Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(5): 1386–1402.
DOI:10.3934/math.2019.5.1386
Received: 29 July 2019
Accepted: 16 August 2019
Published: 16 September 2019

Research article

Dynamic modeling of discrete leader-following consensus with impulses

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Abstract: A leader-following consensus of discrete-time multi-agent systems with nonlinear intrinsic dynamics and impulses is investigated. We propose and prove conditions ensuring a leader-following consensus. Numerical examples are given to illustrate effectiveness of the obtained results. Also, the necessity and sufficiency of the obtained conditions are shown.

Keywords: neural networks; discrete models; leader-following consensus; impulses
Mathematics Subject Classification: 39A30, 92B20

1. Introduction

One of the most important topic in multi-agent systems is the consensus algorithm. In these systems, agents interact with each other via a communication topology and only employ local information. As a result, in order to drive them to accomplish tasks, a control law is required. It is connected with the driving a team of agents to reach an agreement on a certain issue by negotiating with their neighbors. In more details, each agent receives information from the set of other agents in the group and then all agents adjust their own information states depending on the information received from other agents. The goal is to reach an agreement. This behavior is widespread in the nature. A consensus algorithm describes the information transfers between agents and varies depending on the application and the model. In the literature, many different consensus algorithms have been proposed (see for example, [4, 11, 17]). The virtual leader is a special agent whose motion is independent of all the other agents and thus is followed by all the other ones. Such a problem is commonly called leader-following consensus problem [12, 24]. Many different types of such kind of problems are studied recently, for instance those based on nearest-neighbor rules [15], bounded confidence [9], a virtual leader [18]. Note most studied models for neural networks are continuous ones which is connected with the continuous behavior of any agent in the neural networks. The continuous time networks are usually discretized when they are used for the sake of computer-based simulation or experimentation. Unfortunately, the dynamic of the continuous-time networks cannot be preserved by discretization, as it is mentioned in [14]. It proves the application of discrete models [7, 9, 13, 16]. The applied technique and methods for the discrete-time case are, however, different from the continuous-time case. Note one of the main property of the solution of difference equations [1–3, 8] is that it is a sequence of numbers defined on the initially given discrete set of points. In the case there are impulses they connect these numbers at the current point and at the previous one (or ones). Some properties of first order difference equations with impulses are studied by some authors, see for example [10].

Sometimes the interactions between multi-agents are changed instantaneously. Then the model is called impulsive model. Some neural networks with instantaneous changes are studied for stochastic neural networks in [21–23], for stochastic differential equations in [19, 22]. In this paper we set up the discrete-time model of a multi-agent system consisting of agents and the leader. Differently than the existing models, such as [9, 13], we consider the case of two interacting topologies, one is determining the interactions between the agents including the leader, the second one is determining the instantaneous switching interactions of the agents with the leader at some initially given times. Sufficient conditions ensuring both local and global leader-following consensus are found. By intensive application of computer simulation the influence of the impulses on the discrete leader-following consensus is illustrated and the necessity and effectiveness of the obtained conditions are shown.

2. Description of the discrete model with impulses

Let $\mathbb{Z}_+$ denote the set of all nonegative integers. Let the increasing sequence $\{n_k\}_{k=1}^{\infty} : n_k \in \mathbb{Z}_+$, $\lim_{k\to\infty} n_k = \infty$, be given and $n_0 = 0$.

In this paper, we consider a discrete-time multi-agent system consisting of N agents and the leader. The dynamics of each agent labeled i, $i = 1, 2, \dots, N$, is given by the difference equation

$$x_i(n+1) = x_i(n) + f(n, x_i(n)) + u_i(n) \quad \text{for} \quad n \in \mathbb{Z}_+, \ i = 1, 2, \dots, N, \tag{1}$$

where $x_i(n)$ and $u_i(n)$ represent the state and the control input at time n, respectively. Function $f : \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R}$ describes the intrinsic, generally nonlinear, dynamics. The leader, labeled as $i = 0$, for multi-agent system (1) is an isolated agent described by

$$x_0(n+1) = x_0(n) + f(n, x_0(n)) \text{ for } n \in \mathbb{Z}_+. \tag{2}$$

Let the control protocol be based on two interaction topologies and be given by:

$$\begin{aligned} u_i(n) = \Big(\gamma \sum_{j=1}^{N} a_{ij}(x_j(n) - x_i(n)) + \gamma d_i(x_0(n) - x_i(n))\Big)\Delta(n - n_k) \\ + B_{i,n}(x_0(n) - x_i(n))\delta(n - n_k) \quad \text{for } n \in \mathbb{Z}_+, \ \ i = 1, 2, \dots, N, \end{aligned} \tag{3}$$

where $\delta(0) = 1$ and $\delta(n) = 0$ for $n \neq 0$, $\Delta(0) = 0$ and $\Delta(n) = 1$ for $n \neq 0$. $a_{ij} \geq 0$, $(i, j = 1, 2, \dots, N)$ are entries of the weighted adjacency matrix A, $d_i \geq 0$, $(i = 1, 2, \dots, N)$ are entries of the adjacency matrix D associated with the graph, modeled the first interaction topology in the multi-agent systems, $B_{in_k} \in \mathbb{R}$ $(i = 1, 2, \dots, N, k = 1, 2, \dots)$ are the diagonal elements of the matrix B_{n_k} associated with the graph, modeled the second interaction topology, and γ is a real constant.

Definition 2.1. *Multi-agent system (1) and (2) under control law (3) is said to achieve*

- *the local leader-following consensus if there exists* $\varepsilon > 0$ *such that for any initial values* $x_i(n_0) \in \mathbb{R} : |x_i(n_0) - x_0(n_0)| \leq \varepsilon,\ i = 0, 1, 2, \dots, N$ *the corresponding solution to (1) and (2) satisfies* $\lim_{n\to\infty} |x_i(n) - x_0(n)| = 0$ *for* $i = 1, 2, \dots, N$.
- *the leader-following consensus if a solution to (1) and (2) satisfies* $\lim_{n\to\infty} |x_i(n) - x_0(n)| = 0$ *for* $i = 1, 2, \dots, N$ *for any initial values* $x_i(n_0) \in \mathbb{R},\ i = 0, 1, 2, \dots, N$.

