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Research article

Some inequalities via ψ-Riemann-Liouville fractional integrals

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Abstract: In this paper, we establish some Hermite-Hadamard type inequalities via ψ-Riemann-Liouville fractional integrals for s-convex functions in second sense and the functions belongs to the class $P(I)$ (that is, a class of non-negative functions $\curlyvee : I \to \mathbb{R}$ which satisfies the condition $\curlyvee(ra_1 + (1-r)a_2) \leq \curlyvee(a_1) + \curlyvee(a_2)$, for all $a_1, a_2 \in I$ and $r \in [0,1]$). Some applications to special means are also investigated.

Keywords: Hermite-Hadamard inequalities; s-convex functions in second sense; non-negative functions $P(I)$
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1. Introduction

It is being known that the Hermite-Hadamard inequality [6,7] for a convex function $\curlyvee : I \to \mathbb{R}$ on an interval I is

$$\curlyvee\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \curlyvee(z)dz \leq \frac{\curlyvee(a_1)+\curlyvee(a_2)}{2}, \tag{1.1}$$

for all $a_1, a_2 \in I$ with $a_1 < a_2$. Inequality (1.1) is then revised for several generalized convex functions. For more precise data see [1,3,9,10,14,16].

Definition 1.1 ([8]). Let $s \in (0,1]$. A function $\curlyvee : I \subset \mathbb{R}_0 \to \mathbb{R}_0$, where $\mathbb{R}_0 = [0,\infty)$, is called s-convex in the second sense, if

$$\curlyvee(ra_1 + (1-r)a_2) \leq r^s \curlyvee(a_1) + (1-r)^s \curlyvee(a_2), \tag{1.2}$$

for all $a_1, a_2 \in I$ and $r \in [0,1]$.

Dragomir et al. [4] proved following inequality for s-convex functions.

Theorem 1.1 ([4]). *Let $s \in (0,1)$ and $\curlyvee : \mathbb{R}_0 \to \mathbb{R}_0$ is s-convex in the second sense. Let $a_1, a_2 \in [0,\infty)$, $a_1 \leq a_2$. If $\curlyvee \in L_1[a_1,a_2]$, then*

$$2^{s-1} \curlyvee \left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \curlyvee(z)dz \leq \frac{\curlyvee(a_1)+\curlyvee(a_2)}{s+1}. \tag{1.3}$$

In 1995, Dragomir et al. [5] defined following class of functions.

Definition 1.2 ([5]). A mapping $\curlyvee : I \to \mathbb{R}$ belongs to the class $P(I)$, if it is non-negative and satisfy the following inequality:

$$\curlyvee (ra_1 + (1-r)a_2) \leq \curlyvee(a_1) + \curlyvee(a_2), \tag{1.4}$$

for all $a_1, a_2 \in I$ and $r \in [0,1]$.

Theorem 1.2 ([5]). *Let $\curlyvee \in P(I)$, $a_1, a_2 \in I$ and $\curlyvee \in L_1[a_1,a_2]$, then*

$$\curlyvee \left(\frac{a_1+a_2}{2}\right) \leq \frac{2}{a_2-a_1}\int_{a_1}^{a_2} \curlyvee(z)dz \leq 2(\curlyvee(a_1)+\curlyvee(a_2)). \tag{1.5}$$

Fractional Hermite-Hadamard inequalities for the Riemann-Liouville, the Hadamard fractional integrals, the conformable, new conformable fractional integrals and the Katugampola fractional integrals have been studied. For examples see [2, 11, 15, 17–19]. Liu et al. [13] established Hermite-Hadamard type inequalities for ψ-Riemann-Liouville fractional integrals via convex functions.

Definition 1.3 ([12, 21]). Let $(a_1,a_2)(-\infty \leq a_1 < a_2 \leq \infty)$ be a finite or infinite real interval and $\beta > 0$. Let $\psi(x)$ is an increasing and positive monotone function on $(a_1,a_2]$ with continuous derivative on (a_1,a_2). Then the left and right-sided ψ-Riemann-Liouville fractional integrals of a function $\curlyvee$ with respect to ψ on $[a_1,a_2]$ are defined by

$$\mathcal{I}^{\beta:\psi}_{a_1+} \curlyvee (t) = \frac{1}{\Gamma(\beta)}\int_{a_1}^{t} \psi'(z)(\psi(t)-\psi(z))^{\beta-1} \curlyvee (z)dz,$$

$$\mathcal{I}^{\beta:\psi}_{a_2-} \curlyvee (t) = \frac{1}{\Gamma(\beta)}\int_{t}^{a_2} \psi'(z)(\psi(z)-\psi(t))^{\beta-1} \curlyvee (z)dz,$$

respectively; with Gamma function $\Gamma(\cdot)$.

