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Research article

On traveling wave solutions of a class of KdV-Burgers-Kuramoto type equations

Yanxia Hu* **and Qian Liu**

School of Mathematics and Physics, North China Electric Power University, Beijing, 102206, China

* **Correspondence:** Email: yxiahu@163.com.

Abstract: In the paper, the traveling wave solutions of a KdV–Burgers-Kuramoto type equation with arbitrary power nonlinearity are considered. Lie symmetry analysis method on the equation is performed, which shows that the equation possesses traveling wave solutions. By qualitative analysing the equivalent autonomous system of the traveling wave equation of the equation, the existence of the traveling wave solutions of the equation is presented. Through analysing the associated determining system, the non-trivial infinitesimal generator of Lie symmetry admitted by the traveling wave solutions equation under the certain parametric conditions is found. The traveling wave solutions of the KdV–Burgers-Kuramoto type equation by solving the invariant surface condition equation under the certain parametric conditions are obtained.

Keywords: KdV-Burgers-Kuramoto type equation; qualitative analysis; traveling wave solutions; Lie symmetry group
Mathematics Subject Classification: 34A05, 34A34

1. Introduction

Many phenomena in physics, engineering and other subjects can be described by nonlinear partial differential equations. In order to well understand various nonlinear phenomena in nature, the exact solutions for the nonlinear partial differential equations has to be studied. Finding effective methods to solve and analyse these equations has been an interesting subject in the field of differential equations. Sometimes, It is difficult to express various wave solutions of nonlinear partial differential equations explicitly in terms of elementary functions. In many cases it is possible to find and prove the existence of traveling wave solutions by the qualitative theory of differential equations and dynamical systems. From the physical point of view, traveling waves usually describe transition processes. Such as solitons and propagation with a finite speed, and thus they give more insight into the physical aspects of the problems. In order to obtain the traveling wave solutions of nonlinear partial differential equations, several powerful approaches to finding the exact solutions have been proposed, such as Backlund transformation method in [1], Darboux transformation [2–6], Hirota's method [7–9], the Fokas method [10], exp-function method in [11, 12], tanh-sech method in [13], hyperbolic function method in [14], first integral method in [15, 16], homogeneous balance principle in [17], Lie symmetry analysis method in [18, 19] and the references therein [20–22]. Lie group theory plays an important role in studying the solutions of differential equations, for example, see [23–29]. In [25], Lie symmetry analysis is performed on the fifth-order KdV types of equations which arise in modelling many physical phenomena, and the exact analytic solutions are obtained.

In this paper, we consider a class of KdV-Burgers-Kuramoto (KdV-BK) type equation with arbitrary power nonlinearity,

$$u_t + u^p u_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{1.1}$$

where $\alpha, \beta, \gamma \neq 0$ and $p \neq 0$ are real constants. It contains dispersive terms u_{xx}, u_{xxx} and u_{xxxx} and also the nonlinear term $u^p u_x$. Here the choice of $p = 1$ leads equation (1.1) to the KdV-Burgers-Kuramoto(KBK) equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0. \tag{1.2}$$

Equation (1.1) is referred to as the generalized KdV-Burgers-Kuramoto(KBK) equation. In [30], equation (1.1) was used for explanation of the origin of persistent wave propagation through medium of reaction-diffusion type. In [31, 32], the motion of a viscous incompressible flowing down an inclined plane is studied by the application of equation (1.1). Mathematical model for consideration dissipative waves in plasma physics by means of equation (1.1) was presented in [33]. The equation (1.2) is simultaneously involved in nonlinearity, dissipation, dispersion and instability, and is suggested by Kuramoto [34, 35]. The KBK equation is a classical nonlinear partial differential equation, which effectively describes the turbulent motion and other unstable motions. It is well known that there are many works to deal with equation (1.2) in recent years, see [34, 35], [36–39] and the references therein. For example, In [36, 37], The travelling wave periodic solutions of equation (1.2) were considered and obtained.

In this paper, we are mainly interested in equation (1.1). The rest of the paper is organized as follows. In section 2, Lie symmetries for (1.1) are found by differentiating the symmetry condition, and the partial differential equation has traveling wave solutions in this sense is shown. In section 3, the existence of traveling wave solutions of the equation is presented by analysing the corresponding autonomous system. Under the certain parametric conditions, the non-trivial infinitesimal generator of Lie groups admitted by deduced ordinary differential equation is obtained through analysing the determining system in section 4. Based on the Lie symmetry theory, a class of traveling wave solutions of (1.1) is presented by solving the invariant surface condition equation under the certain parametric conditions in section 5. Section 6 is conclusions.

2. Lie symmetry to the KdV-Burgers-Kuramoto type equation

In this section, we consider Lie symmetry to equation (1.1). Based on Lie symmetry analysis theory [19], the generator X of Lie symmetry admitted by (1.1) has the expression

$$X = \xi(x, t)\partial_x + \tau(x, t)\partial_t + \eta(x, t, u)\partial_u,$$

where $\eta(x,t,u) = g(x,t)u+h(x,t)$, and $\xi(x,t)$, $\tau(x,t)$, $g(x,t)$, $h(x,t)$ are function need to be determined. The surface $u = u(x,t)$ is invariant, provided that

$$\eta - \xi u_x - \tau u_t = 0, \tag{2.1}$$

when $u = u(x,t)$. (2.1) is called the invariant surface condition. The linearized symmetry condition is as follows:

$$X^{(4)}\Delta = 0, \tag{2.2}$$

when $\Delta = 0$, where

$$\Delta = u_{xxxx} + \frac{1}{\gamma}(u_t + u^p u_x + \alpha u_{xx} + \beta u_{xxx}),$$

$$X^{(4)} = \xi\partial_x + \tau\partial_t + \eta\partial_u + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} + \eta^{xx}\partial_{u_{xx}} + \eta^{xxx}\partial_{u_{xxx}} + \eta^{xxxx}\partial_{u_{xxxx}},$$

