Mathematics

AIMS Mathematics, 4(5): 1430–1449.
DOI: 10.3934/math.2019.5.1430
Received: 02 March 2019
Accepted: 24 June 2019
Published: 20 September 2019

http://www.aimspress.com/journal/Math

Research article

Existence and uniqueness solutions of fuzzy integration-differential mathematical problem by using the concept of generalized differentiability

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Abstract: In this study, we demonstrate studies on two types of solutions linear fuzzy functional integration and differential equation under two kinds Hukuhara derivative by using the concept of generalized differentiability. Various types of solutions to are generated by applying of two separate concepts of fuzzy derivative in formulation of differential problem. Some patterns are presented to describe these results.

Keywords: existence and uniqueness solutions; integration-differential mathematical problem; generalized differentiability
Mathematics Subject Classification: 34A07

1. Introduction

The theory of calculus, which deals with the investigation and applications of derivatives and integrals of arbitrary order has a long history. The theory of calculus developed mainly as a pure theoretical field of mathematics, in the last decades it has been used in various fields as rheology, viscoelasticity, electrochemistry, diffusion processes, etc [33,34]. Calculus have undergone expanded study in recent years as a considerable interest both in mathematics and in applications. One of the recently influential works on the subject of calculus is the monograph of Podlubny [50] and the other is the monograph of Kilbas et al. [34]. The differential equations have great application potential in modeling a variety of real world physical problems, which deserves further investigations. Among these we might include the modeling of earthquakes, the fluid dynamic traffic model with derivatives, the measurement of viscoelastic material properties, etc. Consequently, several research papers were done to investigate the theory and solutions of differential equations (see [18,21,36,38] and references therein).

The concept of solution for differential equations with uncertainty was introduced by Agarwal, Lakshmikantham and Nieto [1]. They considered Riemann-Liouville differentiability concept based on the Hukuhara differentiability to solve fuzzy differential equations. Arshad and Lupulescu in [12] proved some results on the existence and uniqueness of solution to fuzzy differential equation under Hukuhara Riemann-Liouville differentiability. Some existence results for nonlinear fuzzy differential equations of order involving the Riemann-Liouville derivative have been proposed in [30,31]. The solutions of fuzzy differential equations are investigated by using the fuzzy Laplace transforms in [52]. Recently, the concepts of derivatives for a fuzzy function are either based on the notion of Hukuhara derivative [25] or on the notion of strongly generalized derivative. The concept of Hukuhara derivative is old and well known, but the concept of strongly generalized derivative was recently introduced by Bede and Gal [13]. Using this new concept of derivative, the classes of fuzzy differential equations have been extend and studied in some papers such as: Ahmad et al. [4], Allahviranloo et al. [9–11,49], Bede et al. [14–17], Gasilov [20], Khastan et al. [27–29], Malinowski [42–44] and Nieto [46]. Furthermore, by using this new concept of derivative, Allahviranloo et al. in [7,8] have studied the concepts about generalized Hukuhara Riemann-Liouville and Caputo differentiability of fuzzy valued functions. Later, authors have proved the existence and uniqueness of solution for fuzzy differential equation by using different methods. Alikhani et al. in [6] have proved the existence and uniqueness results for nonlinear fuzzy integral and integration and differential equations by using the method of upper and lower solutions. Mazandarani et al. [45] studied the solution to fuzzy initial value problem under Caputo-type fuzzy derivatives by a modified Euler method. Besides, authors studied some results on the existence and uniqueness of solution to fuzzy differential equation under Caputo type-2 fuzzy derivative and the definition of Laplace transform of type-2 fuzzy number-valued functions [46–48]. Salahshour et al. [48,51] proposed some new results toward existence and uniqueness of solution of fuzzy differential equation. According to the concept of Caputo-type fuzzy derivative in the sense of the generalized fuzzy differentiability, Fard et al. [19] extended and established some definitions on fuzzy calculus of variation and provide some necessary conditions to obtain the fuzzy Euler-Lagrange equation for both constrained and unconstrained fuzzy variational problems. Ahmad et al. [5] proposed a new interpretation of fuzzy differential equations and present their solutions analytically and numerically. The proposed idea is a generalization of the interpretation given in [3,4], where the authors used Zadeh's extension principle to interpret fuzzy differential equations.

In real world systems, delays can be recognized everywhere and there has been widespread interest in the study of delay differential equations for many years. Therefore, delay differential equations (or, as they are called, functional differential equations) play an important role in an increasing number of system models in biology, engineering, physics and other sciences. There exists an extensive amount of literature dealing with delay differential equations and their applications; the reader is referred to the monographs [22,35], and the references therein. The study of fuzzy delay differential equations is expanding as a new branch of fuzzy mathematics. Both theory and applications have been actively discussed over the last few years. In the literature, the study of fuzzy delay differential equations has several interpretations. The first one is based on the notion of Hukuhara derivative. Under this interpretation, Lupulescu established the local and global existence and uniqueness results for fuzzy delay differential equations. The second interpretation was suggested by Khastan et al. [29] and Hoa et al. [24].

In this setting, Khastan et al. proved the existence of two fuzzy solutions for fuzzy delay differential equations using the concept of generalized differentiability. Hoa et al. established the global existence and uniqueness results for fuzzy delay differential equations using the concept of generalized differentiability. Moreover, authors have extended and generalized some comparison theorems and stability theorem for fuzzy delays differential equations with definition a new Lyapunov-like function. Besides that, some very important extensions of the fuzzy delay differential equations In [2,21,28,36,54], the authors considered the fuzzy differential equation with initial value

$$x^{'}(t) = f\big(t.x(t)\big). \quad x(t_0) = x_0 \in E^d \tag{1.1}$$

where $f : [0, \infty) \times E^d \to E^d$ and the symbol $'$ denotes the first type Hukuhara derivative (classic Hukuhara derivative). O. Kaleva also discussed the properties of differentiable fuzzy mappings in [28] and showed that if f is continuous and $f(t, x)$ satisfies the Lipschitz condition with respect to x, then there exists a unique local solution for the fuzzy initial value problem (1.1). V. Lupulescu proved several theorems stating the existence, uniqueness and boundedness of solutions to fuzzy differential equations with the concept of inner product on the fuzzy space under classic Hukuhara derivative in [36].

In [35], V. Lupulescu considered the fuzzy functional differential equation

$$\begin{cases} x^{'}(t) = f(t.x_t). \quad t \geq t_0 \\ x(t) = \varphi(t - t_0) \in E^d . t_0 \geq t \geq t_0 - \sigma \end{cases} \tag{1.2}$$

Where $f : [0, \infty) \times C_\sigma \to E^d$ and the symbol $'$ denotes the first type Hukuhara derivative (classic Hukuhara derivative). Author studied the local and global existence and uniqueness results for (1.2) by using the method of successive approximations and contraction principle.

In this paper, we consider fuzzy functional integration and differential equations under form

$$\begin{cases} \mathrm{D}_H^g\, x(t) = f(t.x_t) + \int_{t_0}^{t} g\,(t.S.X_S)ds.t \geq t_0 \\ x(t) = \varphi(t - t_0) = \varphi_0 \in C_\sigma . t_0 \geq t \geq t_0 - \sigma \end{cases} \tag{1.3}$$

We establish the local and global existence and uniqueness results for (1.3) by using the method of successive approximations and contraction principle. This direction of research is motivated by the results of B. Bede and S. G. Gal [17], Chalco-Cano and Roman-Flores [23], Marek T. Malinowski [38–41], Bashir Ahmad, S. Sivasundaram [1], T. Allahviranloo et al. [5–7].

The paper is organized as follows. In Section 2, we collect the fundamental notions and facts about fuzzy set space, fuzzy differentiation and integration. In Section 3, we discuss the FFIDEs with a two kinds of fuzzy derivative. Some examples of this class having two different solutions were presented in Section 4.

2. Preliminaries and notation

In this section, we give some notations and properties related to fuzzy set space, and summarize the major results for integration and differentiation of fuzzy set-valued mappings. We recall some notations and concepts presented in detail in recent series works of Professor V. Lakshmikantham, et al. (see [33,34]).