Lemma 1.1 ([13]). *Let $\curlyvee : [a_1,a_2] \to \mathbb{R}$ be a differentiable mapping, for $0 \leq a_1 < a_2$, and $\curlyvee \in L_1[a_1,a_2]$. Let $\psi(x)$ is an increasing and positive monotone function on $(a_1,a_2]$, with continuous derivative $\psi'(x)$ on (a_1,a_2) and $\beta \in (0,1)$. Then the following equality for fractional integral holds:*

$$\begin{aligned}
&\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2} - \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee \circ \psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee \circ \psi)(\psi^{-1}(a_1))] \\
&= \frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} [(\psi(z)-a_1)^\beta - (a_2-\psi(z))^\beta](\curlyvee' \circ \psi)(z)\psi'(z)dz.
\end{aligned} \tag{1.6}$$

Lemma 1.2 ([13]). *Let $\curlyvee : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping, for $0 \le a_1 < a_2$, and $\curlyvee \in L_1[a_1, a_2]$. Let $\psi(x)$ is an increasing and positive monotone function on $(a_1, a_2]$, with continuous derivative $\psi'(x)$ on (a_1, a_2) and $\beta \in (0, 1)$. Then the following equality for fractional integral holds:*

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee \circ \psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee \circ \psi)(\psi^{-1}(a_1))] - \curlyvee\left(\frac{a_1+a_2}{2}\right) \\
&= \int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} k(\curlyvee' \circ \psi)(z)\psi'(z)dz \qquad (1.7)\\
&\quad + \frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} [(\psi(z)-a_1)^\beta - (a_2-\psi(z))^\beta](\curlyvee' \circ \psi)(z)\psi'(z)dz,
\end{aligned}$$

where

$$k = \begin{cases} \frac{1}{2}, & \psi^{-1}\left(\frac{a_1+a_2}{2}\right) \le z \le \psi^{-1}(a_2), \\ -\frac{1}{2}, & \psi^{-1}(a_1) < z < \psi^{-1}\left(\frac{a_1+a_2}{2}\right). \end{cases}$$

In this paper, we studied some inequalities for functions belonging to $P(I)$ and s-convex functions in second sense via ψ-Riemann-Liouville fractional integrals.

2. Inequalities for s-convex functions

First we establish the Hermite-Hadamard inequality via ψ-Riemann-Liouville fractional integrals.

Theorem 2.1. *Let $\curlyvee : [a_1, a_2] \to \mathbb{R}$ be a positive function, for $0 \le a_1 < a_2$, and $\curlyvee \in L_1[a_1, a_2]$. Let $\psi(z)$ is an increasing and positive monotone function on $(a_1, a_2]$, with continuous derivative $\psi'(z)$ on (a_1, a_2). Let $\curlyvee$ is s-convex function, then following inequalities for fractional integral hold:*

$$\begin{aligned}
&2^{s-1}\curlyvee\left(\frac{a_1+a_2}{2}\right) \\
&\le \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee \circ \psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee \circ \psi)(\psi^{-1}(a_1))] \qquad (2.1)\\
&\le \left(\frac{\beta}{\beta+s} + \beta B(\beta, s+1)\right)\left(\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2}\right),
\end{aligned}$$

where B is Beta function defined as $B(a_1, a_2) = \int_0^1 z^{a_1-1}(1-z)^{a_2-1}dz$.

Proof. Since $\curlyvee$ is s-convex, we have

$$\curlyvee\left(\frac{u+v}{2}\right) \le \frac{\curlyvee(u)+\curlyvee(v)}{2^s}.$$

Let $u = ra_1 + (1-r)a_2$ and $v = ra_2 + (1-r)a_1$, we get

$$2^s\curlyvee\left(\frac{a_1+a_2}{2}\right) \le \curlyvee(ra_1 + (1-r)a_2) + \curlyvee(ra_2 + (1-r)a_1). \qquad (2.2)$$

Multiplying by $r^{\beta-1}$ on both sides of inequality (2.2), and then integrating with respect to r over $[0,1]$, implies

$$\frac{2^s}{\beta} \curlyvee \left(\frac{a_1+a_2}{2}\right) \leq \int_0^1 r^{\beta-1} \curlyvee (ra_1+(1-r)a_2)dr + \int_0^1 r^{\beta-1} \curlyvee (ra_2+(1-r)a_1)dr. \tag{2.3}$$