Then the system (1), (2) where $u_i(n)$ is given by (3) could be written as a system of impulsive difference equations

$$\begin{aligned}
&x_0(n+1) = x_0(n) + f(n, x_0(n)) \text{ for } n \in \mathbb{Z}_+,\\
&x_i(n+1) = x_i(n) + f(n, x_i(n)) + \gamma \sum_{j=1}^{N} a_{ij}(x_j(n) - x_i(n)) + \gamma d_i(x_0(n) - x_i(n))\\
&\qquad\qquad \text{for } n \in \mathbb{Z}_+,\ n \neq n_k,\ k = 1, 2, \dots,\ i = 1, 2, \dots, N,\\
&x_i(n_k+1) = x_i(n_k) + f(n_k, x_i(n_k)) + B_{i,n_k}(x_0(n_k) - x_i(n_k)),\ k \in \mathbb{Z},\ i = 1, 2, \dots, N,\\
&x_i(0) = x_i^0,\ \ i = 0, 1, 2, \dots, N.
\end{aligned} \tag{4}$$

Remark 2.1. *The model (4) is a generalization of the studied in [13] discrete leader consensus problem with one interacting topology. The system (4) models the case of two interacting topologies: The first one (described by the first and the second equations in (4)) is determining the interactions between the agents including the leader; the second one (described by the third equation in (4)) is determining the instantaneous switching interactions of the agents with the leader at some initially given times.*

Denote $e_i(n) = x_i(n) - x_0(n),\ i = 1, 2, \dots, N$, and rewrite the system of difference equations (3) in the form

$$\begin{aligned}
&e_i(n+1) = F(n, x_i(n)) + \gamma \sum_{j=1}^{N} a_{ij} e_j(n) - e_i(n)\gamma \sum_{j=1}^{N} a_{ij} + (1 - \gamma d_i) e_i(n),\\
&\qquad\qquad \text{for } n \in \mathbb{Z}_+,\ n \neq n_k,\ k = 1, 2, \dots,\ i = 1, 2, \dots, N,\\
&e_i(n_k+1) = (1 - B_{i,n_k}) e_i(n_k) + F(n_k, x_i(n_k)),\ \ k \in \mathbb{Z}_+,\ i = 1, 2, \dots, N,\\
&e_i(0) = e_i^0,\ \ i = 0, 1, 2, \dots, N,
\end{aligned} \tag{5}$$

where $F(n, u) = f(n, u) - f(n, x_0(n))$ and $e_i^0 = x_i^0 - x_0^0$.

Denote

$$S(\varepsilon) = \{x \in \mathbb{R} :\ |x - x_0(n)| \leq \varepsilon \text{ for all } \ n \in \mathbb{Z}\}.$$

Introduce the quadratic n dimensional matrix $\mathcal{L}$ with elements $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij},\ i \neq j$ and the quadratic n dimensional matrix D with elements $d_{ii} = d_i$ and $d_{ij} = 0,\ i \neq j$.

In our further study we will use the following discrete inequality which is an analog to the classical Gronwall's inequality:

Lemma 2.1. *([5]) Let* $\{x_n\}_{n=n_0}^{\infty}$ *and* $\{b_n\}_{n=n_0}^{\infty}$ *be sequences of real numbers with* $b_n \geq 0$ *which satisfies*

$$x_n \leq a + \sum_{k=n_0}^{n-1} b_k x_k, \quad n = n_0,\ n_0 + 1,\ n_0 + 2,\ \dots. \tag{6}$$

Then for any integer $n \geq n_0$ the inequality $x_n \leq a \prod_{k=n_0}^{n-1} (1 + b_k)$ holds.

2.1. Leader-following consensus of (5)

Theorem 2.1. *Let the following conditions be satisfied:*

1. *There exists $\varepsilon > 0$ such that function $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ and $|f(n,u) - f(n,v)| \leq L|u - v|$ uniformly in n for $u, v \in S(\varepsilon)$.*
2. *The inequality $L + M \leq 1$ holds where $M = |\gamma| \sqrt{|\gamma l_{max}|}$, γ_{max} is the eigenvalue of matrix $C\, C^T$ with the maximal modulus, $C = \mathcal{L} + D - \frac{1}{\gamma} I$, I is the unit n dimensional matrix.*
3. *For all $k = 1, 2, \ldots$ the inequality $\max_{i=1,2,\ldots,N} |1 - B_{i,n_k}| \leq 1 - L$ holds.*

Then under control law (3) multi-agent system (1) and (2) with initial values $x_i^0 \in S(\varepsilon)$, $i = 1, \ldots, N$, achieves the local leader-following consensus.

Remark 2.2. *Under the condition 1 of Theorem 2.1 it follows that for any $u \in S(\varepsilon)$ the inequality $|F(n,u)| = |f(n,u) - f(n, x_0(n))| \leq L|u - x_0(n)|$ holds.*

Proof. Let the initial values $x_i^0 \in S(\varepsilon)$, $i = 1, \ldots, N$, i.e., $|e_i(0)| \leq \varepsilon$, $i = 1, 2, \ldots, N$. Then according to (5) and Remark 2.2 we get

$$|e_i(1)| \leq |F(0, x_i^0)| + \gamma \sum_{j=1}^{N} a_{ij} |e_j(0) - e_i(0)| + |1 - \gamma d_i|\, |e_i(0)| \leq (L + M)\varepsilon \leq \varepsilon,$$

i.e., $e_i(1) \in S(\varepsilon),\ i = 1, 2, \ldots, N$.