Let $K_c(R^d)$ denote the collection of all nonempty compact and convex subsets of R^d and scalar multiplication in $K_c(R^d)$ as usual, i.e. for $A, B \in K_c(R^d)$ and $\lambda \in R$.

$$A + B = \{a + b \mid a \in A,\ b \in B\},\ \lambda A = \{\lambda a \mid a \in A\}.$$

The Hausdorff distance d_H in $K_c(R^d)$ is defined as follows

$$d_H(A.B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|\ R^n \,.\, \sup_{a \in A} \inf_{b \in B} \| a - b \|\ R^n \right.$$

where $A, B \in (K_c, R^d)$, $\|.\|_{R^n}$ denotes the Euclidean norm in R^d. It is known that (K_c, R^d), d_H is a complete metric space. Denote $E^d = \{\omega : R^d \to [0, 1]$ such that $\omega(z)$ satisfies (i)-(iv) stated below$\}$

i. ω is normal, that is, there exists $z_0 \in R^d$ such that $\omega(z0) = 1$;
ii. ω is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$\omega(\lambda z1 + (1 - \lambda)z2) \geq \min\{\omega(z1), \omega(z2)\}$,
for any $z1, z2 \in R^d$;

iii. ω is upper semi continuous;
iv. $[\omega]^0 = cl\{z \in R^d : \omega(z) > 0\}$ is compact, where cl denotes the closure in $(R^d, \|.\|)$.

Although elements of E^d are often called the fuzzy numbers [57], we shall just call them the fuzzy sets.

For $\alpha \in (0, 1]$, denote $[\omega]^\alpha = \{z \in R^d \mid \omega(z) \geq \alpha\}$. We will call this set an α-cut (α- level set) of the fuzzy set ω. For $\omega \in E^d$ one has that $[\omega]^\alpha \in Kc(Rd)$ for every $\alpha \in [0, 1]$. For two fuzzys $\omega 1, \omega 2 \in E^d$, we denote $\omega 1 \leq \omega 2$ if and only if $[\omega 1]^\alpha \subset [\omega 2]^\alpha$.

If $g : R^d \times R^d \to R^d$ is a function then, according to Zadeh's extension principle[47,57], one can extend g to $E^d \times E^d \to E^d$ by the formula $g(\omega 1, \omega 2)(z) = \sup z{=}g(z1,z2) \min\{\omega 1(z1), \omega 2(z2)\}$. It is well known that
if g is continuous then $[g(\omega 1, \omega 2)]^\alpha = g([\omega 1]^\alpha, [\omega 2]^\alpha)$ for all $\omega 1, \omega 2 \in E^d$, $\alpha \in [0, 1]$. Especially, for addition and scalar multiplication in fuzzy set space Ed, we have $[\omega 1 + \omega 2]^\alpha = [\omega 1]^\alpha + [\omega 2]^\alpha$, $[\lambda\omega 1]^\alpha = \lambda[\omega 1]^\alpha$. The notation $[\omega]^\alpha = [\omega(\alpha), \omega(\alpha)]$. We refer to ω and ω as the lower and upper branches of ω, respectively.

For $\omega \in E^d$, we define the length of ω as $len(\omega) = \omega(\alpha) - \omega(\alpha)$ In the case $d = 1$, we have $len(\omega) = \omega(\alpha) - \omega(\alpha)$. Let us denote $D_0\ [\omega 1, \omega 2] = \sup\{d_H\ ([\omega 1]^\alpha, [\omega 2]^\alpha) : 0 \leq \alpha \leq 1\}$ the distance between $\omega 1$ and $\omega 2$ in E^d, where $d_H([\omega 1]^\alpha, [\omega 2]^\alpha)$ is Hausdorff distance between two set$[\omega 1]^\alpha$, $[\omega 2]^\alpha$ of (K_c, R^d). Then (E^d, d_H) is a completespace. Some properties of metric D are as follows.

$D_0\ [\omega 1 + \omega 3, \omega 2 + \omega 3] = D_0\ [\omega 1, \omega 2]$, $D_0\ [\lambda\omega 1, \lambda\omega 2] = |\lambda|\ D_0\ [\omega 1, \omega 2]$, $D_0\ [\omega 1, \omega 2] \leq D_0\ [\omega 1, \omega 3] + D_0\ [\omega 3, \omega 2]$, for all $\omega 1, \omega 2, \omega 3 \in E^d$ and $\lambda \in R$. Let $\omega 1, \omega 2 \in E^d$. If there exists $\omega 3 \in E^d$ such that $\omega 1 = \omega 2 + \omega 3$ then $\omega 3$ is called the difference of $\omega 1, \omega 2$ and it is denoted $\omega 1 \ominus \omega 2$. Let us remark that $\omega 1 \ominus \omega 2 \neq \omega 1 + (-1)\,\omega 2$.

Remark 2.1. If for fuzzy numbers $\omega 1, \omega 2, \omega 3 \in E^d$ there exist Hukuhara difference $\omega 1 \ominus \omega 2$, $\omega 1 \ominus \omega 3$ then $D_0\ [\omega 1 \ominus \omega 2, 0] = D_0\ [\omega 1, \omega 2]$ and $D_0\ [\omega 1\ \omega 2, \omega 1 \ominus \omega 3] = D_0\ [\omega 2, \omega 3]$.
The strongly generalized differentiability was introduced in [17] and studied in [18,23,26, 32,37,42,55,56].

Definition 2.1. (See [17,48,49]) Let $x : (a, b) \to E^d$ and $t \in (a, b)$. We say that x is strongly generalized differentiable at t, if there exists $D_H{}^g\, x(t) \in E^d$, such that either
(i) for all $h > 0$ sufficiently small, the differences $x(t + h) \ominus x(t)$, $x(t) \ominus x(t - h)$ exist and the limits (in the metric D_0)

$$\lim_{h \searrow 0^+} \frac{x(t+h) \ominus x(t)}{h} = \lim_{h \searrow 0^+} \frac{x(t+h) \ominus x(t)}{h} = \mathrm{D}_H^g\, x(t)$$

or

(ii) for all $h > 0$ sufficiently small, the difference $x(t) \ominus x(t + h)$, $x(t - h) \ominus x(t)$ exist and the limits

$$\lim_{h \searrow 0^+} \frac{x(t) \ominus x(t+h)}{-h} = \lim_{h \searrow 0^+} \frac{x(t-h) \ominus x(t)}{-h} = \mathrm{D}_H^g\, x(t)$$

or

(iii) for all $h > 0$ sufficiently small, the difference $x(t + h) \ominus x(t)$, $\exists x(t - h) \ominus x(t)$ exist and the limits

$$\lim_{h \searrow 0^+} \frac{x(t+h) \ominus x(t)}{h} = \lim_{h \searrow 0^+} \frac{x(t-h) \ominus x(t)}{-h} = \mathrm{D}_H^g\, x(t)$$

(iv) for all $h > 0$ sufficiently small, the difference $x(t) \ominus x(t + h)$, $\exists x(t) \ominus x(t - h)$ exist and the limits

$$\lim_{h \searrow 0^+} \frac{x(t) \ominus x(t+h)}{-h} = \lim_{h \searrow 0^+} \frac{x(t) \ominus x(t-h)}{h} = \mathrm{D}_H^g\, x(t).$$

In this definition, case (i) ((i)-differentiability for short) corresponds to the classic derivative, so this differentiability concept is a generalization of the Hukuhara derivative. In Ref. [17], B. Bede and S.G. Gal consider four cases for derivative. In this paper we consider only the two first of Definition.

In the other cases, the derivative is trivial because it is reduced to a crisp element
Lemma 2.1. (B, Bede and S. G. Gal [17]) *If* $x(t) = (z1(t), z2(t), z3(t))$ *is triangular number valued function, then*
(i) if x is (i)-differentiable (i.e. Hukuhara differentiable) then DH g $x(t) = (z'1(t), z'2(t), z'3(t))$;
(ii) if x is (ii)-differentiable then DH g $x(t) = (z'3(t), z'2(t), z'1(t))$.

Lemma 2.2. (see [23]) *Let* $x \in E^1$ *and put* $[x(t)]\alpha = [x(t, \alpha), x(t, \alpha)]$ *for each* $\alpha \in [0, 1]$.
(i) If x is (i)-differentiable then $x(t, \alpha)$, $x(t, \alpha)$ *are differentiable functions and we have*

$$[\mathrm{D}_H^g\, x(t)]^\alpha = \left[\underline{x}'(t.\alpha).\ \overline{x}'\ (t.\alpha)\right]\cdot \tag{2.1}$$

(ii) If x is (ii)-differentiable then $x(t, \alpha)$, $x(t, \alpha)$ *are differentiable functions and we have:*
(iii)

$$[\mathrm{D}_H^g\, x(t)]^\alpha = \left[\overline{x}'(t.\alpha).\ \underline{x}'(t.\alpha)\right]\cdot \tag{2.2}$$

Definition 2.2. [49,53] We say that a point $t \in (a, b)$, is a switching for the differentiability of x, if in any neighborhood V of t there exist points $t1 < t < t2$ such that
(**type I**) at $t1$ (2.1) holds while (2.2) does not hold and at $t2$ (2.2) holds and (2.1) not hold, or
(**type II**) at $t1$ (2.2) holds while (2.1) does not hold and at $t2$ (2.1) holds and (2.2) not hold.