Now consider,

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2))+\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\\
&=\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta\Gamma(\beta)}\Big[\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}\psi'(z)(a_2-\psi(z))^{\beta-1}(\curlyvee\circ\psi)(z)dz\\
&\quad+\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}\psi'(z)(\psi(z)-a_1)^{\beta-1}(\curlyvee\circ\psi)(z)dz\Big]\\
&=\frac{\beta}{2}\Big[\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}\left(\frac{a_2-\psi(z)}{a_2-a_1}\right)^{\beta-1}\curlyvee(\psi(z))\frac{\psi'(z)}{a_2-a_1}dz\\
&\quad+\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}\left(\frac{\psi(z)-a_1}{a_2-a_1}\right)^{\beta-1}\curlyvee(\psi(z))\frac{\psi'(z)}{a_2-a_1}dz\Big]\\
&=\frac{\beta}{2}\left[\int_0^1\curlyvee(ra_1+(1-r)a_2)dr+\int_0^1\curlyvee(ra_2+(1-r)a_1)dr\right]\\
&\geq 2^{s-1}\curlyvee\left(\frac{a_1+a_2}{2}\right),
\end{aligned} \tag{2.4}$$

by using (2.3). Thus first inequality of (2.1) is proved.

For next, we again use s-convexity of $\curlyvee$, that is,

$$\curlyvee(ra_1+(1-r)a_2)\leq r^s\curlyvee(a_1)+(1-r)^s\curlyvee(a_2),$$

and

$$\curlyvee(ra_2+(1-r)a_1)\leq r^s\curlyvee(a_2)+(1-r)^s\curlyvee(a_1).$$

By adding

$$\curlyvee(ra_1+(1-r)a_2)+\curlyvee(ra_2+(1-r)a_1)\leq(r^s+(1-r)^s)(\curlyvee(a_1)+\curlyvee(a_2)). \tag{2.5}$$

Multiplying by $r^{\beta-1}$ on both sides of inequality (2.5), and then integrating with respect to r over $[0,1]$, implies

$$\begin{aligned}
&\int_0^1 r^{\beta-1}\curlyvee(ra_1+(1-r)a_2)dr+\int_0^1 r^{\beta-1}\curlyvee(ra_2+(1-r)a_1)dr\\
&\leq\left(\frac{1}{\beta+s}+B(\beta,s+1)\right)(\curlyvee(a_1)+\curlyvee(a_2)).
\end{aligned}$$

By multiplying $\frac{\beta}{2}$ on both sides of above inequality we get,

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2))+\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\\
&\leq\left(\frac{\beta}{\beta+s}+\beta B(\beta,s+1)\right)\left(\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2}\right).
\end{aligned}$$

Hence the proof is completed. □

Remark 2.1. *Under the similar assumptions of Theorem* 2.1.
1. *For* $s = 1$, *we get Theorem* 2.1 *in* [13].
2. *For* $\psi(z) = z$, *we get Theorem* 3 *in* [20].
3. *For* $\psi(z) = z$ *and* $\beta = 1$, *the inequality* (2.1) *becomes inequality* (1.3).
4. *For* $\psi(z) = z$ *and* $\beta = s = 1$, *the inequality* (2.1) *becomes inequality* (1.1).

For the next two results we use Lemma 1.1 and Lemma 1.2, respectively.

Theorem 2.2. *Let* $\curlyvee : [a_1, a_2] \to \mathbb{R}$ *be a differentiable mapping, for* $0 \le a_1 < a_2$. *Let* $\psi(z)$ *is an increasing and positive monotone function on* $(a_1, a_2]$, *with continuous derivative* $\psi'(z)$ *on* (a_1, a_2) *and* $\beta \in (0, 1)$. *If* $|\curlyvee'|^q$ *is s-convex for some fixed* $s \in (0, 1)$ *and* $q \ge 1$, *then the following inequality for fractional integral holds:*

$$\begin{aligned}&\left|\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2} - \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\right| \\ &\le \frac{a_2-a_1}{2}\left[\frac{2}{\beta+1}\left(1-\frac{1}{2^\beta}\right)\right]^{\frac{q-1}{q}} \\ &\quad\times\left\{B_{\frac{1}{2}(s+1,\beta+1)} - B_{\frac{1}{2}(\beta+1,s+1)} + \frac{2^{\beta+s}-1}{(\beta+s+1)2^{\beta+s}}\right\}^{\frac{1}{q}} (|\curlyvee'(a_1)|^q + |\curlyvee'(a_2)|^q)^{\frac{1}{q}},\end{aligned} \tag{2.6}$$

where B_u *is incomplete Beta function defined as:*

$$B_u(a_1, a_2) = \int_0^u z^{a_1-1}(1-z)^{a_2-1}dz, \quad u \in (0, 1).$$

Proof. First note that, for every $z \in (\psi^{-1}(a_1), \psi^{-1}(a_2))$, we have $a_1 < \psi(z) < a_2$. Let $r = \frac{a_2-\psi(z)}{a_2-a_1}$, then we have $\psi(z) = ra_1 + (1-r)a_2$. Applying Lemma 1.1 and s-convexity of $|\curlyvee'|$, we obtain

