

associated with the initial value problem $\dot{x} = \epsilon f^{\pm}(x, t, \epsilon)$, $x(0) = a \in \Omega_-$, where g is given by (3.1). *Let τ_0 be such that $0 < \tau_0 < L$, $w(\tau) \in \Omega_-$ for $\tau \in [0, \tau_0)$ and $w(\tau) \in \Omega_0$ for $\tau \in [\tau_0, L]$. Furthermore, let there be an open set V such that $w([\tau_0, L]) \subset V \subset \Omega$ and ∇G is continuously differentiable and Lipschitz continuous on V.*

Then there is a constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, there is a unique solution x_ϵ of the problem $\dot{x} = \epsilon f^{\pm}(x, t, \epsilon)$, $x(0) = a \in \Omega_-$ defined on $[0, \frac{L}{\epsilon}]$ and sliding from time t_ϵ. In addition, there are constants $C, A > 0$ such that $|t_\epsilon - \frac{\tau_0}{\epsilon}| \leq A$ and

$$|x_\epsilon(t) - w(\epsilon t)| \leq C\epsilon$$

for all $t \in [0, \frac{L}{\epsilon}]$.

Proof. The uniform sliding condition implies that

$$\bar{f}^-(w) \cdot \nabla G(w) > 0.$$

Hence there is a solution w^- of $\dot{w} = \bar{f}^-(w)$, $w(0) = a$ defined on $[0, L^-]$ for some $L^- > \tau_0$ such that $w^-(\tau) \in \Omega_+$ for all $\tau \in (\tau_0, L^-]$.

The averaging theorem for smooth systems implies that there exist $\epsilon_0, C > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, there is a unique solution x_ϵ^- of $\dot{x} = \epsilon f^-(x, t, \epsilon)$, $x(0) = a$ defined on $[0, \frac{L^-}{\epsilon}]$ and it holds

$$|x_\epsilon^-(t) - w^-(\epsilon t)| \leq C\epsilon \tag{7.1}$$

for every $t \in [0, \frac{L^-}{\epsilon}]$. Lemma 5.2 implies that there are constants A, B such that for all $\epsilon \in (0, \epsilon_0]$, the solution x_ϵ^- crosses Ω_0 and we have

$$\left| t_\epsilon - \frac{\tau_0}{\epsilon} \right| \leq A, \quad |x_\epsilon^-(t_\epsilon) - x_0| \leq B\epsilon,$$

where t_ϵ is the time of the first contact of x_ϵ^- with Ω_0.

For $\tau \in [\tau_0, L]$, we define $w^0(\tau) = w(\tau)$. Thus the function w^0 is the solution of the initial value problem $\dot{w} = \bar{g}(w)$, $w(\tau_0) = x_0$.

Due to Lemma 7.1, there is an open set $U \subset V$ such that g is T-PCDLB on U. The uniform sliding condition implies that the solution x_ϵ^- can be uniquely extended as a solution x_ϵ^0 of the initial value problem $\dot{x} = \epsilon g(x, t, \epsilon)$, $x(t_\epsilon) = x_\epsilon^-(t_\epsilon)$. Trajectories of solutions of this system lie in Ω_0 and the local solution x_ϵ^0 can be extended at least as long as the solution does not approach the boundary of U. Suppose that x_ϵ^0 can be defined at least up to some $s > t_\epsilon$.

Assume that $s \le \frac{\tau_0}{\epsilon} + A$. For every $t \in [t_\epsilon, s]$, we estimate

$$|x_\epsilon^0(t) - w^0(\epsilon t)| \le |x_\epsilon^0(t) - x_\epsilon^0(t_\epsilon)| + \left| x_\epsilon^-(t_\epsilon) - x_\epsilon\left(\frac{\tau_0}{\epsilon} - A\right)\right|$$
$$+ \left| x_\epsilon^-\left(\frac{\tau_0}{\epsilon} - A\right) - w^-(\tau_0 - \epsilon A)\right| + |w^-(\tau_0 - \epsilon A) - w^-(\tau_0)| + |w^0(\tau_0) - w^0(\epsilon t)|.$$

Using estimate (7.1) and the Lipschitz continuity of solutions x_ϵ^0, x_ϵ^- and w^-, w^0, we arrive at

$$|x_\epsilon^0(t) - w^0(\epsilon t)| \le C\epsilon$$

for $t_\epsilon \le t \le s \le \frac{\tau_0}{\epsilon} + A$. Observe that C does not depend on s, if $s \le \frac{\tau_0}{\epsilon} + A$ and x_ϵ^0 is separated from ∂U for ϵ_0 sufficiently small. Since s was arbitrary, the solution x_ϵ^0 can be defined up to $\frac{\tau_0}{\epsilon} + A$.

At this point, we can use Lemma 2.4 on the initial value problem $\dot{x} = \epsilon g(x, t, \epsilon)$, $x(\frac{\tau_0}{\epsilon} + A) = x_\epsilon^0(\frac{\tau_0}{\epsilon} + A)$ with $w^0 : [\tau_0, L] \to \Omega$ and $L_1 = \tau_0$, $a = w^0(\tau_0)$, $L_2 = L$. The assumptions of Lemma 2.4 are satisfied since

$$\left| x_\epsilon^0\left(\frac{\tau_0}{\epsilon} + A\right) - w^0(\tau_0)\right| \le \left| x_\epsilon^0\left(\frac{\tau_0}{\epsilon} + A\right) - w^0(\tau_0 + \epsilon A)\right| + |w^0(\tau_0 + \epsilon A) - w^0(\tau_0)| \le C\epsilon.$$

Hence for all $\epsilon \in (0, \epsilon_0]$, the solution x_ϵ^0 can be extended to the interval $[\frac{\tau_0}{\epsilon}, \frac{L}{\epsilon}]$, and it holds

$$|x_\epsilon^0(t) - w^0(\epsilon t)| \le C\epsilon$$

for all $t \in [\frac{\tau_0}{\epsilon} + A, \frac{L}{\epsilon}]$. It is clear that the function

$$x_\epsilon(t) = \begin{cases} x_\epsilon^-(t), & t \in [0, t_\epsilon], \\ x_\epsilon^0(t), & t \in (t_\epsilon, \frac{L}{\epsilon}] \end{cases}$$

is the desired unique solution. □

8. Conclusion

We have contributed to the study of piecewise-smooth differential equations by extending the results of averaging methods for periodic systems to the piecewise-smooth setting. Our main contribution is the analysis of how the boundary behavior influences averaging, especially in case of sliding. We have also proposed some sufficient conditions that allowed us to prove results on averaging in case of transversal crossing and sliding. Moreover, we demonstrated how to achieve results for solutions of piecewise-smooth systems by dividing them into solutions of smooth systems and patching together results from the smooth setting. Future opportunities for study include weakening of the assumptions in the transversal and sliding case and extension of the results to situations, in which $f^+ \cdot \nabla G$ or $f^- \cdot \nabla G$ vanish, which includes grazing solutions. Moreover, it would be interesting to develop an averaging method for differential equations with impacts [2].

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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