Lemma 2.3. *Let a(t),b(t) and c(t) be real valued nonnegative continuous functions defined on R+, d* ≥ 0 *is a constant for which the inequality*

$$\mathrm{a}\ (\mathrm{t}) \leq d + \int_0^t \left[b\ (S)a\ (S) + b(S)\int_0^t c(r)a\ (r)dr\right] ds$$

hold for all t ∈ *R+. Then*

$$\mathrm{a}\ (\mathrm{t}) \leq d + \left[1 + \int_0^t b\ (s)exp\left(\int_0^s (b(r) + c\ (r))dr\right)ds\cdot\right]$$

3. Main results

For $\sigma > 0$ let $C\sigma = C([-\sigma, 0], Ed)$ denote the space of continuous mappings from $[-\sigma, 0]$ to Ed. Define a metric $D\sigma$ in $C\sigma$ by

$$D_\sigma\ [x.y] = \sup_{t\in[-\tau.0)} D_0\ [x(t).y(t)]\cdot$$

Let $p > 0$. Denote $I = [t_0, t_0 + p]$, $J = [t_0 - \sigma, t_0] \cup I = [t_0 - \sigma, t_0 + p]$. For any $t \in I$ denote by the element of $C\sigma$ defined by $x_t(s) = x\ (t + s)$ for $s \in [-\sigma, 0]$.

Let us consider the fuzzy functional integration and differential equations (FFIDEs) with generalized Hukuhara derivative under form

$$\begin{cases} \mathrm{D}_H^g\, x(t) = f\ (t.x_t) + \int_{t_0}^t g(t.s.x_s)ds.t\ \geq\ t_0 \\ x(t) = \varphi(t - t_0) =\ \varphi_0\ \in\ C_\sigma.t_0\ \geq t \geq t_0 - \sigma \end{cases} \tag{3.1}$$

Where $f : I \times C\sigma \to E^d$, $g : I \times I \times C\sigma \to E^d$, $x \in C\sigma$ and the symbol $D_H{}^g$ denotes the generalized Hukuhara derivative from Definition (2.1). By a solution to equation (3.1) we mean a fuzzy mapping $x \in C\ (J, E^d)$, that satisfies:

$X\ (t) = \varphi\ (t - t0)$ for $t \in [t_0 -\sigma, t_0]$, x is differentiable on $[t_0, t_0 + p]$ and

$$D_H^g\ x(t)\ = f\ (t.x_t) + \int_{t_0}^t g(t.s.x_s)ds.for\ t\ \in\ I$$

Lemma 3.1. *Assume that* $f \in C\ (I \times C\sigma, E^d)$, $g \in C\ (I \times I \times C\sigma\ , E^d)$ *and* $x \in C(J, E^d)$. *Then the fuzzy mapping*

$$t\ \to f\ (t.x_t) + \int_{t_0}^t g(t.s.x_s)ds$$

Belongs to $C(I, E^d)$.

Remark 3.1. Under assumptions of the lemma above we have the mapping

$$t\ \to f\ (t.x_t) + \int_{t_0}^t g(t.s.x_s)ds$$

Is integrable over the interval *I*.

Remark 3.2. If $f : I \times C\sigma \to E^d$, $g : I \times I \times C\sigma \to E^d$ are jointly continuous functions and $x \in C(J, Ed)$,then the mapping

$$t\ \to f\ (t.x_t) + \int_{t_0}^t g(t.s.x_s)ds$$

Is bounded on each compact interval *I*. Also, the function

$$t \to f\,(t.x_t) + \int_{t_0}^{t} g(t.s.x_s)ds$$

is bounded on *I*.

Lemma 3.2. *A fuzzy mapping $x : J \to E^d$ is called to be a local solution to the problem (3.1) on J if and only*
if x is a continuous fuzzy mapping and it satisfies to one of the following fuzzy integral equations

$$\begin{cases} x(t) = \varphi(t.t_0)\ for\ t\ \in [t_0 - \sigma.t_0] \\ x(t) = \varphi(0)\ \int_{t_0}^{t}\left(f(s.x_s) + \int_{t_0}^{t} g(t.s.x_s)ds\right)ds \quad t \in I. \end{cases} \tag{3.2}$$

if x is (i)-differentiable or (iii)-differentiable.

$$\begin{cases} x(t) = \varphi(t - t_0)\ for\ t\ \in [t_0 - \sigma.t_0] \\ x(t) = \varphi(0)\ \ominus (-1) \\ \times \int_{t_0}^{t}\left(f(s.x_s) + \int_{t_0}^{t} g(t.s.x_s)ds\right)ds \quad t \in I. \end{cases} \tag{3.3}$$

if x is (ii)-differentiable or (iv)-differentiable. Let us remark that in (3.3) it is hidden the following statement: there exists Hukuhara difference

$$(-1)\int_{t_0}^{t}\left(f(s.x_s) + \int_{t_0}^{t} g(t.s.x_s)ds\right)ds$$

Definition 3.1. Let $x : J \to E^d$ be a fuzzy function such that (i)-differentiable. If x and its derivative satisfy problem (3.1), we say that x is a (i)-solution of problem (3.1).

Definition 3.2. Let $x : J \to E^d$ be a fuzzy function such that (ii)-differentiable. If x and its derivative satisfy problem (3.1), we say that x is a (ii)-solution of problem (3.1).

Definition 3.3. A solution $x : J \to E^d$ is unique if it holds $D\,[x(t), y(t)] = 0$, for any $y : J \to E^d$ which is a solution of (3.1).

Theorem 3.1. *Let* $\varphi\ (t - t_0) \in C\sigma$ *and suppose that* $f \in C\,(I \times C\sigma, E^d)$,$g \in C\,(I \times I \times C\sigma, E^d)$ *satisfy the condition: there exists a constant* $L > 0$ *such that for every* $\xi, \psi \in C\sigma$ *it holds*

$$\max\ \{D_0\ [f(t.\xi).f(t.\psi)].D_0\ [g(t.s.\xi).g(t.s.\psi)]\}\ \le L\sigma\ [\xi.\psi]$$

Moreover, there exists a $M > 0$ *such that* $\max\{D_0[f(t,\xi),0],D_0[g(t,s,\xi),0\}<=M$

Assume that the sequence $\{xn\}^{\infty}$ n=0, $x^n : J \to E^d$ *given by*

$$x^0\ (t) = \begin{cases} \varphi(t - t_0).t\ \in [t_0 - \sigma.t_0] \\ \varphi(0).t\ \in I. \end{cases}$$

and for $n = 1, 2, ...$

$x^{n+1}\ (t)$=

$$\begin{cases} \varphi(t - t_0).t \in [t_0 - \sigma.t_0] \\ \varphi(0)\ \ominus (-1)\int_{t_0}^{t}\left(f(s.x_s^n) + \int_{t_0}^{t} g(s.\tau.x_\tau^n)d\tau\right)ds \quad t \in I. \end{cases} \tag{3.4}$$

is well defined, i.e. the foregoing Hukuhara difference do exist. Then the FFIDE (3.1) has a unique for each case ((i)-differentiable or (ii)-differentiable).