$$\begin{aligned}&\left|\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2} - \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\right| \\ &\le \frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} |(\psi(z)-a_1)^\beta - (a_2-\psi(z))^\beta||(\curlyvee'\circ\psi)(z)|d\psi(z) \\ &= \frac{a_2-a_1}{2}\int_0^1 |(1-r)^\beta - r^\beta||\curlyvee'(ra_1+(1-r)a_2)|dr \\ &\le \frac{a_2-a_1}{2}\int_0^1 |(1-r)^\beta - r^\beta|[r^s|\curlyvee'(a_2)| + (1-r)^s|\curlyvee'(a_2)|]dr \\ &= \frac{a_2-a_1}{2}\Big[\int_0^{\frac{1}{2}} [(1-r)^\beta - r^\beta][r^s|\curlyvee'(a_2)| + (1-r)^s|\curlyvee'(a_2)|]dr \\ &\quad + \int_{\frac{1}{2}}^1 [r^\beta - (1-r)^\beta][r^s|\curlyvee'(a_2)| + (1-r)^s|\curlyvee'(a_2)|]dr\Big].\end{aligned} \tag{2.7}$$

Since

$$\int_0^{\frac{1}{2}} r^s(1-r)^\beta dr = \int_{\frac{1}{2}}^1 r^\beta(1-r)^s dr = B_{\frac{1}{2}}(s+1, \beta+1),$$

$$\int_0^{\frac{1}{2}} r^\beta(1-r)^s dr = \int_{\frac{1}{2}}^1 r^s(1-r)^\beta dr = B_{\frac{1}{2}}(\beta+1, s+1),$$

$$\int_0^{\frac{1}{2}} r^{\beta+s} dr = \int_{\frac{1}{2}}^1 (1-r)^{\beta+s} dr = \frac{1}{2^{\beta+s+1}(\beta+s+1)},$$

and

$$\int_0^{\frac{1}{2}} (1-r)^{\beta+s} dr = \int_{\frac{1}{2}}^1 r^{\beta+s} dr = \frac{1}{\beta+s+1} - \frac{1}{2^{\beta+s+1}(\beta+s+1)}.$$

By substituting above integral values in (2.7) and after some simplification we get the required inequality (2.6) for $q = 1$.

Now consider the case when $q > 1$. Again using Lemma 1.1, power mean inequality and the s-convexity of $|\curlyvee'|^q$ on $[a_1, a_2]$, we get

$$\begin{aligned}
&\left|\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2} - \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}\left[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))\right]\right| \\
&\leq \frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} |(\psi(z)-a_1)^\beta - (a_2-\psi(z))^\beta||(\curlyvee'\circ\psi)(z)|d\psi(z) \\
&= \frac{a_2-a_1}{2}\int_0^1 |(1-r)^\beta - r^\beta||\curlyvee'(ra_1+(1-r)a_2)|dr \\
&= \frac{a_2-a_1}{2}\left(\int_0^1 |(1-r)^\beta - r^\beta|dr\right)^{1-\frac{1}{q}}\left(\int_0^1 |(1-r)^\beta - r^\beta||\curlyvee'(ra_1+(1-r)a_2)|^q dr\right)^{\frac{1}{q}} \\
&= \frac{a_2-a_1}{2}\left(\int_0^1 |(1-r)^\beta - r^\beta|dr\right)^{\frac{q-1}{q}} \\
&\quad\times\left(\int_0^1 |(1-r)^\beta - r^\beta|[r^s|\curlyvee'(a_2)|^q + (1-r)^s\curlyvee'(a_2)|^q]dr\right)^{\frac{1}{q}} \\
&= \frac{a_2-a_1}{2}\left[\frac{2}{\beta+1}\left(1-\frac{1}{2^\beta}\right)\right]^{\frac{q-1}{q}} \\
&\quad\times\left\{B_{\frac{1}{2}(s+1,\beta+1)} - B_{\frac{1}{2}(\beta+1,s+1)} + \frac{2^{\beta+s}-1}{(\beta+s+1)2^{\beta+s}}\right\}^{\frac{1}{q}}(|\curlyvee'(a_1)|^q + |\curlyvee(a_2)|^q)^{\frac{1}{q}}.
\end{aligned} \tag{2.8}$$

This completes the proof. □

Remark 2.2. *Under the similar assumptions of Theorem* 2.2*.*
1. *For* $s = 1$ *and* $q = 1$*, we get Theorem* 3.4 *in* [13].
2. *For* $\psi(z) = z$*, we get Theorem* 4 *in* [20].