and

$$\begin{aligned}
\eta^x &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t,\\
\eta^t &= \eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x,\\
\eta^{xx} &= \eta_{xx} + (2\eta_{ux} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt},\\
\eta^{xxx} &= \eta_{xxx} + (3\eta_{uxx} - \xi_{xxx})u_x - \tau_{xxx}u_t + (3\eta_{ux} - 3\xi_{xx})u_{xx} - 3\tau_{xx}u_{xt} + (\eta_u - 3\xi_x)u_{xxx} - 3\tau_x u_{txx},\\
\eta^{xxxx} &= \eta_{xxxx} + (4\eta_{uxxx} - \xi_{xxxx})u_x - \tau_{xxxx}u_t + (6\eta_{uxx} - 4\xi_{xxx})u_{xx} - 4\tau_{xxx}u_{tx}\\
&\quad +(4\eta_{ux} - 6\xi_{xx})u_{xxx} - 6\tau_{xx}u_{txx} + (\eta_u - 4\xi_x)u_{xxxx} - 4\tau_x u_{txxx}.
\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), one can have

$$\begin{aligned}
&(gu + h)pu^{p-1}u_x + [(g_x u + h_x) + (g - \xi_x)u_x - \tau_x u_t]u^p +\\
&\alpha[(g_{xx}u + h_{xx}) + (2g_x - \xi_{xx})u_x - \tau_{xx}u_t + (g - 2\xi_x)u_{xx} - 2\tau_x u_{xt}]\\
&+\beta[g_{xxx}u + h_{xxx} + (3g_{xx} - \xi_{xxx})u_x - \tau_{xxx}u_t + (3g_x - 3\xi_{xx})u_{xx} - 3\tau_{xx}u_{xt} + (g - 3\xi_x)u_{xxx} - 3\tau_x u_{txx}]\\
&+\gamma[g_{xxxx}u + h_{xxxx} + (4g_{xxx} - \xi_{xxxx})u_x - \tau_{xxxx}u_t + (6g_{xx} - 4\xi_{xxx})u_{xx} - 4\tau_{xxx}u_{tx}\\
&+(4g_x - 6\xi_{xx})u_{xxx} - 6\tau_{xx}u_{txx} + (g - 4\xi_x)u_{xxxx} - 4\tau_x u_{txxx}] + g_t u + h_t + (g - \tau_t)u_t - \xi_t u_x = 0.
\end{aligned} \tag{2.4}$$

The linearized symmetry condition gives us a systematic approach to finding Lie point symmetries. We can obtain u_{xxxx} from the restriction $\Delta = 0$ and substitute u_{xxxx} into (2.4). According to their dependence on derivative of u, we can get a linear system of determining equation for ξ, τ g, h :

$$\begin{aligned}
&\tau_x = 0;\\
&-3\beta\tau_x - 6\gamma\tau_{xx} = 0;\\
&\beta(g - 3\xi_x) + \gamma(4g_x - 6\xi_{xx}) - \beta(g - 4\xi_x) = 0;\\
&-2\alpha\tau_x - 3\beta\tau_{xx} - 4\gamma\tau_{xxx} = 0;\\
&\gamma(6g_{xx} - 4\xi_{xxx}) + \beta(3g_x - 3\xi_{xx}) + \alpha(g - 2\xi_x) - \alpha(g - 4\xi_x) = 0;\\
&-\gamma\tau_{xxxx} - \beta\tau_{xxx} - \alpha\tau_{xx} + g - \tau_t - (g - 4\xi_x) = 0;\\
&\gamma(4g_{xxx} - \xi_{xxxx}) + \beta(3g_{xx} - \xi_{xxx}) + \alpha(2g_x - \xi_{xx}) - \xi_t = 0;\\
&pg + (g - \xi_x) - (g - 4\xi_x) = 0;\\
&hp = 0;\\
&g_x = 0;\\
&h_x = 0;\\
&\alpha g_{xx} + \beta g_{xxx} + \gamma g_{xxxx} + g_t = 0;\\
&\alpha h_{xx} + \beta h_{xxx} + \gamma h_{xxxx} + h_t = 0.
\end{aligned} \tag{2.5}$$

We solve the first equation of (2.5) to obtain

$$\tau = A(t),$$

where $A(t)$ is an arbitrary function.

Owing to $p \neq 0$, the ninth equation of (2.5) yields

$$h = 0,$$

and the tenth equation of (2.5) yields

$$g = B(t),$$

where $B(t)$ is an arbitrary function. The twelfth equation of (2.5) tells us

$$B(t) = c_3,$$

where c_3 is an arbitrary constant. So,

$$g = c_3.$$

From the eighth equation of (2.5), we have

$$\xi_x = -\frac{1}{3}pc_3.$$

The sixth equation of (2.5) gives us the result

$$\tau_t = -\frac{4}{3}pc_3.$$

Substituting the expression of ξ_x into the third equation of (2.5) yields $c_3 = 0$. Thus, $g = 0, \xi_x = 0,\ \tau_t = 0$. The fifth equation of (2.5) is satisfied apparently. One can get $\xi_t = 0$ from the seventh equation of (2.5). So, the generator of Lie symmetry is

$$X = c_1\partial_x + c_2\partial_t,$$

where $c_1,\ c_2$ are arbitrary constants.

The derived above generator X implies that (1.1) has a invariant solution in the form

$$u = u(v),\quad v = x - ct,$$

where c is an arbitrary constant.

3. The analysis of the traveling wave solutions of the KdV-Burgers-Kuramoto type equation

Substituting the traveling wave solution of the form $u = u(v),\ v = x - ct$ into (1.1), we can get the traveling wave solution equation as follows,

$$-cu' + u^p u' + \alpha u'' + \beta u''' + \gamma u^{(4)} = 0. \tag{3.1}$$

For (3.1) with $p \neq -1$, performing the integration once, one has the following traveling wave solutions equation

$$-cu + \frac{u^{p+1}}{p+1} + \alpha u' + \beta u'' + \gamma u''' = k, \tag{3.2}$$

where k is an integration constant. Let $R = \frac{du}{dv}$, $w = \frac{dR}{dv}$, $k = 0$, (3.2) can be rewritten as

$$\begin{cases} u' = R \\ R' = w \\ w' = \frac{1}{\gamma}(cu - \frac{u^{p+1}}{p+1} - \alpha R - \beta w). \end{cases} \tag{3.3}$$

As we know, a solitary wave solution of (1.1) corresponds to a heteroclinic orbit of system (3.3). We will prove that the heteroclinic orbit of system (3.3) does exist when the parametric conditions are satisfied correspondingly.