Proof. From assumptions of this Theorem we have

$D_0[x^1(t), x_0(t)] = D_0\ [x^1\ (t). x^0\ (t)]$

$$= D_0\ [\varphi(0) \ominus (-1)$$

$$\times \int_{t_0}^{t} \left(f(s. x_s^0) + \int_{t_0}^{s} g(s.\tau. x_\tau^0) d\tau \right) ds. \varphi(0)]$$

$$\leq \int_{t_0}^{t} \left(D_0\left[f(s. x_s^0). \hat{0}\right] + \int_{t_0}^{s} D_0\left[g(s.\tau. x_\tau^0). \hat{0}\right] d\tau \right) ds$$

$$\leq M\ (t - t_0) + M\ \frac{(t - t_0)^2}{2!}.$$

for $t \in I$. Further for every $n \geq 2$ and $t \in I$ we get $D_0\ [x^{n+1}(t), x^n(t)]$

$$= D_0\ [\ominus (-1) \int_{t_0}^{t} \left(f(s. x_s^n) + \int_{t_0}^{s} g(s.\tau. x_\tau^0) d\tau \right) ds.$$

$$\ominus (-1) \int_{t_0}^{t} \left(f(s. x_s^{n-1}) + \int_{t_0}^{s} g(s.\tau. x_\tau^{n-1}) d\tau \right) ds]$$

$$\leq L \int_{t_0}^{t} \left(D_\sigma [x_s^n . x_s^{n-1}] + \int_{t_0}^{s} D_\sigma [x_\tau^n . x_\tau^{n-1}] d\tau \right) ds.$$

$$\leq L \int_{t_0}^{t} \Big(\sup_{\theta \in [-\sigma. 0]} D_0\ [x^n\ (s + \theta). x^{n-1}\ (s + \theta)]$$

$$+ \int_{t_0}^{s} \sup_{\theta \in [-\sigma. 0]} D_0\ [x^n\ (\tau + \theta). x^{n-1}\ (\tau + \theta)] d\tau) ds$$

$$= L \int_{t_0}^{t} \Big(\sup_{r \in [s - \sigma. s]} D_0\ [x^n\ (r). x^{n-1}\ (r)]$$

$$+ \int_{t_0}^{s} \sup_{\nu \in [\tau - \sigma. \tau]} D_0\ [x^n\ (\nu). x^{n-1}\ (\nu)] d\nu) dr$$

In particular, from (3.4), we get

$$D_0\ [x^2\ (t). x^1\ (t)]$$

$$\leq LM \left(\frac{(t - t_0)^2}{2!} + 2\ \frac{(t - t_0)^3}{3!} + \frac{(t - t_0)^4}{4!} \right)$$

Therefore, by mathematical induction, for every $n \in$ N and $t \in I$

$$D_0\ [x^{n+1}\ (t). x^n\ (t)]$$

$$\leq LM^n \left(\frac{(t-t_0)^{n+1}}{(n+1)!} + {}^n\lambda_1\ \frac{(t-t_0)^{n+2}}{(n+2)!} + \cdots + {}^n\lambda_n\ \frac{(t-t_0)^{2n+1}}{(2n+1)!} + \frac{(t-t_0)^{2n+2}}{(2n+2)!} \right) \tag{3.5}$$

In the inequality (3.5), $\lambda_1, . . . , \lambda n$ are balancing constants. We observe that for every $n \in \{0, 1, 2, . . .\}$, the function $x^n(\cdot) : J \rightarrow E^d$ are continuous. Indeed, since $\varphi \in C\sigma$, $x^0(t)$ is continuous on $t \in [-\sigma, t_0 + p]$. We see that

$$D_0\, [x^1\, (t+h).x^1\, (t)]$$

$$= D_0 \left[\begin{array}{c} \varphi(0) \ominus (-1) \\ \times \int_{t_0}^{t+h} \left(f(s.x_s^0) + \int_{t_0}^{s} (g(s.\tau.x_\tau^0)d\tau\right)ds. \\ \varphi(0) \ominus (-1) \\ \times \int_{t_0}^{t} (f(s.x_s^0) + \int_{t_0}^{s} (g(s.\tau.x_\tau^0)d\tau)ds \end{array} \right]$$

Thus, by mathematical induction, for every $n \geq 2$, we deduce that

$D_0\,[x^n\,(t+h), x^n(t)] \rightarrow 0$

as $h \rightarrow 0^+$. A similar inequality is obtained for $D_0\,[x^n(t - h), x^n(t)] \rightarrow 0$ as $h \rightarrow 0^+$. In the sequel we shall show that for the $\{x^n(t)\}$ the Cauchy convergence condition is satisfied uniformly in t, and as a consequence$\{x^n(\cdot)\}$ is uniformly convergent. For $n > m > 0$, from (3.5) we obtain

$$\begin{aligned} &\sup_{t \in I} D_0\,[x^n\,(t).x^m\,(t)] \\ &= \sup_{t \in J} D_0\,[x^n\,(t).x^m\,(t)] \\ &\leq \sum_{k=m}^{n-1} \sup_{t \in J} D_0\,[x^{K+1}\,(t).x^K\,(t)] \end{aligned}$$

$$\leq M \sum_{k=m}^{n-1} \left(\frac{(t-t_0)^{k+1}}{(k+1)!} +^n \lambda_1\, \frac{(t-t_0)^{k+2}}{(k+2)!} + \cdots +^k \lambda_k\, \frac{(t-t_0)^{2k+1}}{(2k+1)!} + \frac{(t-t_0)^{2k+2}}{(2k+2)!} \right)$$

The convergence of this series implies that for any $\varepsilon > 0$ we find $n_0 \in$ N large enough such that for n, $m > n_0$

$$D_0\,[x^n\,(t).x^m\,(t)] < \varepsilon \tag{3.6}$$

Since (E^d, D_0) is a complete metric space and (3.6) holds, the sequence $\{x^n(\cdot)\}$ is uniformly convergent to a mapping $x \in C(J, E^d)$. We shall that x is a solution to (3.1). Since $x^n(t) = \varphi\ (t - t_0)$ for every $n = 0, 1, 2, ...$ and every $t \in [t_0 - \sigma, t_0]$, we easily have $x(t) = \varphi\ (t - t_0)$. For $s \in I$ and $n \in$ N

$$D_0 \left[\int_{t_0}^{t} (f(s.x_s^n)ds. \int_{t_0}^{t} (f(s.x_s)ds) \right]$$

$$\leq L \int_{t_0}^{t} \sup_{\theta \in [s-\sigma.s]} D_0\,[x^n\,(\theta).x\,(\theta)]d\theta \rightarrow 0$$

And

$$D_0\, [\int_{t_0}^{t} (\int_{t_0}^{s} g(s.\tau.x_\tau^n).d\tau)ds\ . \int_{t_0}^{t} (\int_{t_0}^{s} g(s.\tau.x_\tau).d\tau)ds\ ]$$

$$\leq L \int_{t_0}^{t} (\int_{t_0}^{t} \sup_{\theta \in [\tau - \sigma. \tau]} D_0 [x^n (v). x (v)] dv) ds \to 0$$

As $n \longrightarrow \infty$ for any $t \in I$. Consequently, we have

$$D_0[\varphi(0). x(t) + (-1) \int_{t_0}^{t} (f(s. x_s) + \int_{t_0}^{t} (g(t. s. x_s) ds) ds]$$

$$\leq D_0 [x^n (t). x (t)]$$

$$+ \int_{t_0}^{t} (D_0 [f(s. x_s^{n-1}) + f(s. x_s)]$$

$$+ \int_{t_0}^{t} (D_0 [g(s. \tau. x_\tau^{n-1}). g(s. \tau. x_\tau) d\tau]) ds$$

We infer that

$$D_0 \left[\varphi(0). x(t) + (-1) \int_{t_0}^{t} (f(s. x_s) + \int_{t_0}^{t} (g(t. s. x_s) ds) ds \right] = 0$$

for every $t \in I$. Therefore x is the solution of (3.3), due to Lemma (3.2) we have that x is a (ii)-solution of (3.1). For the uniqueness of the solution x let us assume that $x, y \in C (J, E^d)$ are two solutions of (3.3). By definition of the solution we have $x(t) = y(t)$ if $t \in [t_0 - \sigma, t_0]$.Note that for $t \in$ I

$$D_0 [x (t). y (t)]$$

$$\leq L \int_{t_0}^{t} (\sup_{\theta \in [s - \sigma. s]} D_0[x(\theta). y(\theta)]$$

$$+ \int_{t_0}^{t} \left(\sup_{\nu \in [\tau - \sigma. \tau]} D_0[x(\nu). y(\nu)] d\tau \right) ds$$

If we let $a(s) = \sup_{r \in [s-\sigma, s]} D0 [x(r), y(r)]$, $s \in [t0, t] \subset [t_0, t_0 + p]$, then we have

$$a(t) \leq L \int_{t_0}^{t} \left(a(s) + \int_{t_0}^{s} a(\tau) d\tau \right) ds$$

and by Lemma 2.3 we obtain that $a(t) = 0$ on I. This prove the uniqueness of the solution for (3.1).

Remark 3.3. The existence and uniqueness theorem for the problem (3.1) can be obtained using the contraction principle.