Theorem 2.3. *Let* $\curlyvee : [a_1, a_2] \to \mathbb{R}$ *be a differentiable mapping, for* $0 \leq a_1 < a_2$*. Let* $\psi(z)$ *is an increasing and positive monotone function on* $(a_1, a_2]$*, with continuous derivative* $\psi'(z)$ *on* (a_1, a_2) *and* $\beta \in (0, 1)$*. If* $|\curlyvee'|$ *is s-convex for some fixed* $s \in (0, 1)$*, then the following inequality for fractional* *integral holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2))+\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]-\curlyvee\left(\frac{a_1+a_2}{2}\right)\right|\\
&\le\frac{\curlyvee(a_2)-\curlyvee(a_1)}{2}\\
&\quad+\frac{a_2-a_1}{2}\left\{B_{\frac{1}{2}(s+1,\beta+1)}-B_{\frac{1}{2}(\beta+1,s+1)}+\frac{2^{\beta+s}-1}{(\beta+s+1)2^{\beta+s}}\right\}(|\curlyvee'(a_1)|+|\curlyvee'(a_2)|).
\end{aligned}\tag{2.9}$$

Proof. From Lemma 1.2 and the s-convexity of $|\curlyvee'|$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2))+\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]-\curlyvee\left(\frac{a_1+a_2}{2}\right)\right|\\
&=\Bigg|\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}k(\curlyvee'\circ\psi)(z)\psi'(z)dz\\
&\quad+\frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}[(\psi(z)-a_1)^\beta-(a_2-\psi(z))^\beta](\curlyvee'\circ\psi)(z)\psi'(z)dz\Bigg|\\
&\le\left|\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}k(\curlyvee'\circ\psi)(z)\psi'(z)dz\right|\\
&\quad+\left|\frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}[(\psi(z)-a_1)^\beta-(a_2-\psi(z))^\beta](\curlyvee'\circ\psi)(z)\psi'(z)dz\right|\\
&:=H_1+H_2,
\end{aligned}\tag{2.10}$$

where

$$H_1:=\left|\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}k(\curlyvee'\circ\psi)(z)\psi'(z)dz\right|,$$

$$H_2:=\left|\frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}[(\psi(z)-a_1)^\beta-(a_2-\psi(z))^\beta](\curlyvee'\circ\psi)(z)\psi'(z)dz\right|,$$

and k is defined as in Lemma 1.2. Note that

$$H_1=\frac{\curlyvee(a_2)-\curlyvee(a_1)}{2},\tag{2.11}$$

and from Theorem 2.2 for the case $q=1$, we have

$$H_2\le\frac{a_2-a_1}{2}\left\{B_{\frac{1}{2}(s+1,\beta+1)}-B_{\frac{1}{2}(\beta+1,s+1)}+\frac{2^{\beta+s}-1}{(\beta+s+1)2^{\beta+s}}\right\}(|\curlyvee'(a_1)|+|\curlyvee(a_2)|).\tag{2.12}$$

Hence by using (2.11) and (2.12) in (2.10), we get (2.9). □

Remark 2.3. *By taking* $s=1$ *in* (2.9)*, we get inequality* (9) *in* [13].

3. Inequalities for class of non-negative functions $P(I)$

First we establish the Hermite-Hadamard inequality via ψ-Riemann-Liouville fractional integrals.

Theorem 3.1. *Let $\curlyvee : [a_1, a_2] \to \mathbb{R}$ be a positive function, for $0 \le a_1 < a_2$, and $\curlyvee \in L_1[a_1, a_2]$. Let $\psi(z)$ is an increasing and positive monotone function on $(a_1, a_2]$, with continuous derivative $\psi'(z)$ on (a_1, a_2). Let $\curlyvee \in P(I)$ is, then following inequalities for fractional integral hold:*

$$\begin{aligned}
&\frac{1}{2}\curlyvee\left(\frac{a_1+a_2}{2}\right)\\
&\le \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\\
&\le [\curlyvee(a_1)+\curlyvee(a_2)].
\end{aligned}\tag{3.1}$$

Proof. Since the function $\curlyvee$ belongs to the class $P(I)$, we have

$$\curlyvee\left(\frac{u+v}{2}\right) \le \curlyvee(u) + \curlyvee(v).$$

Let $u = ra_1 + (1-r)a_2$ and $u = ra_2 + (1-r)a_1$, we get

$$\curlyvee\left(\frac{a_1+a_2}{2}\right) \le \curlyvee(ra_1+(1-r)a_2) + \curlyvee(ra_2+(1-r)a_1).\tag{3.2}$$