For convenience, we consider p being a positive integer. When p is an even number and $(p+1)c > 0$, (3.3) has three equilibrium points

$$u_1^* = (0,0,0),\ u_{2,3}^* = (\pm \sqrt[p]{(1+p)c}, 0, 0).$$

When p is an odd number, (3.3) has two equilibrium points

$$u_1^* = (0,0,0),\ u_2^* = (\sqrt[p]{(1+p)c}, 0, 0).$$

The coefficient matrix A_1 of the linearization system of (3.3) about the equilibrium point u_1^* is

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{c}{\gamma} & -\frac{\alpha}{\gamma} & -\frac{\beta}{\gamma} \end{bmatrix}.$$

Then, the characteristic equation for A_1 is

$$f_1(\lambda) = \lambda^3 + \frac{\beta}{\gamma}\lambda^2 + \frac{\alpha}{\gamma}\lambda - \frac{c}{\gamma}. \tag{3.4}$$

Similarly, one can get the coefficient matrix A_2 of the linearization system of (3.3) about the equilibrium points $u_{2,3}^*$. The matrix is

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{pc}{\gamma} & -\frac{\alpha}{\gamma} & -\frac{\beta}{\gamma} \end{bmatrix}.$$

Then, the characteristic equation for A_2 is

$$f_2(\lambda) = \lambda^3 + \frac{\beta}{\gamma}\lambda^2 + \frac{\alpha}{\gamma}\lambda + \frac{pc}{\gamma}. \tag{3.5}$$

We have the following result for (3.3).

Theorem 3.1. If α,β,γ and c are of same sign, p is a positive integer and $pc\gamma - \alpha\beta < 0$, then the equilibrium point A_1 of (3.3) has a one-dimensional unstable manifold and the equilibrium point $A_{2,3}$ (or A_2) of (3.3) has a three-dimensional stable manifold.

Proof. The proof is based on Argument Principle. For proving the equilibrium point A_1 of (3.3) possessing a one-dimensional unstable manifold, one needs to prove the characteristic equation (3.4) has only one root in the right half complex plane, and for proving the equilibrium point $A_{2,3}$ (or A_2) of (3.3) possessing a three-dimensional stable manifold, one needs to prove the characteristic equation (3.5) has three roots in the left half complex plane.

Let us to consider $f_1(\lambda)$ firstly. Since $f_1(\lambda)$ is analytic in complex plane, the number of roots in the right half complex plane is

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_C arg f_1(z), \tag{3.6}$$

where C is composed of $\Gamma_R: \quad z = Re^{i\theta},\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, a straight line $\overrightarrow{(Ri,-Ri)}$ is on the imaginary axis and $\Delta_C arg f_1(z)$ denotes the total change quantity in the argument of $f_1(z)$ along C. Apparently, (3.6) equals

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg f_1(z) + \frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\overrightarrow{(Ri,-Ri)}} arg f_1(z).$$

The first part of the above formula is

$$\begin{aligned}
&\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg z^3 + \frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg\Big(1+\frac{\frac{\beta}{\gamma}z^2+\frac{\alpha}{\gamma}z-\frac{c}{\gamma}}{z^3}\Big)\\
&= \frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg(R^3e^{3\theta i})\\
&= \frac{3}{2}.
\end{aligned}$$

The second part of that formula is

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\overrightarrow{(R,-R)}} arg f_1(iy),$$

where $f_1(iy) = (-\frac{\beta}{\gamma}y^2 - \frac{c}{\gamma}) + (\frac{\alpha}{\gamma}y - y^3)i$, and $f_1(0) = -\frac{c}{\gamma}$. Because β,γ and c are of same sign, $\frac{\beta}{\gamma} > 0,\ \frac{c}{\gamma} > 0$. So $Re(f_1(iy)) < 0$ and $f(0) < 0$, the image $f_1(iy)$ only lies on the left complex plane. For $|R| \to \infty$, $f_1(iy)$ has the asymptotic behavior,

$$Re(f_1(iy)) \sim -\frac{\beta}{\gamma}y^2 < 0,\ Im(f_1(iy)) \sim -y^3.$$

So,

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\overrightarrow{(R,-R)}} arg(f_1(iy)) = \frac{\frac{\pi}{2}-\frac{3\pi}{2}}{2\pi} = -\frac{1}{2}.$$

The number of roots of $f_1(\lambda) = 0$ in the right half complex plane is 1. Therefore, the equilibrium point A_1 of (3.3) has a one-dimensional unstable manifold.

Similarly, we consider the number of roots of $f_2(\lambda) = 0$ in the left half complex plane. Since $f_2(\lambda)$ is analytic in complex plane, the number of roots in the left half complex plane is

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_C arg f_2(z), \tag{3.7}$$

where C is composed of $\Gamma_R: \quad z = Re^{i\theta}, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, a straight line $\overrightarrow{(-Ri, Ri)}$ on the imaginary axis and $\Delta_C arg f_2(z)$ denotes the total change quantity in the argument of $f_2(z)$ along C. Apparently, (3.7) equals

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg f_2(z) + \frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\overrightarrow{(-Ri,Ri)}} arg f_2(z).$$

The first part of the above formula is

$$\begin{aligned} &\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg z^3 + \frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg(1 + \frac{\frac{\beta}{\gamma}z^2 + \frac{\alpha}{\gamma}z + \frac{pc}{\gamma}}{z^3}) \\ &= \frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\Gamma_R} arg(R^3 e^{3\theta i}) \\ &= \frac{3}{2}. \end{aligned}$$

The second part of that formula is

$$\frac{1}{2\pi}\lim_{R\to\infty}\Delta_{\overrightarrow{(-R,R)}} arg f_2(iy),$$

where $f_2(iy) = (-\frac{\beta}{\gamma}y^2 + \frac{pc}{\gamma}) + (\frac{\alpha}{\gamma}y - y^3)i$ and $f_2(0) = \frac{pc}{\gamma}$.