Now, we shall prove existence and uniqueness results for (3.1) by using the contraction principle, which studied in [34]. In the following, for a given $k > 0$, we consider the set S_k of all continuous fuzzy functions

$x \in C ([t_0 - \sigma, \infty), E^d)$ such that $x(t) = \varphi \ (t - t_0) = x_0$ on $[t_0 - \sigma, t_0]$ and

$\sup_{t \geq t0-\sigma} \{D0[x(t, \omega), 0] \exp(\hat{} -kt) < \infty$.

On S_k we can define the following metric

$$D_k[x.y] = \sup_{t \geq t_0 - \sigma} \{D_0[x(t).y(t)] \exp(-kt)\} \quad (3.7)$$

Where $k > 0$ is chosen suitably later. We easily prove that the space [*Sk*, *Dk*] of continuous fuzzy functions
$x : [t_0, \infty) \to E^d$ is a complete metric space with distance (3.7).

Theorem 3.2. *Assume that*
(i) $f \in C([t_0, \infty) \times C\sigma, E^d)$, $g \in C([t_0, \infty) \times [t_0, \infty) \times C\sigma, E^d)$ *and there exists a constant* $L > 0$ *such that*
(ii) $max \{D_0[f(t.\xi).f(t.\psi)].\{D_0[g(t.s.\xi).g(t.s.\psi)] \leq LD_\sigma[\xi.\psi]$
for all $\xi, \psi \in C\sigma$ *and* $t, s \geq t0$;
(iii) there exists constants $M > 0$ *and* $b > 0$ *such that*

$$max \{D_0[f(t.\hat{0}).\hat{0}].D_0\; g(t.s.\hat{0}).\hat{0}]\} \leq M \exp(bt)$$

for all $t \geq t_0$*, where* $b < k$*. Then the FFIDE (3.1) has a unique solution for each case on* $[t_0, \infty)$*.*

Proof. Since the way of the proof is similar for all four cases, we only consider case (ii)-differential for x. In this case, we consider the complete metric space (S_k, D_k), and define an operator
T : $Sk \to Sk$
$x \to \mathrm{T}x$
given by

$$(\mathbb{T}x)(t) = \begin{cases} \{\varphi(t - t_0)\; if\; t \in [t_0 - \sigma.t_0] \\ \varphi(0) \ominus (-1) \\ \times \int_{t_0}^{t} (f(s.x_s) + \int_{t_0}^{t} (g(s.\tau.x_s)d\tau)ds \end{cases}$$

We can choose a big enough value for k such that T is a contraction, so the Banach fixed point theorem provides the existence of a unique fixed point for T, that is, a unique solution for (3.1).
Step 1: We shall prove that $\mathrm{T}(S_k) \subset S_k$ with assumption $k > b$. Indeed, let $x \in S_k$. For each $t \geq t_0$, we get

$$D_k\;[\,(\mathbb{T}x)(t).\hat{0}]$$

$$= \sup_{t \geq t_0} \{D_0[\varphi(0) \ominus (-1) \int_{t_0}^{t} (f(s.x_s)$$

$$+ \int_{t_0}^{s} g(s.\tau.x_\tau)d\tau)ds.\hat{0}]\; esp\;(-kt)\}$$

$$\leq \sup_{t \geq t_0} \{(D_0[\varphi(0).\hat{0} + \int_{t_0}^{t} \{D_0[f(s.x_s).f(s.\hat{0})]$$

$$+D_0[f(s.\hat{0}).\hat{0}]\}ds$$

$$+ \int_{t_0}^{t} (\int_{t_0}^{s} \{D_0[g(s.\tau.x_\tau).g(s.\tau.\hat{0})]$$

$$+D_0[g(s.\tau.\hat{0}).\hat{0}]\}\,d\tau)ds)\exp(-kt)\}$$

$$\leq \sup_{t\geq t_0}\{(D_0[\varphi(0).\hat{0}+L\int_{t_0}^{t}\{D_\sigma[x_s.\hat{0}]ds$$

$$+\frac{M}{b}\exp(bt)+L\int_{t_0}^{t}\left(\int_{t_0}^{s}D_\sigma[x_\tau.\hat{0}]d\tau\right)ds$$

$$+\frac{M}{b^2}\exp(bt))\exp(-kt)\}$$

Since $x \in S_k$, there exists ρ such that $\sup_{t\geq t0-\sigma}\{D_0\,[x(t), 0]\exp(-kt)\} < \rho < \infty$. Therefore, for all $t \geq t_0$, we obtain $D_k\,[(\mathbb{T}x)(t).0]$

$$\leq \sup_{t\geq t_0}\{(D_0[\varphi(0).\hat{0}+\left(1+\frac{1}{k}\right)\frac{pL}{k}\exp(kt)$$

$$+\left(1+\frac{1}{b}\right)\frac{M}{b}\exp(bt))\ \exp(-kt)\}$$

$$\leq D_0[\varphi(0).\hat{0}]+\left(1+\frac{1}{b}\right)\frac{1}{b}(M+pL)$$

$$\leq K+\left(1+\frac{1}{b}\right)\frac{1}{b}(M+pL)<\ \infty$$

We infer that T$x \subset S_k$.

Step 2: The following steps, we shall prove that T is a contraction by metric D_k. The first, we consider Let $x, y \in S_k$. Then for $-\sigma \leq s \leq 0$, $D_0\,[\,(\mathrm{T}x)\,(t_0+s), (\mathrm{T}y)(t_0+s)] = 0$. For each $t \geq t_0$, we have $D_k\,[(\mathbb{T}x)(t).(\mathbb{T}y)(t)]$

$$\leq \sup_{t\geq t_0}\{D_0[(\mathbb{T}x)(t).(\mathbb{T}y)(t)]\exp(-kt)$$

$$\leq \sup_{tt_0}\{D_0[\varphi(0)\ominus(-1)\int_{t_0}^{t}(f(s.x_s)$$

$$+\int_{t_0}^{s}g(s.\tau.x_\tau)d\tau)ds.$$

$$\varphi(0)\ominus(-1)\int_{t_0}^{t}(f(s.y_s)+\int_{t_0}^{s}g(s.\tau.y_\tau)d\tau)ds]$$

$$\times\exp(-kt)\}$$

$$\leq \sup_{t\geq t_0}\{(L\int_{t_0}^{t}(D_\sigma[x_s.y_s]+\int_{t_0}^{s}D_\sigma[x_\tau.y_\tau]d\tau)ds)$$

$$\times\exp(-kt)\}$$

$$=\sup_{tt_0}\{(L\int_{t_0}^{t}\sup_{\theta\epsilon[-\sigma.0]}D_0[x(s+\theta).y(s+\theta)]ds$$

$$+L\int_{t_0}^{t}\int_{t_0}^{s}\sup_{\theta\epsilon[-\sigma.0]} D_0[x(\tau+\theta).y(\tau+\theta)])ds)$$
$$\times\exp(-kt)\}$$
$$=\sup_{tt_0}\{(L\int_{t_0}^{t}\sup_{r\epsilon[s-\sigma.s]} D_0[x(r).y(r)]dr$$
$$+L\int_{t_0}^{t}(\int_{t_0}^{s}\sup_{v\epsilon[\tau-\sigma.\tau]} D_0[x(v).y(v)]dv)ds)$$
$$\times\exp(-kt)\}$$
$$\leq LD_k\ [x.y]\sup_{t\geq t_0}(\int_{t_0}^{t}(exp\ (k(r-t))$$
$$+\int_{t_0}^{s}\exp(k\ (v-t))\ dv)dr)$$
$$\leq\frac{(1+k)LD_k\ [x.y]}{k^2}$$

Choosing $k > b$ and $(1 + k)\ L/k2 < 1$, we have the operator T on S_k is a contraction by using Banach fixed point theorem provides the existence of a unique fixed point for T and the unique fixed of T is in the space S_k, that is a unique solution for (3.1) in case (ii)-differentiable and for each case.