Using induction we prove $e_i(n) \in S(\varepsilon),\ n = 1, 2, \ldots, n_1$.

Then from the second equation in (5) with $k = 1$ we obtain

$$|e_i(n_1 + 1)| = |1 - B_{i,n_1}|\, |e_i(n_1)| + |F(n_1, x_i(n_1))| \leq (L + |1 - B_{i,n_1}|)\varepsilon \leq \varepsilon.$$

By induction we prove that $e_i(n) \in S(\varepsilon)$ for all $n \in \mathbb{Z}$, $i = 1, 2, \ldots, N$.

Then from (5) for any $n = 1, 2, \ldots, n_1$ we get

$$e(n) = \sum_{j=0}^{n-1} (-\gamma B)^{n-j-1} F(j, e(j)) + (-\gamma B)^n e(0), \tag{7}$$

where $e = (e_1, e_2, \ldots, e_N)$ and $F(n,e) = (F_1(n, e_1),\ F_2(n, e_2),\ \ldots,\ F_N(n, e_N))$. Therefore,

$$\|e(n)\| \leq \sum_{j=0}^{n-1} M^{n-j-1} L \|e(j)\| + M^n \|e(0)\|, \tag{8}$$

or

$$M^{-n} \|e(n)\| \leq \sum_{j=0}^{n-1} M^{-j-1} L \|e(j)\| + \|e(0)\|. \tag{9}$$

Thus, by the discrete analogue of Gronwall's inequality (see, Lemma 2.1), we get

$$\|e(n)\| \le M^n \|e(0)\| \prod_{j=0}^{n-1} (1 + M^{-1}L) \le \|e(0)\|(M+L)^n, \quad n = 1, 2, \ldots, n_1. \tag{10}$$

From the second equation in (5) with $k = 1$ we get

$$\|e(n_1+1)\| = (L + \max_{i=1,2,\ldots,N} |1 - B_{i,n_1}|)\, \|e(n_1)\| \le \|e(0)\|(L + \max_{i=1,2,\ldots,N} |1 - B_{i,n_1}|)(M+L)^{n_1}. \tag{11}$$

Similarly, for $n = n_1 + 1, n_1 + 2, \ldots, n_2$ we get

$$\begin{aligned} \|e(n)\| &\le M^{n-n_1} \|e(n_1)\| \prod_{j=n_1}^{n-1} (1 + M^{-1}L) \le \|e(n_1)\|(M+L)^{n-n_1} \\ &\le \|e(0)\|(L + \max_{i=1,2,\ldots,N} |1 - B_{i,n_1}|)(M+L)^n \ \text{ for } \ n = n_1+1, n_1+2, \ldots, n_2 - 1. \end{aligned} \tag{12}$$

From the second equation in (5) with $k = 2$ we get

$$\begin{aligned} \|e(n_2+1)\| &= (L + \max_{i=1,2,\ldots,N} |1 - B_{i,n_2}|)\, \|e(n_2)\| \\ &\le \|e(0)\|(L + \max_{i=1,2,\ldots,N} |1 - B_{i,n_2}|)(L + \max_{i=1,2,\ldots,N} |1 - B_{i,n_1}|)(M+L)^{n_2}. \end{aligned} \tag{13}$$

Applying induction and the conditions 2 and 3 of Theorem 2.1 we obtain $\lim_{n\to\infty} \|e(n)\| = 0$.

□

Theorem 2.2. *Let the conditions 2 and 3 of Theorem 2.1 be satisfied and the function* $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ *be Lipshitz with a constant L w.r.t. its second argument in* $\mathbb{R}$.

Then under control law (3) multi-agent system (1) and (2) achieves the leader-following consensus.

The proof of Theorem 2.2 is similar to the one of Theorem 2.1 and we omit it.

3. Examples

In this section we will present several examples to illustrate the effectiveness and the necessity of the obtained conditions. All examples are computer realized. The algorithms for calculating the state trajectories lead to delays in computations if CAS Wolfram Mathematica is used. Thus, by the help of the programming language C++ we obtain the values of the state trajectories in quicker way. The graphs are generated by CAS Wolfram Mathematica.

Now we will study a group of 4 followers and the leader with two interacting topologies. The first one $\mathcal{G}$ and the second one $\mathcal{F}$ determining the switching interactions with the leader at times n_k, $k = 1, 2, \ldots$, where the weighted adjacency matrix A, the diagonal matrix D, giving the leader adjacency matrix associated with $\bar{\mathcal{G}}$ and the diagonal matrix B_k, giving the leader adjacency switching matrix associated with $\bar{\mathcal{F}}$ are given by

$$A = \begin{bmatrix} 0 & 1 & 0.5 & 1 \\ 1 & 0 & 0 & 0 \\ 1.5 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} 1.9 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1.9 \end{bmatrix},$$

the intrinsic dynamics is described by $f(n, x) = 0.1 \ln(1 + x^2)$ and the constant $\gamma = 0.2$. Then $L = 0.1$ and $M \approx 0.84$, i.e., conditions 1 and 2 of Theorem 2.1 are satisfied.

We will study different case for this system illustrating the above theory.

Example 1. (*The multi agent system without impulses*). Let the initial values be $x_0^0 = 10$, $x_1^0 = 3$, $x_2^0 = 15$, $x_3^0 = 5$, $x_4^0 = 20$. Consider

$$\begin{aligned}
&x_0(n+1) = x_0(n) + 0.1 \ln(1 + x_0(n)^2) \text{ for } n = 0, 1, 2, \ldots, 30\\
&x_i(n+1) = x_i(n) + 0.1 \ln(1 + x_i(n)^2) + 0.2 \sum_{j=1}^{4} a_{ij}(x_j(n) - x_i(n))\\
&\qquad\qquad + 0.2 d_i(x_0(n) - x_i(n)) \text{ for } n = 0, 1, 2, \ldots, \quad i = 1, 2, 3, 4\\
&x_i(0) = x_i^0, \quad i = 0, 1, 2, 3, 4.
\end{aligned} \tag{14}$$

According to Theorem 2.2 the leader-following consensus is achieved. The state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ are shown in Figure 1 (discretely) and Figure 2 (continuously) and its values for $n = 1, 2, \ldots, 12$ are shown in Table 1. From Table 1 and Figures 1 and 2 it could be seen that the state trajectory $x_i(n)$ of any agent approaches the state trajectory $x_0(n)$ of the leader.