We first compute the quantity $\Delta_{(-\infty,0)} arg(f_2(iy))$. Note that

$$Re(f_2(iy)) = -\frac{\beta}{\gamma}y^2 + \frac{pc}{\gamma}, \ Im(f_2(iy)) = \frac{\alpha}{\gamma}y - y^3,$$

it is obvious that as y increases from $-\infty$ to 0, $Re(f_2(iy))$ increases monotonously from $-\infty$ to $\frac{pc}{\gamma}$, and $Im(f_2(iy))$ decreases monotonously from $+\infty$ to $-\frac{2\alpha}{3\gamma}\sqrt{\frac{\alpha}{3\gamma}}$ afterwards increases monotonously to 0. Owing to $\sqrt{\frac{\alpha}{\gamma}} > \sqrt{\frac{pc}{\beta}}$ from the assumption $pc\gamma - \alpha\beta < 0$, as y increases from $-\infty$ to 0, the image $f_2(iy)$ starts in second quadrant of complex plane, intersects the minus Re-axis, passes through the third quadrant, intersects the Im-axis at a certain point, passes through the forth quadrant and finally ends up the point $(\frac{pc}{\gamma}, 0)$ of the positive Re-axis. So, $\Delta_{(-\infty,0)} arg(f_2(iy)) = \frac{3\pi}{2}$. Similarly, we can get $\Delta_{(0,+\infty)} arg(f_2(iy)) = \frac{3\pi}{2}$. Therefore, The number of roots of $f_2(\lambda) = 0$ in the left half complex plane is 3. Therefore, the equilibrium point $A_{2,3}$ (or A_2) of (3.3) has a three-dimensional stable manifold. This completes the proof.

The local trajectory in the neighborhood of these equilibrium points of (3.3) are shown in Figures 1 and 2. In Figure 1, we let $\alpha = 710, \beta = 200, \gamma = 170, p = 4, c = 1$, and the initial value is $(0.000001, -0.000001, -0.000001)$. In Figure 2, we let $\alpha = 710, \beta = 200, \gamma = 170, p = 4, c = 1$, and the initial value is $(-0.000001, 0.000001, 0.000001)$. In these Figures, there are the images in coordinate plane $(t, u), (t, R), (t, w), (u, R), (u, w), (R, w)$ and in coordinate space (u, R, w), which show that the equilibrium point $(0, 0, 0)$ is unstable and the equilibrium points $(\pm\sqrt[p]{(p+1)c}, 0, 0)$ is stable.

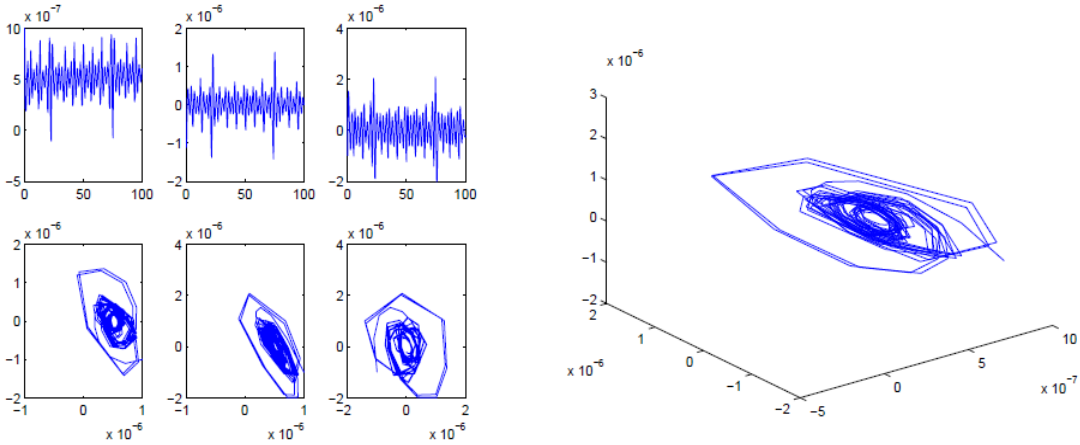

Figure 1. The local trajectory.
$\alpha = 710, \beta = 200, \gamma = 170, p = 4, c = 1$, and the initial value is $(0.000001, -0.000001, -0.000001)$

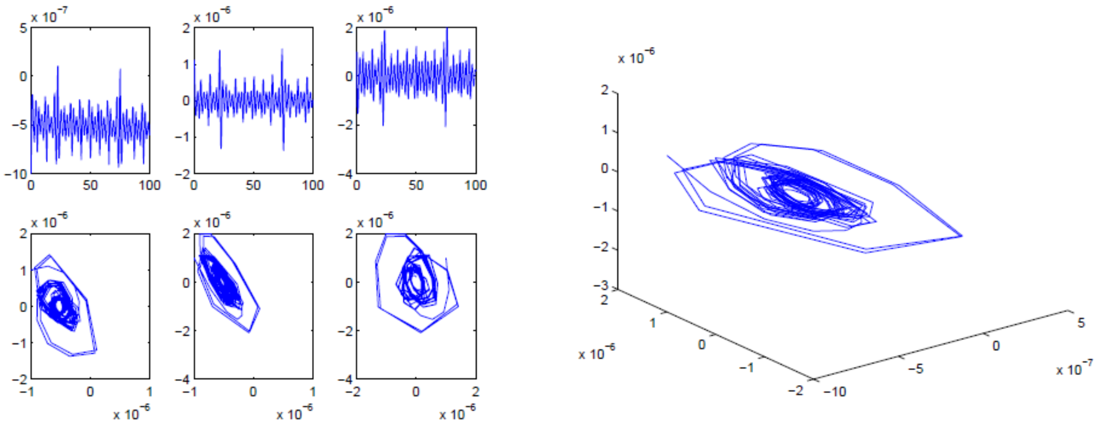

Figure 2. The local trajectory.
$\alpha = 710, \beta = 200, \gamma = 170, p = 4, c = 1$, and the initial value is $(-0.000001, 0.000001, 0.000001)$

Theorem 3.2. When α, β, γ, c are of same sign, p is a positive integer and $pc\gamma - \alpha\beta < 0$, (3.3) has potentially a heteroclinic orbit .