Multiplying by $r^{\beta-1}$ on both sides of inequality (2.2), and then integrating with respect to r over $[0, 1]$, implies

$$\frac{1}{2\beta}\curlyvee\left(\frac{a_1+a_2}{2}\right) \le \frac{1}{2}\left[\int_0^1 r^{\beta-1}\curlyvee(ra_1+(1-r)a_2)dr + \int_0^1 r^{\beta-1}\curlyvee(ra_2+(1-r)a_1)dr\right].\tag{3.3}$$

Now consider,

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\\
&= \frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta\Gamma(\beta)}\Big[\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} \psi'(z)(a_2-\psi(z))^{\beta-1}(\curlyvee\circ\psi)(z)dz\\
&\quad + \int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} \psi'(z)(\psi(z)-a_1)^{\beta-1}(\curlyvee\circ\psi)(z)dz\Big]\\
&= \frac{\beta}{2}\Big[\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}\left(\frac{a_2-\psi(z)}{a_2-a_1}\right)^{\beta-1}\curlyvee(\psi(z))\frac{\psi'(z)}{a_2-a_1}dz\\
&\quad + \int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}\left(\frac{\psi(z)-a_1}{a_2-a_1}\right)^{\beta-1}\curlyvee(\psi(z))\frac{\psi'(z)}{a_2-a_1}dz\Big]\\
&= \frac{\beta}{2}\Big[\int_0^1 \curlyvee(ra_1+(1-r)a_2)dr + \int_0^1 \curlyvee(ra_2+(1-r)a_1)dr\Big]\\
&\ge \frac{1}{2}\curlyvee\left(\frac{a_1+a_2}{2}\right),
\end{aligned}\tag{3.4}$$

by using (3.3). Thus first inequality of (3.1) is proved.

For next, again using the property of $\curlyvee \in P(I)$, that is,

$$\curlyvee(ra_1 + (1-r)a_2) \leq \curlyvee(a_1) + \curlyvee(a_2),$$

and

$$\curlyvee(ra_2 + (1-r)a_1) \leq \curlyvee(a_2) + \curlyvee(a_1).$$

By adding

$$\curlyvee(ra_1 + (1-r)a_2) + \curlyvee(ra_2 + (1-r)a_1) \leq 2(\curlyvee(a_1) + \curlyvee(a_2)). \tag{3.5}$$

Multiplying by $r^{\beta-1}$ on both sides of inequality (3.5), and then integrating with respect to r over $[0,1]$, implies

$$\frac{1}{2}\Big[\int_0^1 r^{\beta-1}\curlyvee(ra_1 + (1-r)a_2)dr + \int_0^1 r^{\beta-1}\curlyvee(ra_2 + (1-r)a_1)dr\Big] \leq \frac{1}{\beta}(\curlyvee(a_1) + \curlyvee(a_2)).$$

That is,

$$\frac{\Gamma(\beta+1)}{2(a_2-a_1)^{\beta}}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))] \leq [\curlyvee(a_1) + \curlyvee(a_2)].$$

This completes the proof. □

Remark 3.1. *By taking $\beta = 1$ and $\psi(z) = z$ in* (3.1)*, we get inequality* 3.2 *of Theorem* 3.1 *in* [5].

Theorem 3.2. *Let $\curlyvee : [a_1, a_2] \to \mathbb{R}$ be a differentiable mapping, for $0 \leq a_1 < a_2$. Let $\psi(z)$ is an increasing and positive monotone function on $(a_1, a_2]$, with continuous derivative $\psi'(z)$ on (a_1, a_2) and $\beta \in (0,1)$. If $|\curlyvee'| \in P(I)$, then the following inequality for fractional integral holds:*

$$\begin{aligned}&\left|\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2} - \frac{\Gamma(\beta+1)}{2(a_2-a_1)^{\beta}}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\right| \\ &\leq \frac{a_2-a_1}{2}\left[\frac{2}{\beta+1}\left(1-\frac{1}{2^{\beta}}\right)\right](|\curlyvee'(a_1)| + |\curlyvee'(a_2)|).\end{aligned} \tag{3.6}$$

Proof. Again using Lemma 1.1 and the property of $|\curlyvee'|$ on $[a_1, a_2]$, we get

$$\begin{aligned}&\left|\frac{\curlyvee(a_1)+\curlyvee(a_2)}{2} - \frac{\Gamma(\beta+1)}{2(a_2-a_1)^{\beta}}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2)) + \mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]\right| \\ &\leq \frac{1}{2(a_2-a_1)^{\beta}}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)} |(\psi(z)-a_1)^{\beta} - (a_2-\psi(z))^{\beta}||(\curlyvee'\circ\psi)(z)|d\psi(z) \\ &= \frac{a_2-a_1}{2}\int_0^1 |(1-r)^{\beta} - r^{\beta}||\curlyvee'(ra_1+(1-r)a_2)|dr \\ &\leq \frac{a_2-a_1}{2}\int_0^1 |(1-r)^{\beta} - r^{\beta}|[|\curlyvee'(a_2)| + \curlyvee'(a_2)|]dr \\ &= \frac{a_2-a_1}{2}\left[\frac{2}{\beta+1}\left(1-\frac{1}{2^{\beta}}\right)\right](|\curlyvee'(a_1)| + |\curlyvee(a_2)|).\end{aligned} \tag{3.7}$$