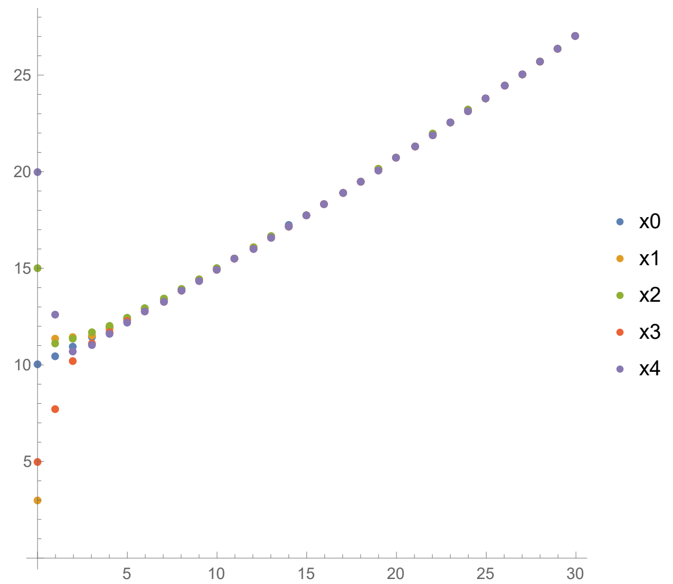

Figure 1. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (discretely).

Let's change the initial conditions, i.e., $x_0^0 = 20, x_1^0 = 13, x_2^0 = 18, x_3^0 = 15, x_4^0 = 27$. The state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ are shown in Figures 3 and 4. Again the state trajectory $x_i(n)$ of any agent approaches the state trajectory $x_0(n)$ of the leader.

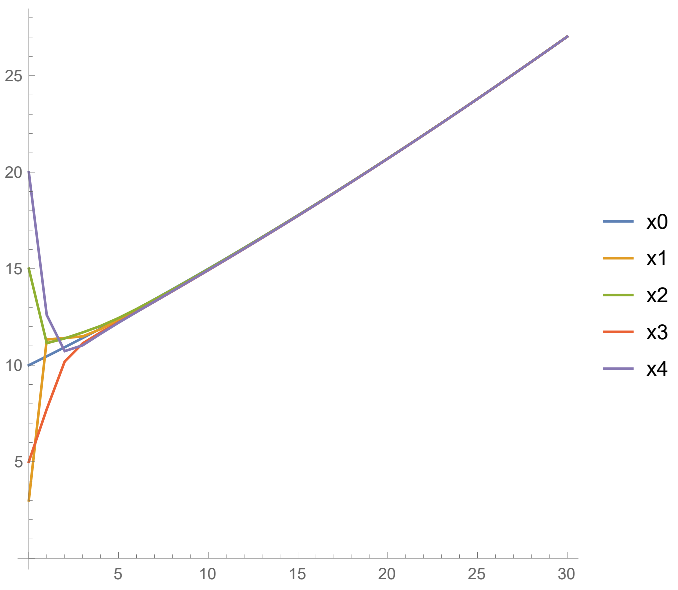

Figure 2. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

Table 1. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 12$.

n	x_0	x_1	x_2	x_3	x_4
0	10	3	15	5	20
1	10.4615	11.3303	11.1421	7.72581	12.5994
2	10.932	11.4116	11.3904	10.1924	10.7297
3	11.4111	11.4929	11.6986	11.1309	11.0307
4	11.8988	11.87	12.0351	11.7022	11.6279
5	12.3948	12.342	12.4458	12.2304	12.2032
6	12.8989	12.8429	12.9096	12.7599	12.7534
7	13.4109	13.358	13.4042	13.2934	13.2949
8	13.9307	13.883	13.9173	13.8311	13.8355
9	14.458	14.4161	14.443	14.3734	14.3787
10	14.9928	14.9565	14.9781	14.9209	14.9261
11	15.5347	15.5035	15.5214	15.4736	15.4784
12	16.0838	16.0571	16.072	16.0318	16.0361

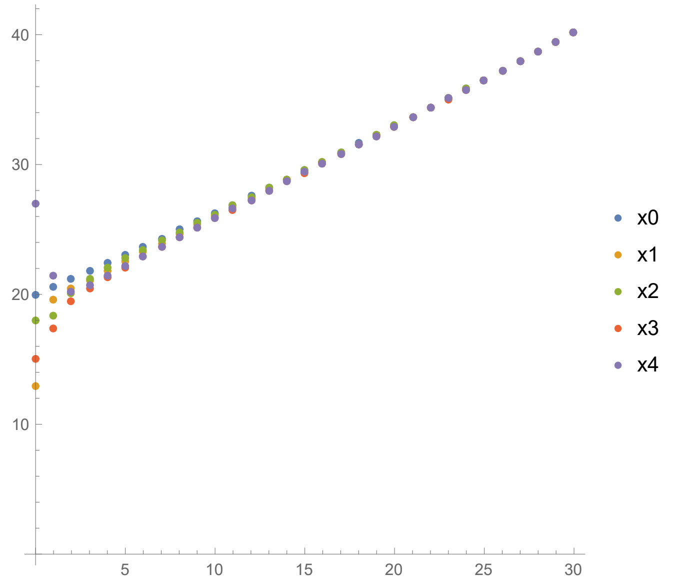

Figure 3. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (discretely).