Proof. Owing to Theorem 3.1, the sum of the dimension of the unstable manifold $W^u(A_1)$ and the stable manifold $W^s(A_{2,3})$ is four. The dimension of the phase plane of (3.3) is three. Therefore, these two manifolds potentially intersect in R^3 along one-dimension curve, which is a heteroclinic orbit of (3.3).

As we know, the heteroclinic orbit of (3.3) corresponds a traveling wave solution of (1.1). Motivated by the above results, we will consider to obtain traveling wave solutions of (1.1) by using Lie symmetry method.

4. Lie symmetry to the traveling wave solutions equation

In the section, we consider to search for Lie symmetries admitted by (3.2).

4.1. $p \neq -1$

We suppose that $V = \xi(x,y)\dfrac{\partial}{\partial x} + \eta(x,y)\dfrac{\partial}{\partial y}$ is the infinitesimal generator of Lie symmetry admitted by (3.2). Here, for convenience, symbols y and x are in place of u and v, respectively.

Therefore,

$$\begin{aligned} V^{(3)} &= \xi(x,y)\tfrac{\partial}{\partial x} + \eta(x,y)\tfrac{\partial}{\partial y} + \eta^{(1)}(x,y,y')\tfrac{\partial}{\partial y'} \\ &+\eta^{(2)}(x,y,y',y'')\tfrac{\partial}{\partial y''} + \eta^{(3)}(x,y,y',y'')\tfrac{\partial}{\partial y'''} \end{aligned}$$

is the 3th-extended infinitesimal generator, where

$$\begin{aligned} \eta^{(1)}(x,y,y') &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\ \eta^{(2)}(x,y,y',y'') &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\ &+(\eta_y - 2\xi_x - 3\xi_y y')y'', \\ \eta^{(3)}(x,y,y',y'',y''') &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - 2\xi_{xxy})y'^2 \\ &+(\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 + 3(\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2)y'' \\ &-3\xi_y y''^2 + (\eta_y - 3\xi_x - 4\xi_y y')y'''. \end{aligned} \tag{4.1}$$

The linearized symmetry condition

$$V^{(3)}(y''' - f(x,y,y',y'')) = 0, \quad y''' = f(x,y,y',y''), \tag{4.2}$$

where $f(x,y,y',y'') = -\dfrac{y^{p+1}}{\gamma(p+1)} + \dfrac{c}{\gamma}y + \dfrac{k}{\gamma} - \dfrac{\alpha}{\gamma}y' - \dfrac{\beta}{\gamma}y''$. Plugging (4.1) to (4.2) and replacing y''' by $f(x,y,y',y'')$, This yields

$$\begin{aligned} &\eta(\frac{y^p}{\gamma} - \frac{c}{\gamma}) + \frac{\alpha}{\gamma}(\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2) \\ &+\frac{\beta}{\gamma}(\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'') \\ &+\eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - 2\xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 \\ &+3(\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2)y'' - 3\xi_y y''^2 \\ &+(\eta_y - 3\xi_x - 4\xi_y y')(-\frac{y^{p+1}}{\gamma(p+1)} + \frac{c}{\gamma}y + \frac{k}{\gamma} - \frac{\alpha}{\gamma}y' - \frac{\beta}{\gamma}y'') = 0. \end{aligned} \tag{4.3}$$

Both ξ and η are independent of y' and y'', After setting the coefficients of the powers $(y')^i(y'')^j(i,j = 0,1,2,3,4)$ in (4.3) to zero, one can get the determining equations system,

$$\xi_{yyy} = 0, \tag{4.4}$$

$$-\frac{\beta}{\gamma}\xi_{yy} + \eta_{yyy} - 3\xi_{xyy} = 0, \tag{4.5}$$

$$-\frac{\alpha}{\gamma}\xi_y + \frac{\beta}{\gamma}(\eta_{yy} - 2\xi_{xy}) + 3(\eta_{xyy} - 2\xi_{xxy}) + \frac{4\alpha}{\gamma}\xi_y = 0, \tag{4.6}$$

$$\xi_y = 0, \tag{4.7}$$

$$-3\frac{\beta}{\gamma}\xi_y + 3(\eta_{yy} - 3\xi_{xy}) + \frac{4\beta}{\gamma}\xi_y = 0, \tag{4.8}$$

$$\xi_{yy} = 0, \tag{4.9}$$

$$\frac{\beta}{\gamma}(\eta_y - 2\xi_x) + 3(\eta_{xy} - \xi_{xx}) - \frac{\beta}{\gamma}(\eta_y - 3\xi_x) = 0, \tag{4.10}$$

$$\frac{\alpha}{\gamma}(\eta_y - \xi_x) + \frac{\beta}{\gamma}(2\eta_{xy} - \xi_{xx}) + (3\eta_{xxy} - \xi_{xxx}) - \frac{\alpha}{\gamma}(\eta_y - 3\xi_x) = 0, \tag{4.11}$$

$$\eta(\frac{y^p}{\gamma} - \frac{c}{\gamma}) + \frac{\alpha}{\gamma}\eta_x + \frac{\beta}{\gamma}\eta_{xx} + \eta_{xxx} + (\eta_y - 3\xi_x)(-\frac{y^{p+1}}{(p+1)\gamma} + \frac{c}{\gamma}y + \frac{k}{\gamma}) = 0. \tag{4.12}$$