4. Illustrations

In this section, we shall present some examples being simple illustrations of the theory of FFIDE. We will consider the FFIDE (3.1) with (i) and (ii) derivative, respectively. Let us start the illustrations with considering the following fuzzy functional integration and differential equation:

$$\begin{cases} \mathrm{D}_H^g\ x(t)=f(t.x_t)+\int_{t_0}^{t}k(t.s)x_s ds.\ t\ \geq t_0 \\ x(t)=\ \varphi(t-t_0)\in\ t\ .t\in[-\sigma.t_0]. \end{cases} \quad (4.1)$$

Where $f : I \times E^1 \rightarrow E^1$, $k(t, s) : I \times I \rightarrow$ R. Let $[x(t)]^\alpha = [x(t,\ \alpha), x(t,\ \alpha)]$. By using Zadeh's extension principle, we obtain $[f\ (t, xt)\ ]^\alpha = [f\ (t,\ \alpha,\ xt(\alpha), xt(\alpha)), f\ (t,\ \alpha,\ xt(\alpha), xt(\alpha))]$, for $\alpha \in [0, 1]$. By using Lemma 2.2, we have the following two cases. If $x(t)$ is (i)-differentiable, then $[D_H{}^g\ x(t)]^\alpha = [x\ (t,\ \alpha), x\ (t,\ \alpha)]$ and (4.1) is translated into the following delay integration and differential system:

$$\begin{cases} \underline{x}'(t.\alpha)=\ \underline{f}\left(t.\alpha.\underline{x}_t(\alpha).\overline{x}_t(\alpha)\right)+\ \int_{t_0}^{t}\underline{k(t.s)x_s(\alpha)}\ ds. \\ \qquad t\geq t_0 \\ \underline{x}\ (t.\alpha)=\ \underline{\varphi}(t-t_0.\alpha).-\sigma\ \leq t\leq t_0 \\ \overline{x}\ (t.\alpha)=\ \overline{\varphi}(t-t_0.\alpha).-\sigma\ \leq t\leq t_0 \end{cases} \quad (4.2)$$

If $x(t)$ is (ii)-differentiable, then $[D_H{}^g\ x(t)]^\alpha = [x\ (t,\ \alpha), x\ (t,\ \alpha)]$ and (4.1) is translated into the following delay integration and differential system:

$$\begin{cases} \underline{x}'(t.\alpha) = \underline{f}\left(t.\alpha.\underline{x}_t(\alpha).\overline{x}_t(\alpha)\right) + \int_{t_0}^{t} \underline{k(t.s)x_s(\alpha)}\ ds. \\ t \geq t_0 \\ \underline{x}'\ (t.\alpha) = \overline{f}\left(t.\alpha.\underline{x}_t(\alpha).\overline{x}_t(\alpha)\right) + \int_{t_0}^{t} \underline{k(t.s)x_s(\alpha)}\ ds. \\ t \geq t_0 \\ \underline{x}\ (t.\alpha) = \underline{\varphi}(t - t_0.\alpha). -\sigma \leq t \leq t_0 \\ \overline{x}\ (t.\alpha) = \overline{\varphi}(t - t_0.\alpha). -\sigma \leq t \leq t_0 \end{cases} \tag{4.3}$$

$$\underline{k(t.s)x_s(\alpha)} = \begin{cases} k(t.s)\ \underline{x_s(\alpha)}\ .\ k(t.s) \geq 0\ . \\ k(t.s)\ \overline{x_s(\alpha)}. k(t.s). < 0 \end{cases}$$

$$\overline{k(t.s)x_s(\alpha)} = \begin{cases} k(t.s)\ \overline{x_s(\alpha)}\ .\ k(t.s) \geq 0\ . \\ k(t.s)\ \underline{x_s(\alpha)}. k(t.s). < 0 \end{cases}$$

Example 4.1. Let us consider the linear fuzzy functional integration and differential equation under two kinds Hukuhara derivative

$$\begin{cases} \mathrm{D}_H^g\ x(t) = (t - \frac{1}{2}) + \lambda \int_0^t e^{(s-t)}\, x\left(s - \frac{1}{2}\right) ds \\ x(t) = \varphi(t). t \in \left[-\frac{1}{2}\ .0\right]. \end{cases} \tag{4.4}$$

Where $k\ (t,\ s) = \lambda\ e^{(s-t)}$, $\varphi\ (t) = (1 - t,\ 2 - t,\ 3 - t)$, $\lambda \in \mathrm{R}\backslash\{0\}$. In this example we shall solve (4.4) on [0,1⁄2].

Case 1: ($\lambda > 0$ or $k\ (t,\ s) > 0$) From (4.2), we get

$$\begin{cases} \underline{x}'(t.\alpha) = \underline{x}\left(t.-\frac{1}{2}.\alpha\right) + \lambda \int_0^t e^{(s-t)}\, \underline{x}\left(s - \frac{1}{2}.\alpha\right) ds. \\ t \geq 0 \\ \overline{x}'\ (t.\alpha) = \overline{x}\left(t.-\frac{1}{2}.\alpha\right) + \lambda \int_0^t e^{(s-t)}\ \overline{x}\left(s - \frac{1}{2}.\alpha\right) ds. \\ t \geq 0 \\ \underline{x}\ (t.\alpha) = 1 + \alpha - t\ . -\frac{1}{2} \leq t \leq 0 \\ \overline{x}\ (t.\alpha) = 3 - \alpha - t\ .\frac{-1}{2} \leq t \leq 0 \end{cases} \tag{4.5}$$

Where $\alpha \in [0,\ 1]$. By solving delay integration and differential systems (4.5), we obtain (i)-solution

$$[x(t)]^{\alpha} = [1 + \alpha + (1 + \alpha)t - \frac{t^2}{2} - \lambda\, e^{(-t)}\ (2 + \alpha)$$

$$+\lambda(2 + \alpha - t). 3 - \alpha + (3 - \alpha)t - \frac{t^2}{2}$$

$$-\lambda\, e^{(-t)}\ (4 - \alpha) + \lambda\ (4 - \alpha - t)].$$

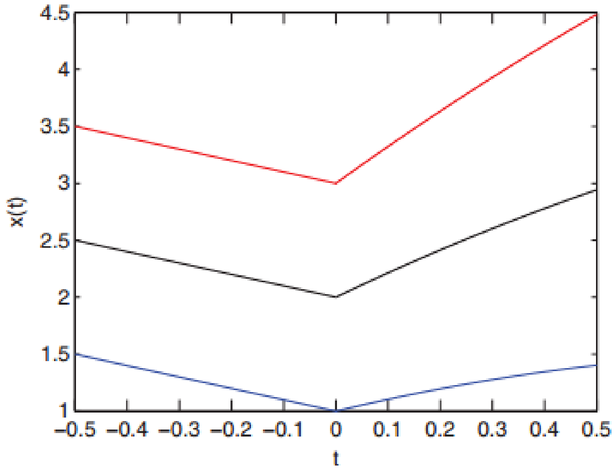

Figure 1. Graphs of *x(t)* for $t \in [-\frac{1}{2}, \frac{1}{2}]$, λ = 0.1.

$t \in [0, 1/2]$. The (i)-solution of (4.4) on [-1/2, 1/2] are illustrated in Figure 1. From (4.3), we obtain

$$\begin{cases} \underline{x}'(t.\alpha) = \underline{x}\left(t.-\frac{1}{2}.\alpha\right) + \lambda \int_0^t e^{(s-t)}\, \underline{x}\left(s-\frac{1}{2}.\alpha\right) ds. \\ t \geq 0 \\ \overline{x}\,(t.\alpha) = 1 + \alpha - t.\frac{-1}{2} \leq t \leq 0 \\ t \geq 0 \\ \underline{x}\,(t.\alpha) = 1 + \alpha - t\,.\frac{-1}{2} \leq t \leq 0 \\ \overline{x}\,(t.\alpha) = 3 - \alpha - t\,.\frac{-1}{2} \leq t \leq 0 \end{cases} \tag{4.6}$$

By solving delay integration and differential systems (4.6), we obtain (ii)-solution

$$[x(t)]^{\alpha} = [1 + \alpha + (3-\alpha)t - \frac{t^2}{2} - \lambda\, e^{(-t)}\,(4-\alpha)$$
$$+\lambda(4-\alpha-t).3 - \alpha + (1+\alpha)t - \frac{t^2}{2}$$
$$-\lambda\, e^{(-t)}\,(2+\alpha) + \lambda\,(2+\alpha-t)].$$

$t \in [0, 1/2]$. The (ii)-solution of (4.4) on [-1/2, 1/2] are illustrated in Figure 2.

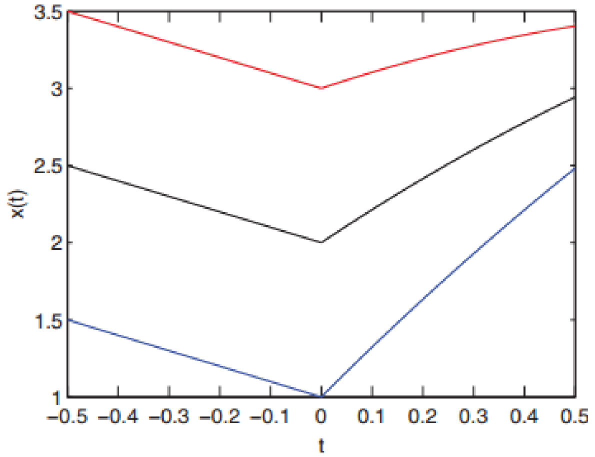

Figure 2. Graphs of *x(t)* for $t \in [-\frac{1}{2}, \frac{1}{2}]$, λ = 0.1.