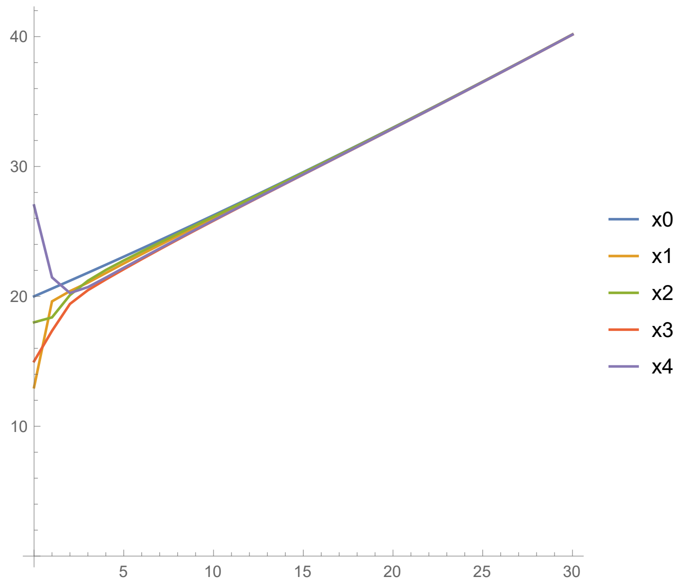

Figure 4. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

$\square$

Example 2. (*Instantaneous changes of the behavior of the followers with small jumps*). Now we will consider the case when the followers at same times change their behavior instantaneously, i.e., in a form of impulses. Let the points of impulses be $n_k = 4k - 1$, $k = 1, 2, \ldots$ and the model be

$$
\begin{aligned}
&x_0(n+1) = x_0(n) + 0.1\ln(1 + x_0(n)^2) \text{ for } n = 0, 1, 2, \dots, 30\\
&x_i(n+1) = x_i(n) + 0.1\ln(1 + x_i(n)^2) + 0.2\sum_{j=1}^{4} a_{ij}(x_j(n) - x_i(n)) + 0.2d_i(x_0(n) - x_i(n))\\
&\qquad\qquad \text{for } \ n = 0,\ 1,\ 2,\ \dots,\ n \neq 4k-1,\ k = 1,\ 2,\ \dots,\ i = 1,\ 2,\ 3,\ 4\\
&x_i(4k) = (1 - B_{i,k})x_i(4k-1) + 0.1\ln(1 + x_i(4k-1)^2) + B_{i,k}x_0(4k-1)\\
&\qquad\qquad \text{for } \ k = 1,\ 2,\ \dots,\ i = 1,\ 2,\ 3,\ 4\\
&x_i(0) = x_i^0, \quad i = 0,\ 1,\ 2,\ 3,\ 4,
\end{aligned}
\tag{15}
$$

where $B_{i,k} = 1.9$, $i = 1, 4$, $B_{i,k} = 0.9$, $i = 2, 3$, $k = 1, 2, 3, \dots$ and the initial values are $x_0^0 = 10$, $x_1^0 = 3$, $x_2^0 = 15$, $x_3^0 = 5$, $x_4^0 = 20$.

Then $L + |1 - B_{i,k}| = 1$, i.e., condition 3 of Theorem 2.1 is satisfied. According to Theorem 2.1 the leader-following consensus is achieved. The state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 30$ are shown in Figure 5 (discretely) and Figure 6 (continuously) and its values for $n = 1, 2, \dots, 12$ are shown in Table 2. From Table 2 and Figures 5 and 6 it could be seen that the state trajectory $x_i(n)$ of any agent approaches the state trajectory $x_0(n)$ of the leader.

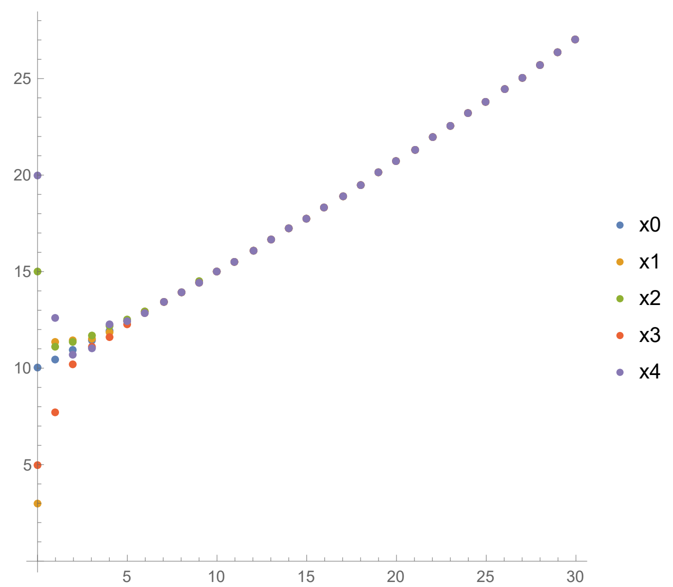

Figure 5. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 30$ (discretely).

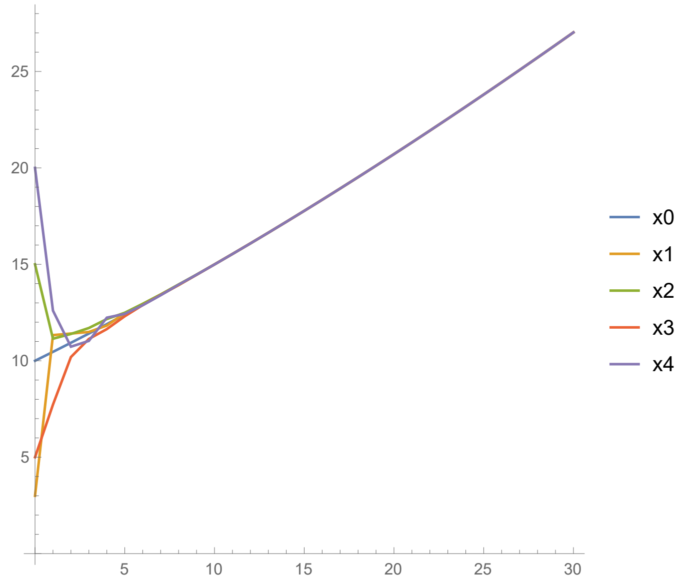

Figure 6. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

Table 2. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 12$.