The equation (4.7) gives

$$\xi = \xi(x). \tag{4.13}$$

After putting (4.13) into equation (4.8) yields

$$\eta = a_1(x)y + a_2(x), \tag{4.14}$$

where $a_1(x)$ and $a_2(x)$ are functions of x. Equation (4.4), (4.5), (4.6) and (4.9) are satisfied by (4.13) and (4.14). Substituting (4.13) and (4.14) into (4.10), (4.11) and (4.12), we have the system

$$\begin{aligned}
&\beta\xi' + 3\gamma a_1'(x) - 3\gamma\xi'' = 0,\\
&2\alpha\xi' + 2\beta a_1'(x) - \beta\xi'' + 3\gamma a_1''(x) - \gamma\xi''' = 0,\\
&(a_1(x)y + a_2(x))(\frac{y^p}{\gamma} - \frac{c}{\gamma}) + \frac{\alpha}{\gamma}(a_1'(x)y + a_2'(x)) + \frac{\beta}{\gamma}(a_1''(x)y + a_2''(x)) + (a_1'''(x)y + a_2'''(x))\\
&+(a_1(x) - 3\xi')(-\frac{y^{p+1}}{(p+1)\gamma} + \frac{c}{\gamma}y + \frac{k}{\gamma}) = 0.
\end{aligned} \tag{4.15}$$

The third equation of (4.15) is a polynomial of y with degree $p+1$ which is zero if and only if each variable coefficient is set to zero,

$$pa_1(x) + 3\xi' = 0, \tag{4.16}$$

$$a_2(x) = 0, \tag{4.17}$$

$$\alpha a_1'(x) + \beta a_1''(x) + \gamma a_1'''(x) - 3c\xi' = 0, \tag{4.18}$$

$$(a_1(x) - 3\xi')k = 0. \tag{4.19}$$

So the above set of differential equations of $a_1(x)$ and $a_2(x)$ can be solved according to the following two cases,

Case 1: $k = 0$

Substituting ξ' derived from (4.16) into the first equation of (4.15), under the condition $p \neq -3$, one has

$$a_1(x) = c_1 e^{\frac{\beta p x}{3(\gamma p + 3\gamma)}}$$

and

$$\xi' = -\frac{pc_1}{3}e^{\frac{\beta px}{3(\gamma p + 3\gamma)}},$$

where c_1 is an integration constant. Substituting ξ' and $a_1(x)$ into the second equation of (4.15), one can get a parametric condition

$$\beta^2(9+p)p + 3\beta^2(6+p)(3+p) - 18\alpha\gamma(3+p)^2 = 0. \tag{4.20}$$

The other parametric condition can be obtained by pulling $a_1(x)$ to (4.18) and it is

$$\beta^3(4p^2+9p) + 9\alpha\beta\gamma(3+p)^2 + 27c\gamma^2(3+p)^3 = 0. \tag{4.21}$$

Integrating ξ' and using $a_1(x), a_2(x)$, we have

$$\xi = -\frac{c_1\gamma(p+3)}{\beta}e^{\frac{\beta px}{3\gamma(p+3)}} + c_2,$$

and

$$\eta = c_1 e^{\frac{\beta px}{3\gamma(p+3)}}y,$$

where c_2 is an arbitrary constant. The infinitesimal generator

$$X = [-\frac{c_1\gamma(p+3)}{\beta}e^{\frac{\beta px}{3\gamma(p+3)}} + c_2]\partial_x + c_1 e^{\frac{\beta px}{3\gamma(p+3)}}y\partial_y. \tag{4.22}$$

If $p = -3$, we can get $a_1(x)$ and $\xi'(x)$ being all zeros form substituting ξ' into the first equation of (4.15). So, $\xi = c, \eta = 0$, where c is an arbitrary constant. The infinitesimal generator is $X = c\partial_x$.

Case 2: $k \neq 0$,

In this case, from (4.16) and (4.19), we can have $a_1(x) = 0, \xi'(x) = 0$. Furthermore, (4.15) and (4.18) hold. We can obtain

$$\xi = c, \eta = 0,$$

where c is an arbitrary constant. The infinitesimal generator is $X = c\partial_x$.

4.2. $p = -1$

For equation (3.1) with $p = -1$, performing the integration once, one has

$$-cu + \ln u + \alpha u' + \beta u'' + \gamma u''' = k, \tag{4.23}$$

Similarly, we can obtain the system satisfied by generators of the Lie group admitted by (4.23) by the linearized symmetry condition (4.2).

Thus, we can get

$$a_1(x) = 0,\ a_2(x) = 0,\ \xi = c,$$

where c is an arbitrary constant. In this case, the infinitesimal generator of Lie symmetry is $X = c\partial_x$.

Therefore, we can have the following result about the Lie symmetries admitted by (3.2).

Theorem 4.1.
Case 1. $p \neq -1, p \neq -3$ and $k = 0$.
when (4.20) and (4.21) are satisfied, (3.2) accepts Lie symmetry with the infinitesimal generator (4.22).
Case 2. $p \neq -1, p \neq -3$ and $k \neq 0$.
(3.2) accepts Lie symmetry with the infinitesimal generator $X = c\frac{\partial}{\partial x}$, where c is an arbitrary constant.
Case 3. $p = -1$ or $p = -3$.
(3.2) accepts Lie symmetry with the infinitesimal generator is $X = c\frac{\partial}{\partial x}$, where c is an arbitrary constant.

5. The invariant solutions of the KdV-Burgers-Kuramoto type equation

In the section, we consider to obtain the traveling wave solutions of the KdV-Burgers-Kuramoto type equation under the parametric conditions in Section 4 using the invariant curve condition

$$Q = \eta - y'\xi = 0. \tag{5.1}$$

If the infinitesimal generator of Lie symmetry admitted by (3.2) is

$$X = c\partial_x,$$

then

$$Q = y'c = 0,$$

that is, a trivial solution $y = constant$ is obtained. The trivial solution has no new useful meaning. So, we consider the following case.