Since

$$\int_0^{\frac{1}{2}}(1-r)^\beta dr=\int_{\frac{1}{2}}^1 r^\beta dr=\frac{1}{\beta+1}-\frac{1}{(\beta+1)2^{\beta+1}},$$

$$\int_0^{\frac{1}{2}} r^\beta dr=\int_{\frac{1}{2}}^1(1-r)^\beta dr=\frac{1}{(\beta+1)2^{\beta+1}}.$$

This completes the proof. □

Theorem 3.3. *Let* $\curlyvee : [a_1,a_2]\to\mathbb{R}$ *be a differentiable mapping, for* $0\le a_1<a_2$. *Let* $\psi(z)$ *is an increasing and positive monotone function on* $(a_1,a_2]$, *with continuous derivative* $\psi'(z)$ *on* (a_1,a_2) *and* $\beta\in(0,1)$. *If* $|\curlyvee'|\in P(I)$, *then the following inequality for fractional integral holds:*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2))+\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]-\curlyvee\left(\frac{a_1+a_2}{2}\right)\right|\\&\le\frac{\curlyvee(a_2)-\curlyvee(a_1)}{2}+\frac{a_2-a_1}{2}\left[\frac{2}{\beta+1}\left(1-\frac{1}{2^\beta}\right)\right](|\curlyvee'(a_1)|+|\curlyvee'(a_2)|).\end{aligned}\tag{3.8}$$

Proof. From Lemma 1.2 and the property of $|\curlyvee'|$ on $[a_1,a_2]$, we have

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{2(a_2-a_1)^\beta}[\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_1)+}(\curlyvee\circ\psi)(\psi^{-1}(a_2))+\mathcal{I}^{\beta:\psi}_{\psi^{-1}(a_2)-}(\curlyvee\circ\psi)(\psi^{-1}(a_1))]-\curlyvee\left(\frac{a_1+a_2}{2}\right)\right|\\&=\left|\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}k(\curlyvee'\circ\psi)(z)\psi'(z)dz\right.\\&\quad\left.+\frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}[(\psi(z)-a_1)^\beta-(a_2-\psi(z))^\beta](\curlyvee'\circ\psi)(z)\psi'(z)dz\right|\\&\le\left|\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}k(\curlyvee'\circ\psi)(z)\psi'(z)dz\right|\\&\quad+\left|\frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}[(\psi(z)-a_1)^\beta-(a_2-\psi(z))^\beta](\curlyvee'\circ\psi)(z)\psi'(z)dz\right|\\&:=H_3+H_4,\end{aligned}\tag{3.9}$$

where

$$H_3:=\left|\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}k(\curlyvee'\circ\psi)(z)\psi'(z)dz\right|,$$

$$H_4:=\left|\frac{1}{2(a_2-a_1)^\beta}\int_{\psi^{-1}(a_1)}^{\psi^{-1}(a_2)}[(\psi(z)-a_1)^\beta-(a_2-\psi(z))^\beta](\curlyvee'\circ\psi)(z)\psi'(z)dz\right|,$$

and k is defined as in Lemma 1.2. Note that

$$H_3=\frac{\curlyvee(a_2)-\curlyvee(a_1)}{2},\tag{3.10}$$

and from Theorem 3.2, we have

$$H_4\le\frac{a_2-a_1}{2}\left[\frac{2}{\beta+1}\left(1-\frac{1}{2^\beta}\right)\right](|\curlyvee'(a_1)|+|\curlyvee(a_2)|).\tag{3.11}$$

Hence by using (3.10) and (3.11) in (3.9), we get (3.8). □

4. Application to special means

Consider the following special means of real numbers a_1, a_2 such that $a_1 \neq a_2$.