n	x_0	x_1	x_2	x_3	x_4
0	10	3	15	5	20
1	10.4615	11.3303	11.1421	7.72581	12.5994
2	10.932	11.4116	11.3904	10.1924	10.7297
3	11.4111	11.4929	11.6986	11.1309	11.0307
4	11.8988	11.8267	12.1625	11.6417	12.2345
5	12.3948	12.4733	12.4902	12.3074	12.4317
6	12.8989	12.9336	12.9543	12.8847	12.8793
7	13.4109	13.4241	13.4409	13.4101	13.3971
8	13.9307	13.919	13.9581	13.93	13.9429
9	14.458	14.4634	14.4671	14.4566	14.4628
10	14.9928	14.9965	14.9976	14.9946	14.9942
11	15.5347	15.5369	15.5375	15.5371	15.536
12	16.0838	16.0818	16.0863	16.0859	16.0826

□

Example 3. (*Instantaneous changes of the behavior of the followers with at least one large jump*). Now we will consider the case when the followers at same times change their behavior instantaneously, but at least one of them has a large jump.

Consider (15) with the following initial values $x_0^0 = 10, x_1^0 = 3, x_2^0 = 15, x_3^0 = 5, x_4^0 = 20$, $B_{1,k} = 1.9$, $B_{i,k} = 0.1$, $i = 2, 3$, $k = 1, 2, 3, \ldots$ and $B_{4,k} = 4k$.

Then the condition 3 of Theorem 2.1 is not satisfied because $L + \max_{i=1,2,3,4} |1 - B_{i,k}| = 4k - 0.9 > 1$.

The state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ are shown in Figure 7 (discretely) and Figure 8 (continuously) and its values for $n = 1, 2, \ldots, 12$ are shown in Table 3. From Table 3 and Figures 7 and 8 it could be seen that the state trajectory $x_4(n)$ of the agent with large jumps does not approach the state trajectory $x_0(n)$ of the leader.

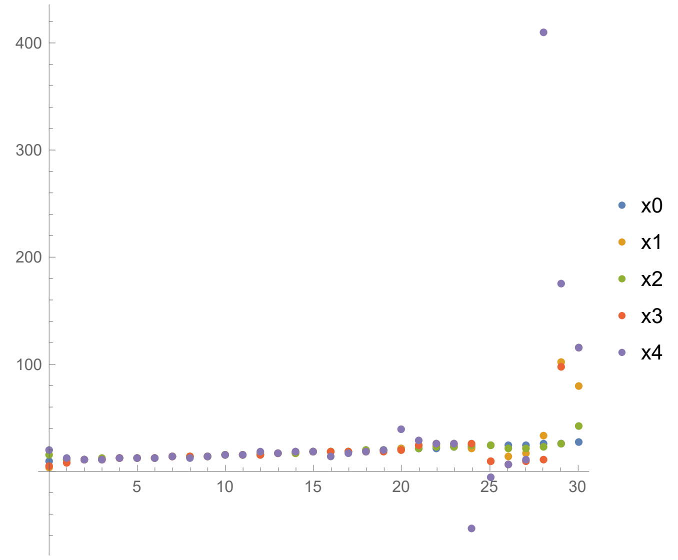

Figure 7. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (discretely).

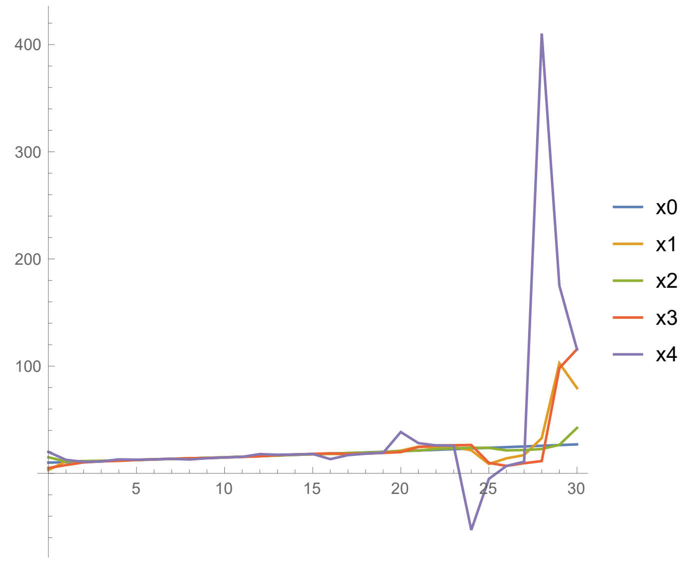

Figure 8. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

Table 3. Values of $x_i(n)$, i = 0, 1, 2, 3, 4 and n = 0, 1, 2, . . . , 12.

n	x_0	x_1	x_2	x_3	x_4
0	10	3	15	5	20
1	10.4615	11.3303	11.1421	7.72581	12.5994
2	10.932	11.4116	11.3904	10.1924	10.7297
3	11.4111	11.4929	11.6986	11.1309	11.0307
4	11.8988	11.8267	12.1625	11.6417	13.0333
5	12.3948	12.6331	12.4902	12.4672	12.7639
6	12.8989	13.0505	12.9862	13.0815	13.0813
7	13.4109	13.5158	13.4775	13.587	13.5597
8	13.9307	13.8379	13.9916	14.0918	12.8913
9	14.458	14.2586	14.4647	14.3051	14.0913
10	14.9928	14.8628	14.9556	14.781	14.7798
11	15.5347	15.4358	15.4934	15.3444	15.362
12	16.0838	16.1715	16.046	15.91	17.9818

Therefore, the condition 3 of Theorem 2.1 is a necessary condition to achieve leader-following consensus.