Under the parameter conditions of (4.20) and (4.21), the infinitesimal generator is (4.22). Inserting $\xi(x)$ and $\eta(x, y)$ of (4.22) into (5.1), one has

$$y' = \frac{y}{\frac{c_2}{c_1}e^{-\frac{\beta p x}{3\gamma(3+p)}} - \frac{\gamma(p+3)}{\beta}}.$$

After solving the above equation, we can have

$$y = c_3|\frac{c_2}{c_1} - \frac{\gamma(p+3)}{\beta}e^{\frac{\beta p x}{3\gamma(3+p)}}|^{-\frac{3}{p}}, \tag{5.2}$$

where c_3 is an integration constant. Using the identity $\frac{e^{2x}}{1+e^{2x}} = \frac{1}{2}\tanh x + \frac{1}{2}$ and choosing $\frac{c_1}{c_2} = -\frac{\beta}{\gamma(p+3)}$, we obtain the invariant solutions of (1.1),

$$u(x, t) = c_3(\pm\frac{\beta}{2\gamma(p+3)})^{\frac{3}{p}}[\tanh(\frac{\beta p(x - ct)}{-6\gamma(p+3)}) + 1]^{\frac{3}{p}},$$

where c_3 is an arbitrary constant.

When $p = 1$, (1.1) becomes (1.2), the parametric conditions (4.20) and (4.21) change to

$$47\beta^2 = 144\alpha\gamma \tag{5.3}$$

and

$$13\beta^3 + 144\alpha\beta\gamma + 1728c\gamma^2 = 0. \tag{5.4}$$

Substituting (5.3) into (5.4) yields the condition

$$5\alpha\beta + 47\gamma c = 0. \tag{5.5}$$

Accordingly, under the conditions (5.3) and (5.5), one can get a traveling wave solution of (1.2),

$$u(x,t) = c_3(\pm\frac{\beta}{8\gamma})^3[\tanh\frac{\beta(x-ct)}{-24\gamma} + 1]^3,$$

where c_3 is an arbitrary constant. After using the identity $sech^2 t = 1 - \tanh^2 t$, the traveling wave solution can be turn to

$$u(x,t) = \mp 3c_3(\frac{\beta}{3\gamma})^3 sech^2\frac{\beta(x-ct)}{24\gamma} \mp c_3(\frac{\beta}{3\gamma})^3 sech^2\frac{\beta(x-ct)}{24\gamma}\tanh\frac{\beta(x-ct)}{24\gamma} \pm 4c_3(\frac{\beta}{3\gamma})^3(\tanh\frac{\beta(x-ct)}{24\gamma}+1).$$

The solution is be equivalent to that in the existing literatures, for example [35, 39].

When $p = 2$, the parametric conditions (4.20) and (4.21) change to

$$71\beta^2 = 225\alpha\gamma \tag{5.6}$$

and

$$34\beta^3 + 225\alpha\beta\gamma + 27\times 5^3 c\gamma^2 = 0. \tag{5.7}$$

Substituting (5.6) into (5.7) yields the condition

$$7\alpha\beta + 639\gamma c = 0. \tag{5.8}$$

Accordingly, under the conditions (5.6) and (5.8), one can get traveling wave solutions of (1.2)

$$u(x,t) = c_3(|\frac{\beta}{10\gamma}|)^{\frac{3}{2}}[\tanh\frac{2\beta(x-ct)}{-30\gamma} + 1]^{\frac{3}{2}}.$$

where c_3 is an arbitrary constant. The solution is a new solution of (1.1) compared with the results in [40].

6. Conclusions

In this paper, we consider the solutions of a KdV-Burgers-Kuramoto type equation with an arbitrary power nonlinearity $u^p u_x$. Firstly, we present the condition of the existence of traveling wave solutions of the equation based on the qualitative theory of differential equations. Then, with p ($p \neq -1$, $p \neq -3$) being an arbitrary constant, when $k = 0$, and the corresponding parametric conditions (4.20) and (4.21) are satisfied, we derive the invariant solutions of the KdV-Burgers-Kuramoto equation by solving the determining system and the invariant curve condition equation. When $p = 2$, the KdV-Burgers-Kuramoto equation with the nonlinearity term $u^2 u_x$ is the equation in [40]. Compared with the results in [40], a new solution is obtained.

For the case $\gamma = 0$ in (1.1), the KdV-Burgers-Kuramoto type equation with an arbitrary power nonlinearity can be turn to the KdV-Burgers type equation. One can not directly get the traveling wave solutions from the traveling wave solutions of (1.1) with $\gamma = 0$ because the equation has singular points. Thus, the dynamical properties and the qualitative analysis to the traveling wave solutions equation have to be studied. These will be investigated in our future work.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. M. R. Miura, *Backlund Transfortion*, New York: Springer−Verlag, 1978.
2. W. Peng, S. Tian, T. Zhang, *Dynamics of breather waves and higher−order rogue waves in a coupled nonlinear Schrödinger equation*, Europhysics Letters, **123** (2018), 50005.
3. X. Wang, T. Zhang, M. Dong, *Dynamics of the breathers and rogue waves in the higher−order nonlinear Schrödinger equation*, Appl. Math. Lett., **86** (2018), 298304.
4. L. Feng, T. Zhang, *Breather wave, rogue wave and solitary wave solutions of a coupled nonlinear Schrödinger equation*, Appl. Math. Lett., **78** (2018), 133–140.
5. D. Guo, S. Tian, T. Zhang, *Integrability, soliton solutions and modulation instability analysis of a (2+1)−dimensional nonlinear Heisenberg ferromagnetic spin chain equation*, Comput. Math. Appl., **77** (2019), 770–778.
6. L. Feng, S. Tian, T. Zhang, *Solitary wave, breather wave and rogue wave solutions of an inhomogeneous fifth−order nonlinear Schrödinger equation from Heisenberg ferromagnetism*, Rocky MT J. Math., **49** (2019), 29–45.
7. W. Peng, S. Tian, T. Zhang, *Breather waves and rational solutions in the (3+1)−dimensional Boiti−Leon−Manna−Pempinelli equation*, Comput. Math. Appl., **77** (2019), 715–723.
8. M. Dong, S. F. Tian, X. W. Yan, et al, *Solitary waves, homoclinic breather waves and rogue waves of the (3+1)−dimensional Hirota bilinear equation*, Comput. Math. Appl., **75** (2018), 957–964.
9. X. Yan, S. Tian, M. Dong, et al. *Characteristics of solitary wave, homoclinic breather wave and rogue wave solutions in a (2+1)−dimensional generalized breaking soliton equation*, Comput. Math. Appl., **76** (2018), 179–186.