Case 2: ($\lambda < 0$ or $k\ (t, s) < 0$) From (4.2), we get

$$\begin{cases} \underline{x}'(t.\alpha) = \underline{x}\left(t.-\frac{1}{2}.\alpha\right) + \lambda \int_0^t e^{(s-t)}\,\overline{x}\left(s-\frac{1}{2}.\alpha\right)ds. \\ \qquad t \geq 0 \\ \overline{x}'(t.\alpha) = \overline{x}\,(t-\frac{1}{2}.\alpha) + \lambda \int_0^t e^{(s-t)}\,\underline{x}\left(s-\frac{1}{2}.\alpha\right)ds. \\ \qquad t \geq 0 \\ \underline{x}\,(t.\alpha) = 1+\alpha-t \qquad .\frac{-1}{2} \leq t \leq 0 \\ \overline{x}\,(t.\alpha) = 3-\alpha-t \qquad .\frac{-1}{2} \leq t \leq 0 \end{cases} \tag{4.7}$$

By solving delay integration and differential systems (4.7), we obtain (i)-solution

$$[x(t)]^\alpha = [1+\alpha+(1+\alpha)t - \frac{t^2}{2} - \lambda\, e^{(-t)}\,(4-\alpha)$$
$$+\lambda(4-\alpha-t).3-\alpha+(3-\alpha)t-\frac{t^2}{2}$$
$$-\lambda\, e^{(-t)}\,(2+\alpha) + \lambda\,(2+\alpha-t)].$$

$t \in [0, 1/2]$. The (i)-solution of (4.4) on [-1/2, 1/2] are illustrated in Figure 3.

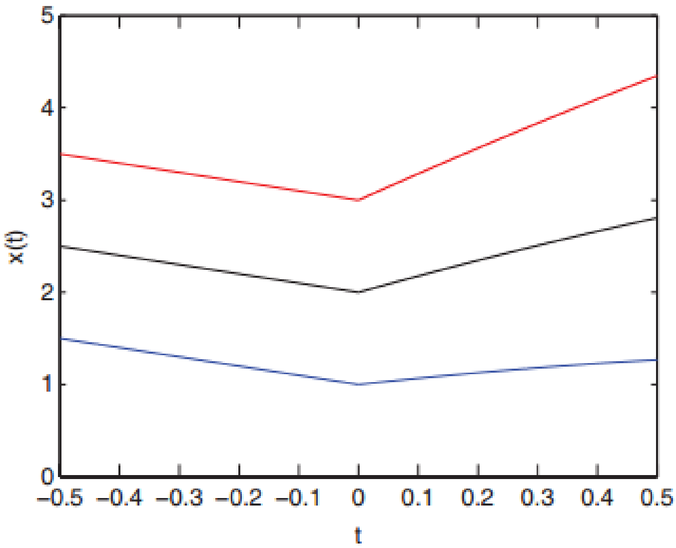

Figure 3. Graphs of $x(t)$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda = 0.1$.

From (4.3), we obtain

$$\begin{cases} \underline{x}'(t.\alpha) = \underline{x}\left(t.-\frac{1}{2}.\alpha\right) + \lambda \int_0^t e^{(s-t)}\,\overline{x}\left(s-\frac{1}{2}.\alpha\right)ds. \\ \qquad t \geq 0 \\ \underline{x}'(t.\alpha) = \overline{x}\,(t-\frac{1}{2}.\alpha) + \lambda \int_0^t e^{(s-t)}\,\underline{x}\left(s-\frac{1}{2}.\alpha\right)ds. \\ \qquad t \geq 0 \\ \underline{x}\,(t.\alpha) = 1+\alpha-t\,.\frac{-1}{2} \leq t \leq 0 \\ \overline{x}\,(t.\alpha) = 3-\alpha-t\,.\frac{-1}{2} \leq t \leq 0 \end{cases} \tag{4.8}$$

By solving delay integration and differential systems (4.7), we obtain (ii)-solution

$$[x(t)]^{\alpha} = [1+\alpha+(3-\alpha)t-\frac{t^2}{2}- \lambda\, e^{(-t)}\,(2+\alpha)$$
$$+\lambda(2+\alpha-t).3-\alpha+(1+\alpha)t-\frac{t^2}{2}$$
$$-\lambda\, e^{(-t)}\,(4-\alpha)+\ \lambda\,(4-\alpha-t)].$$

$t \in [0, 1/2]$. The (ii)-solution of (4.4) on [-1/2, 1/2] are illustrated in Figure 4. From Example 4.1, we notice that, the solutions under classic Hukuhara derivative ((i)-differentiable) have increasing length of its values. Indeed, we can see the Figures 1 and 3.

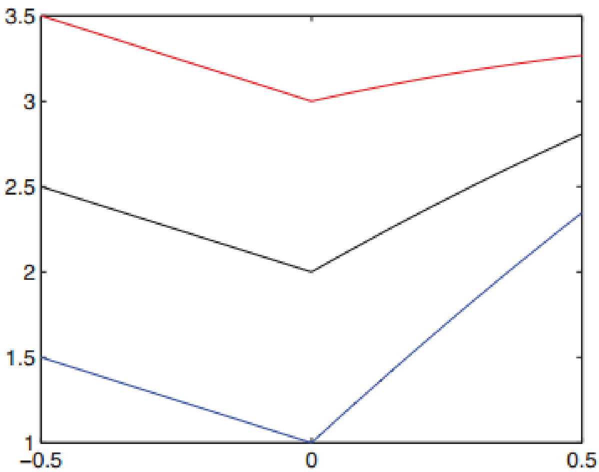

Figure 4. Graphs of $x(t)$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$, $\lambda = 0.1$.

However, if we consider the second type Hukuhara derivative ((ii)- differentiable) the length of solutions change. Under the second type Hukuhara differentiable solutions have non-increasing length of its values (see Figures 2 and 4).

5. Conclusions

In this paper, we have obtained a global existence and uniqueness result for a solution to fuzzy functional integration and differential equations. Also, we have proved a local existence and uniqueness results using the method of successive approximation. Results here might be used in further research on fuzzy functional integration and differential equations. Other possible directions of research could be an approach for fuzzy differential equations using other concepts of calculus for fuzzy functions and derivative for fuzzy functions (see [3,8]).

Acknowledgments

The authors would like to express his gratitude to the anonymous referees for their helpful comments and suggestions, which have greatly improved the paper.

Conflict of interest

The authors declare no conflict of interests.