□

Example 4. (*Necessity of small Lipschitz constant*). Now we change the intrinsic dynamics to $f(n,x) = 0.1\frac{x^2-20x}{15}e^{\frac{-n}{4}}$. Let $x_0^0 = 10$, $C = 4$ and $U^* = [6,14]$. Then $\max\limits_{x\in U^*}\sum\limits_{n=0}^{\infty}|f(n,x)| = 0.1\max\limits_{x\in[6,14]}\left(\frac{x^2-20x}{10}\right)\sum\limits_{n=0}^{\infty}e^{\frac{-n}{4}} \leq 0.1 * \frac{100}{15} * \frac{e^{0.25}}{-1+e^{0.25}} < 4$ and according to Lemma 5 [13] $|x_0(n) - 10| \leq 4$. Now let $\varepsilon = 3.5$ and let $x \in S(3.5)$, i.e., $|x - x_0(n)| \leq 3.5$. Then $|x - 10| \leq |x - x_0(n)| + |x_0(n) - 10| \leq 7.5$ and therefore, $2\frac{x-10}{10} \in [-1,1]$ for $x \in [2.5, 17.5]$. Then $L = 0.1$.

Now, consider the initial values such that $|x_i^0 - 10| \leq 3.5$, i.e., $x_0^0 = 10$, $x_1^0 = 3$, $x_2^0 = 15$, $x_3^0 = 5$ and $x_4^0 = 17$.

The state trajectories $x_i(n)$, i = 0, 1, 2, 3, 4 and n = 0, 1, 2, . . . , 30 are shown in Figure 9 (discretely) and Figure 10 (continuously) and its values for n = 1, 2, . . . , 12 are shown in Table 4. From Table 4 and Figures 9 and 10 it could be seen that the state trajectories $x_i(n)$, i = 1, 2, 3, 4 approach the state trajectory $x_0(n)$ of the leader, i.e., the local leader consensus is achieved and $x_i(n) \in S(3.5)$ for i = 1, 2, 3, 4 and $n \in \mathbb{Z}$.

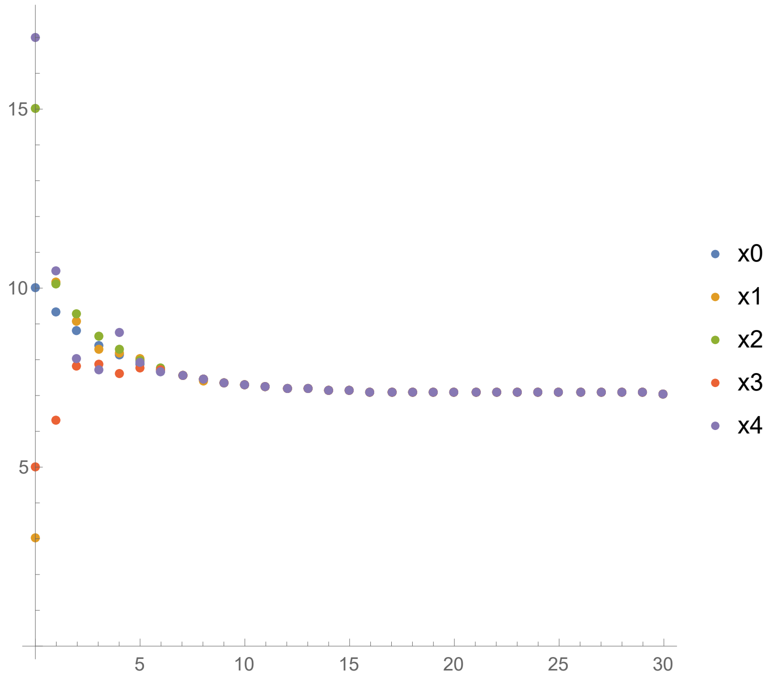

Figure 9. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (discretely).

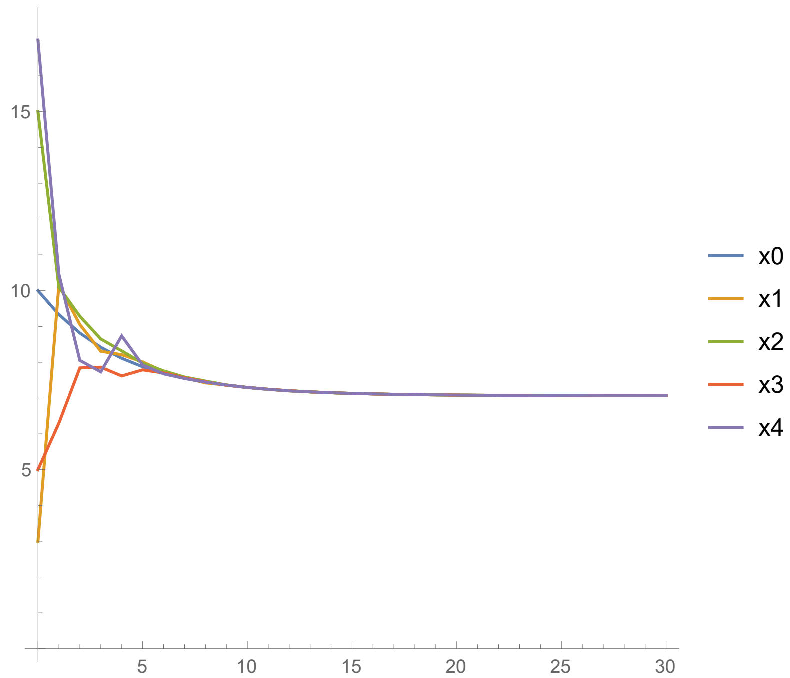

Figure 10. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

Table 4. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 12$.