10. S. Tian, *Initial−boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via theFokas method*, J. Differ. Equations, **262** (2017), 506–558.

11. W. Ma, T. Huang, Y. Zhang, *A multiple exp−function method for nonlinear differential equations and its application*, Phys. Scripta, **82** (2010), 065003.

12. J. He, X. Wu, *Exp−function method for nonlinear wave equations*, Chaos, Solitons and Fractals, **30** (2006), 700–708.

13. A. M. Wazwaz, *Two reliable methods for solving variants of the KdV equation with compact and noncompact structures*, Chaos, Solitons and Fractals, **28** (2006), 454–462.

14. T. Xia, B. Li, H. Zhang, *New explicit and exact solutions for the Nizhnik−Novikov−Vesselov equation*, Applied Mathematics E-Notes, **1** (2001), 139–142.

15. Z. Feng, *The first integral method to study the Burgers−Korteweg−de Vries equation*, Journal of Physics A, **35** (2002), 343–349.

16. Z. Feng, X. Wang, *The first integral method to the two−dimensional Burgers−KdV equation*, Phys. Lett. A, **308** (2002), 173–178.

17. W. Ma, J. H. Lee, *A transfortiom rational function method and exact solutions to (3+1)−dimensional Jimbo−Miwa equation*, Chaos, Solitons and Fractals, **42** (2009), 1356–1363.

18. P. J. Olver, *Applications of Lie groups to differential equations*, New York: Springer−Verlag, 1999.

19. W. G. Bluman, C. S. Anco, *Symmetry and integration methods for differential equations*, New York: Springer−Verlag, 2002.

20. X. Wang, S. Tian and T. Zhang, *Characteristics of the breather and rogue waves in a (2+1)−dimensional nonlinear Schrödinger equation*, proceedings of the american mathematical society, **146** (2018), 3353–3365.

21. S. Tian, T. Zhang, *Long−time asymptotic behavior for the Gerdjikov−Ivanov type of derivative nonlinear Schrödinger equation with time−periodic boundary condition*, proceedings of the american mathematical society, **146** (2018), 1713–1729.

22. J. Li, *Singular Traveling Wave Equations: Bifurcations and Exact Solutions*, Beijing: Science Press, 2013.

23. X. Wang, S. Tian, C. Qin, et al, *A Lie symmetry analysis, conservation laws and exact solutions of the generalized time fractional Burgers equation*, Europhysics Letters, **114** (2016), 20003.

24. Z. Feng, *Traveling waves to a reaction−diffusion equation*, Discret. Contin. Dyn. S., Supp., **12** (2007), 382–390.

25. H. Liu, J. Li, L. Liu, *Lie symmetry analysia, optimal systems and exact solutions to the fifth−order KdV types of equations*, J. Math. Anal. Appl., **368** (2010), 551–558.

26. H. Liu, J. Li, Q. Zhang, *Lie symmetry analysis and exact explicit solutions for general Burgers' equation*, J. Comput. Appl. Math., **228** (2009), 1–9.

27. M. L. Gandarias, C. M. Khalique, *Symmetries, solutions and conservation laws of a class of nonlinear dispersive wave equations*, Commun. Nonlinear Sci., **32** (2016), 114–121.

28. Y. Hu, C. Xue, *One-parameter Lie groups and inverse integrating factors of n−th order autonomous systems*, J. Math. Anal. Appl., **388** (2012), 617–626.

29. Y. Hu, K. Guan, *Techniques for searching first integrals by Lie group and application to gyroscope system*, Sci. China Math., **48** (2005), 1135–1143.

30. Y. Kuramoto, T. Tsuzuki, *Persistent propagation of concentration waves in dissipative media far from thermel equilibrium*, Prog. Theor. Phys., **55** (1976), 356–369.

31. J. Topper, T. Kawahara, *Approximate equation for long nonlinear waves on a viscous fluid*, J. Phys. Soc. Jpn, **44** (1978), 663–666.

32. V. Y. Shkadov, *Solitary waves in a layer of viscous liquid*, Fluid Dynamics, **12** (1977), 52–55.

33. B. I. Cohen, J. A. Krommers, W. M. Tang, et al. *Non−linear saturation of the dissipative trapped−ion mode by mode coupling*, Nucl. Fusion, **16** (1976), 971–992.

34. S. D. Liu, S. K. Liu, Z. Huang, et al. *On a class of nonlinear Schrödinger equation III*, Progress in Natural Science, **9** (1999), 912–918.

35. Z. Fu, S. D. Liu, S. K. Liu, *New exact solutions to the Kdv−Burgers−Kuramoto equation*, Chaos, Solitons and Fractals, **23** (2005), 609–616.

36. Abdul-Majid Wazwaz, *Partial differential equations and solitary waves theory*, Beijing: Higher Education Press, 2009.

37. X. Chen, Z. Fu, S. Liu, *Periodic solutions to KdV−Burgers−Kuramoto equations*, Commun. Theor. Phys., **45** (2006), 815–818.

38. Y. Fu, Z. Liu, *Persistence of travelling fronts of Kdv−Burgers−Kuramoto equation, Applied Mathematics and Computation*, Chaos, Solitons and Fractals, **216** (2010), 2199–2206.

39. J. Nickel, *Travelling wave solutions of the Kuramoto−Sivashinsky equation*, Chaos, Solitons and Fractals, **33** (2007), 1376–1382.

40. Y. Hu, *Lie symmetry analysis and exact solutions to a class of new KdV−Burgers−Kuramoto type equation*, Chinese Control and Decision Conference, (2016), 6705–6709.

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