References

1. B. Ahmad and S. Sivasundaram, *Dynamics and stability of impulsive hybrid setvalued integration and differential equations with delay*, Nonlinear Analysis: Theory, Methods & Applications, **65** (2006), 2082–2093.
2. A. Ahmadian, M. Suleiman, S. Salahshour, et al. *A Jacobi operational matrix for solving a fuzzy linear differential equation*, Adv. Differ. Equ-NY, **2013** (2013), 104.
3. R. P. Agarwal, V. Lakshmikantham and J. J. Nieto, *On the concept of solution for differential equations with uncertainty*, Nonlinear Analysis: Theory, Methods & Applications, **72** (2010), 2859–2862.
4. R. P. Agarwal, D. O'Regan and V. Lakshmikantham, *Viability theory and fuzzy differential equations*, Fuzzy Set. Syst., **151** (2005), 536–580.
5. T. Allahviranloo, S. Abbasbandy, O. Sedaghatfar, et al. *A New Method for Solving Fuzzy Integration and differential Equation Under Generalized Differentiability*, Neural Computing and Applications, **21** (2012), 191–196.
6. T. Allahviranloo, S. Hajighasemi, M. Khezerloo, et al. *Existence and Uniqueness of Solutions of Fuzzy Volterra Integration and differential Equations*, International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, (2010), 491–500.
7. T. Allahviranloo, A. Amirteimoori, M. Khezerloo, et al. *A New Method for Solving Fuzzy Volterra Integration and differential Equations*, Australian Journal of Basic and Applied Sciences, **5** (2011), 154–164.
8. T. Allahviranloo, S. Salahshour and S. Abbasbandy, *Solving fuzzy differential equations by fuzzy Laplace transforms*, Commun. Nonlinear Sci., **17** (2012), 1372–1381.
9. T. Allahviranloo, S. Abbasbandy, O. Sedaghatfar, et al. *A New Method for Solving Fuzzy Integration and differential Equation Under Generalized Differentiability*, Neural Computing and Applications, **21** (2012), 191–196.
10. T. Allahviranloo, M. Ghanbari, E. Haghi, et al. *A note on "Fuzzy linear systems"*, Fuzzy Set. Syst., **177** (2011), 87–92.
11. T. Allahviranloo, S. Abbasbandy, N. Ahmady, et al. *Improved predictor corrector method for solving fuzzy initial value problems*, Inform. Sciences, **179** (2009), 945–955.
12. T. Allahviranloo, N. A. Kiani and N. Motamedi, *Solving fuzzy differential equations by differential transformation method*, Inform. Sciences, **179** (2009), 956–966.
13. T. Allahviranloo, S. Abbasbandy, S. Salahshour, et al. *A new method for solving fuzzy linear differential equations*, Computing, **92** (2011), 181–197.
14. T. Allahviranloo and S. Salahshour, *A new approach for solving first order fuzzy differential equation*, International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, **81** (2010), 522–531.
15. T. Allahviranloo and S. Salahshour, *Euler method for solving hybrid fuzzy differential equation*, Soft Comput., **15** (2011), 1247–1253.
16. L. C. Barros, R. C. Bassanezi and P. A. Tonelli, *Fuzzy modeling in population dynamics*, Ecological Modeling, **128** (2000), 27–33.
17. B. Bede and S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Set. Syst., **151** (2005), 581–599.

18. B. Bede, I. J. Rudas and A. L. Bencsik, *First order linear fuzzy differential equations under generalized differentiability*, Inform. Sciences, **177** (2007), 1648–1662.
19. B. Bede and L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Set. Syst., **230** (2013), 119–141.
20. B. Bede, *A note on "two-point boundary value problems associated with non-linear fuzzy differential equations"*, Fuzzy Set. Syst., **157** (2006), 986–989.
21. J. J. Buckley and T. Feuring, *Fuzzy differential equations*, Fuzzy Set. Syst., **110** (2000), 43–54.
22. S. S. L. Chang and L. Zadeh, *On fuzzy mapping and control*, IEEE T. Syst. Man Cy., **2** (1972), 30–34.
23. Y. Chalco-Cano and H. Roman-Flores, *On new solutions of fuzzy differential equations*, Chaos Soliton. Fract., **38** (2008) ,112–119.
24. Y. Chalco-Cano, A. Rufian-Lizana, H. Román-Flores, et al. *Calculus for interval-valued functions using generalized Hukuhara derivative and applications*, Fuzzy Set. Syst., **219** (2013), 49–67.
25. M. Chen and C. Han, *Some topological properties of solutions to fuzzy differential systems*, Inform. Sciences, **197** (2012), 207–214.
26. D. Dubois and H. Prade, *Towards fuzzy differential calculus part 3: Differentiation*, Fuzzy Set. Syst., **8** (1982), 225–233.
27. L. J. Jowers, J. J. Buckley and K. D. Reilly, *Simulating continuous fuzzy systems*, Inform. Sciences, **177** (2007), 436–448.
28. O. Kaleva, *Fuzzy differential equations*, Fuzzy Set. Syst., **24** (1987), 301–317.
29. A. Khastan, J. J. Nieto and R. Rodríguez-López, *Variation of constant formula for first order fuzzy differential equations*, Fuzzy Set. Syst., **177** (2011), 20–33.
30. A. Khastan, J. J. Nieto, R. Rodríguez-López, *Fuzzy delay differential equations under generalized differentiability*, Inform. Sciences, **275** (2014), 145–167
31. S. Khezerloo, T. Allahviranloo, S. H. Ghasemi, et al. *Expansion Method for Solving Fuzzy Fredholm-Volterra Integral Equations*, International Conference on Information Processing and Management of Uncertainty in KnowledgeBased Systems, **81** (2010), 501–511.
32. V. B. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1992.
33. V. Lakshmikantham and R. N. Mohapatra, *Theory of fuzzy differential equations and inclusions*, Taylor Francis, London, 2003.
34. V. Lakshmikantham and S. Leela, *Fuzzy differential systems and the new concept of stability*, *Nonlinear Dynamics and Systems Theory*, **1** (2001), 111–119.
35. V. Lupulescu, *On a class of fuzzy functional differential equations*, Fuzzy Set. Syst., **160** (2009), 1547–1562.
36. V. Lupulescu, *Initial value problem for fuzzy differential equations under dissipative conditions*, Inform. Sciences, **178** (2008), 4523–4533.
37. J. K. Hale, *Theory of functional differential equations*, Springer, New York, 1997.
38. M. T. Malinowski, *Interval differential equations with a second type Hukuhara derivative*, Applied Mathematics Letters, **24** (2011), 2118–2123.
39. M. T. Malinowski, *Random fuzzy differential equations under generalized Lipschitz condition*, Nonlinear Analysis: Real World Applications, **13** (2012), 860–881.
40. M. T. Malinowski, *Existence theorems for solutions to random fuzzy differential equations*, Nonlinear Analysis: Theory, Methods & Applications**,** **73** (2010), 1515–1532.

41. M. T. Malinowski, *On random fuzzy differential equations*, Fuzzy Set. Syst., **160** (2009), 3152–3165.
42. M. T. Mizukoshi, L. C. Barros, Y. Chalco-Cano, et al. *Fuzzy differential equations and the extension principle*, Inform. Sciences, **177** (2007), 3627–3635.
43. J. J. Nieto, A. Khastan and K. Ivaz, *Numerical solution of fuzzy differential equations under generalized differentiability*, Nonlinear Analysis: Hybrid Systems, **3** (2009), 700–707.
44. J. J. Nieto, *The Cauchy problem for continuous fuzzy differential equations*, Fuzzy Set. Syst., **102** (1999), 259–262.
45. J. J. Nieto and R. Rodríguez-López, *Bounded solutions for fuzzy differential and integral equations*, Chaos Soliton. Fract., **27** (2006), 1376–1386.
46. J. Y. Park and J. U. Jeong, *On existence and uniqueness of solutions of fuzzy integration and differential equations*, Indian J. Pure Appl. Math., **34** (2003), 1503–1512.
47. P. Prakash, J. J. Nieto, J. H. Kim, et al. *Existence of solutions of fuzzy neutral differential equations in Banach spaces*, Dynamical Systems and Applications, **14** (2005), 407–418.
48. M. L. Puri and D. Ralescu, *Differential for fuzzy functions*, J. Math. Anal. Appl., **91** (1983), 552–558.
49. S. Salahshour and T. Allahviranloo, *Applications of fuzzy Laplace transforms*, Soft Computing, **17** (2013), 145–158.
50. S. Salahshour and T. Allahviranloo, *Application of fuzzy differential transform method for solving fuzzy Volterra integral equations*, Appl. Math. Model., **37** (2013), 1016–1027.
51. S. Salahshour and M. Khan, *Exact solutions of nonlinear interval Volterra integral equations*, International Journal of Industrial Mathematics, **4** (2012), 375–388.
52. S. Salahshour, M. Khezerloo, S. Hajighasemi, et al. *Solving Fuzzy Integral Equations of the Second Kind by Fuzzy Laplace Transform Method*, International Journal of Industrial Mathematics, **4** (2012), 21–29.
53. L. Stefanini and B. Bede, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*, Nonlinear Analysis: Theory, Methods & Applications, **71** (2009), 1311–1328.
54. S. Song and C. Wu, *Existence and uniqueness of solutions to Cauchy problem of fuzzy differential equations*, Fuzzy Set. Syst., **110** (2000), 55–67.
55. M. R. Nourizadeh, T. Allahviranloo, N. Mikaeilvand, *Positive solutions of fuzzy fractional Volterra integro-differential equations with the Fuzzy Caputo Fractional Derivative using the Jacobi polynomials operational matrix*, International Journal of Computer Science and Network Security, **18** (2018), 241–252.
56. X. P. Xue and Y. Q. Fu, *On the structure of solutions for fuzzy initial value problem*, Fuzzy Set. Syst., **157** (2006), 212–229.
57. L. A. Zadeh, *Fuzzy sets*, Information and Control**, 8** (1965), 338–353.

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