n	x_0	x_1	x_2	x_3	x_4
0	10	3	15	5	17
1	9.33333	10.16	10.1	6.3	10.46
2	8.81644	9.05493	9.28618	7.84188	8.05256
3	8.41775	8.30711	8.64974	7.86241	7.73205
4	8.11072	8.21144	8.31737	7.61742	8.73617
5	7.87422	8.01055	7.97522	7.78805	7.92225
6	7.69185	7.73923	7.75871	7.69998	7.6762
7	7.55102	7.57123	7.58679	7.56612	7.54812
8	7.44212	7.42382	7.47411	7.45562	7.44475
9	7.3578	7.3625	7.36679	7.35952	7.36424
10	7.29244	7.29662	7.29694	7.29599	7.29568
11	7.24173	7.24446	7.24435	7.2454	7.24444
12	7.20235	7.19989	7.20471	7.20565	7.19991

Note that for $x \notin [2.5, 17.5]$ the function is locally Lipschitz but the Lipschitz constant does not satisfy condition 2 of Theorem 2.1. Let, for example, consider the following initial values $x_0^0 = 10, x_1^0 = 1, x_2^0 = 30, x_3^0 = 40, x_4^0 = 50$.

The state trajectories $x_i(n), i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 30$ are shown in Figure 11 (discretely) and Figure 12 (continuously) and its values for $n = 1, 2, \dots, 12$ are shown in Table 5. From Table 5 and Figures 11 and 12 it could be seen that the state trajectories $x_i(n)$, $i = 1, 2, 3, 4$ are in the enough small tube around the leader only for large values of time.

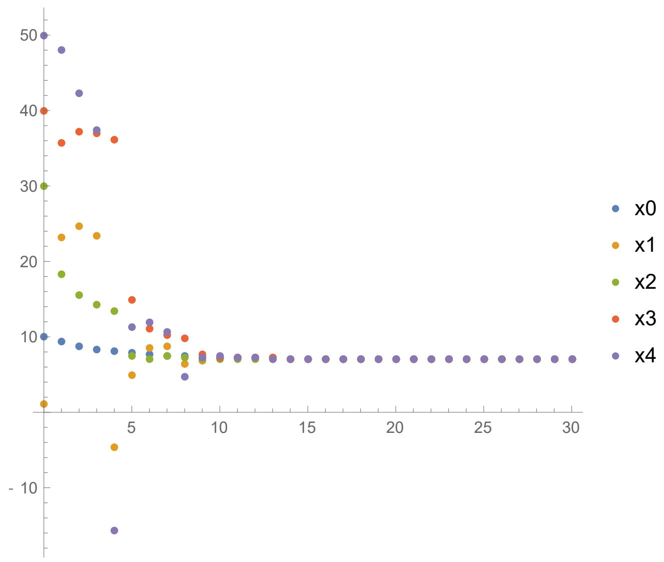

Figure 11. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \dots, 30$ (discretely).

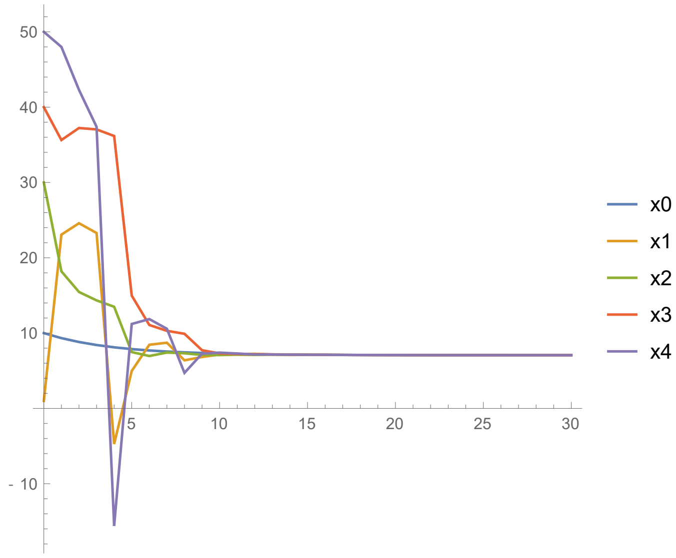

Figure 12. Graph of the state trajectories $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 30$ (continuously).

Table 5. Values of $x_i(n)$, $i = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, \ldots, 12$.

n	x_0	x_1	x_2	x_3	x_4
0	10	1	30	40	50
1	9.33333	23.0733	18.2	35.6333	48
2	8.81644	24.5862	15.4579	37.231	42.2981
3	8.41775	23.2924	14.3431	37.045	37.3886
4	8.11072	-4.72793	13.495	36.1707	-15.6087
5	7.87422	4.9687	7.48142	14.9797	11.2101
6	7.69185	8.44963	6.95711	11.0789	11.8626
7	7.55102	8.72412	7.41453	10.2998	10.5713
8	7.44212	6.38127	7.32008	9.90918	4.71726
9	7.3578	6.82887	7.09739	7.72221	7.27396
10	7.29244	7.15641	7.0835	7.29794	7.40498
11	7.24173	7.19621	7.13159	7.22616	7.28862
12	7.20235	7.24343	7.10349	7.18838	7.16005

□

4. Conclusion

A discrete model of a multi-agent system with a virtual leader, whose motion is independent of all the other agents, is studied. It is modeled the case when at initially known time-points the interactions between multi-agents are changed instantaneously. We consider the case of two interacting topologies, one is determining the interactions between the agents including the leader, the second one is determining the instantaneous switching interactions of the agents with the leader. Several sufficient conditions ensuring both local and global leader-following consensus are obtained. These results are illustrated on particular examples by intensive application of computer simulation. The influence of the impulses on the discrete leader-following consensus is shown and the necessity of some of the obtained conditions is illustrated.

Acknowledgments

K. Stefanova is supported by National Program "Young Scientists and Postdoctoral Candidates" of Ministry of Education and Science, Bulgaria.

Conflict of interest

The authors declare no conflict of interest.

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