(1). The arithmetic mean

$$A = A(a_1, a_2) = \frac{a_1 + a_2}{2}.$$

(2). The p-logarithmic mean

$$L_p = L_p(a_1, a_2) = \left(\frac{a_2^{p+1} - a_1^{p+1}}{(p+1)(a_2 - a_1)} \right)^{\frac{1}{p}}, \ p \in \mathbb{R} \setminus \{-1, 0\}.$$

Proposition 4.1. Let $a_1, a_2 \in \mathbb{R}^+$, $a_1 < a_2$ and $0 < s < 1$, then

$$\left| A(a_1^s, a_2^s) - L_s^s(a_1, a_2) \right| \leq \frac{a_2 - a_1}{2} \left[\frac{s^2 2^s + s}{(s+1)(s+2)2^s} \right] A(a_1^{s-1}, a_2^{s-1}).$$

Proof. Apply Theorem 2.2 with $\vee = z^s$, $\psi(z) = z$ and $\beta = q = 1$, we get the required result. □

Proposition 4.2. Let $a_1, a_2 \in \mathbb{R}^+$, $a_1 < a_2$ and $0 < s < 1$, then

$$\left| L_s^s(a_1, a_2) - A^s(a_1, a_2) \right| \leq A(a_1^s, a_2^s) - \frac{a_2 - a_1}{2} \left[\frac{s(s2^s + 1)}{(s+1)(s+2)2^{s-1}} \right] A(a_1^{s-1}, a_2^{s-1}).$$

Proof. Apply Theorem 2.3 with $\vee = z^s$, $\psi(z) = z$ and $\beta = q = 1$, we get the required result. □

Proposition 4.3. Let $a_1, a_2 \in \mathbb{R}^+$, $a_1 < a_2$, then

$$\left| A(a_1^2, a_2^2) - L_2^2(a_1, a_2) \right| \leq \frac{a_2^2 - a_1^2}{2}.$$

Proof. Apply Theorem 3.2 with $\vee = z^2$, $\psi(z) = z$ and $\beta = 1$, we get the required result. □

Proposition 4.4. Let $a_1, a_2 \in \mathbb{R}^+$, $a_1 < a_2$, then

$$\left| L_2^2(a_1, a_2) - A^2(a_1, a_2) \right| \leq A(a_1^2, a_2^2) - \frac{a_2^2 - a_1^2}{2}.$$

Proof. Apply Theorem 3.3 with $\vee = z^2$, $\psi(z) = z$ and $\beta = 1$, we get the required result. □

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Conflict of interest

The authors declare that there is no interest regarding the publication of this paper.

References

1. M. Avci, H. Kavurmaci, M. E. Ozdemir, *New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications*, Appl. Math. Comput., **217** (2011), 5171–5176.
2. F. Chen, *Extension of the Hermite–Hadamard inequality for harmonically convex functions via fractional integrals*, Appl. Math. Comput., **268** (2015), 121–128.
3. F. Chen, S. Wu, *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., **9** (2016), 705–716.
4. S. S. Dragomir, S. Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math., **32** (1999), 687–696.
5. S. S. Dragomir, J. Pecaric, L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (1995), 335–341.
6. J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., (1893), 171–215.
7. Ch. Hermite, *Sur deux limites d'une intégrale denie*, Mathesis., **3** (1883), 82.
8. H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., **48** (1994), 100–111.
9. I. Iscan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat., **43** (2014), 935–942.
10. I. Iscan, *Hermite-Hadamard type inequalities for p-convex functions*, Int. J. Ana. Appl., **11** (2016), 137–145.
11. M. Jleli, D. O'Regan, B. Samet, *On Hermite–Hadamard type inequalities via generalized fractional integrals*, Turk. J. Math., **40** (2016), 1221–1230.
12. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier Science, 2006.
13. K. Liu, J. Wang, D. O'Regan, *On the Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via convex functions*, J. Inequal. Appl., **2019** (2019), 27.
14. C. P. Niculescu, L. E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, 2 Eds., New York: Springer, 2018.
15. N. Mehreen, M. Anwar, *Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for p-convex functions via new fractional conformable integral operators*, J. Math. Compt. Sci., **19** (2019), 230–240.
16. N. Mehreen, M. Anwar, *Hermite-Hadamard type inequalities via exponentially p-convex functions and exponentially s-convex functions in second sense with applications*, J. Inequal. Appl., **2019** (2019), 92.

17. N. Mehreen, M. Anwar, *Integral inequalities for some convex functions via generalized fractional integrals*, J. Inequal. Appl., **2018** (2018), 208.

18. M. Z. Sarikaya, E. Set, H. Yaldiz, et al., *Hermite-Hadamard's inequlities for fractional integrals and related fractional inequalitis*, Math. Comput. Modelling, **57** (2013), 2403–2407.

19. E. Set, M. Z. Sarikaya, A. Gozpinar, *Some Hermite–Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities*, Creat. Math. Inform., Accepted paper, 2017.

20. E. Set, M. Z. Sarikaya, M. E. Ozdemir, et al., *The Hermite–Hadamard's inequality for some convex functions via fractional integrals and related results*, J. Appl. Math. Stat. Inform., **10** (2014), 69–83.

21. J. V. C. Sousa, E. C. Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simulat., **60** (2018), 72–91.

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