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Research article

A total variable-order variation model for image denoising

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Abstract: In this paper, we explore a new variational model based on the fractional derivative and total variation. Due to some metrics, our approach shows great results compared to other competitive models. In particular, deleting the noise and preserving edges, features and corners are headlights to our approach. For the fractional variable-order derivatives, different discretizations were presented to comparison. The theoretical results are validated by the Primal Dual Projected Gradient (PDPG) Algorithm which is well adapted to the fractional calculus.

Keywords: fractional derivative; total variation; image denoising; primal dual; finite difference
Mathematics Subject Classification: 26A33, 94A08

1. Introduction

1.1. Image denoising general formulation and some related works

In this paper, we present a variational model based on the fractional variable-order as a regularizer for image denoising. It could be also applied in various domain such as image restoration [1,4], image registration [9,15] and image inpainting [7,10,11]...

The general well-posed problem can be written as follow

$$\min_{u} \underbrace{\|u-u_0\|^2_{L^2(\Omega)}}_{fidelity} + \underbrace{\lambda\, R(u),}_{Regularization}$$

where: $u = u(x)$ is the observed image with $x \in \Omega \subset \mathbb{R}^2$, u_0 is the initial image, Ω is the bounded domain of the image, and λ is a regularization term.

In the literature, many regularization terms have been proposed. The well known term is the Total Variation (TV) first discovered by Rudin, Osher and Fatemi (ROF) [18]. This model can preserve the image edges by looking for solutions of piecewise constant functions in the space of bounded variation

(BV). However, it has many drawbacks. It can yield the so-called staircasing phenomenon which is the appearance of virtual edges. In general, the TV model has many other disadvantages including the loss of image contrast.

The main interest in our paper is the fractional derivative. Authors in this domain use PDE [2, 21] or variational optimization in image denoising [4, 17]. For the first type, a higher order-derivative is used or we implement the fractional derivative. For example, Bai and Feng [3] proposed an anisotropic diffusion equations which has the following form

$$\frac{\partial u}{\partial t} = -D_x^{\alpha *}(g(|D^\alpha u|^2)D^\alpha u) - D_y^{\alpha *}(g(|D^\alpha u|^2)D^\alpha u), \tag{1}$$

where α is the order of derivation, $g(.)$ is diffusivity function and $D_x^{\alpha *}$ is the adjoint operator of D_x^α.

Another example is the work of [22]. Authors used the fractional derivative in the sens of Caputo to replace the first order derivative over time.

The latest models utilize a regularization term based on the TV model. For instance, J. Zhang and K. Chen [21] proposed a total fractional-order variation TV^α based on the following model

$$\min_u \left\{E(u) = TV^\alpha(u) + \frac{\lambda}{2}\|u - u_0\|^2_{L^2(\Omega)}\right\}, \tag{2}$$

where α is a constant-order derivative. We shall discuss the regularization term TV^α in the next section.

Another interesting variational model for noise removal is the work of F. Dong and Y. Chen [8]. They used different orders of fractional derivatives in the regularization term.

The total variation model (TV) has made an impact in image processing despite its weaknesses. Fortunately, many models have been developed further on to deal with these flaws. In particular, the work of A. Chambolle and P. Lions in [5] have brought an important insight to these challenges. The authors have considered an inf-convolution functional

$$\min_{u=u_1+u_2}\left\{\frac{1}{2}\|u_0 - u_1 - u_2\|^2_{L^2(\Omega)} + \alpha\int_\Omega |\nabla u_1|\,dx + \beta\int_\Omega \left|\nabla^2 u_2\right| dx\right\},$$

using two convex regularizes where $u_0 = u_1 + u_2 + \mu_0$. The term u_1 is the piecewise constant part of u_0, u_2 the piecewise smooth part and μ_0 is the Gaussian noise. They proved that this model is practically efficient against the staircasing effect and more importantly preserves the image features.

An enhancement of the late model was an idea by [6] which states the replacement of the regularization terms by the Laplacian operator,

$$\min_{u=u_1+u_2}\left\{\frac{1}{2}\|u_0 - u_1 - u_2\|^2_{L^2(\Omega)} + \alpha\int_\Omega |\nabla u_1|\,dx + \beta\int_\Omega |\Delta u_2|\,dx\right\}.$$

Another different approach is suggested in [17]. It concerns the study of a regularized functional by the total variation of the $(l-1)$-th derivative of u. The model reads

$$\min_u\left\{\frac{1}{2}\|u_0 - u\|^2_{L^2(\Omega)} + \alpha\int_\Omega \left|D\nabla u^{l-1} u\right| dx\right\}.$$

1.2. Our contribution

In the literature there exists plenty definitions of the fractional derivative. Yet, in a way they all intersect. Fractional derivative is without a doubt a useful tool in modeling, especially when it comes to image and signal processing, control systems, and complex crowd motion. In our case, we are interested in image denoising. A big part of the existing models including the mentioned above are using an edge detector. That is to identify the edges to delete the noise in the homogeneous areas of the image.

To illustrate, the Figure 1 shows the difference between the extraction of edges using the first derivative and the fractional derivative where $\alpha = 1.5$. We can see clearly that the fractional derivative extracted edges and characteristics are more precise.

Original image	first derivative	fractional α-order derivative

Figure 1. edge detection using the first order derivative and the fractional α-order derivative with α=1.5.

Our concern in this work is to build an approach similar to the model (2). However, our contribution consists of using a **variable**-order fractional derivative. In particular, the order α shall depend on each local position of pixels, i.e. $\alpha(|x|)$ is a space dependent functional where $x \in \mathbb{R}^2$. Notice then that the derivative of the image changes value as the coordinate of a pixel x changes. We note that in previous works, the choice of a constant α have always been numerical. It is exactly the same case for our model where $\alpha(|x|)$ is settled in the interval $[1, 2]$. We will discuss more details in further sections.

The structure of this paper is presented as follows. In Section 2, we present the proposed model and introduce the fractional derivative famous formulas. Both constant and variable-order cases will be presented for comparison. In Section 3, using finite differences method we approach the fractional-order derivatives discretizations with different ways and we write the PDPG algorithm. In Section 4, we illustrate our results with few numerical simulations. In the last Section 5, we bring up a conclusion.

2. The total variable-order variation model

Before we introduce our model, we need to put some base ground first, starting with the next subsection.

2.1. Fractional variable order derivatives

For simplicity reasons, we will apply the fractional derivative at a function f in one-dimensional case, i.e. a domain $[a, b] \subset \mathbb{R}$, with an order α in $\mathbb{R}^+$.

We are going to put under consideration the following definition on the fractional constant-order Grunwald-Letnikov type derivative (see [16] for more):

Definition 2.1. *Fractional constant order derivative is defined as follows:*

$$ {}_{a}^{G}D_{x}^{\alpha}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{[\frac{x-a}{h}]}(-1)^{j}\binom{\alpha}{j}f(x-jh), $$

Where :

$$ \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}. $$

This definition present a fractional constant-order derivative on the left of x, which is usually used in the literature. Now, in the case of a variable-order derivative, different types of definitions can be found, see [19].

We will start with a definition of Variable-, fractional-order backward difference (VFOBD) introduced by P. Ostalczyk in [14].

Definition 2.2. *One defines a VFOBD of a discrete function f_l as a discrete convolution of a function f_l:*

$$ {}_{0}\Delta_{l}^{\alpha_l}f_l = a_l^{\alpha_l} * f_l $$

with a discrete function:

$$ a_l^{\alpha_l} = \begin{cases} 1 & \textit{for } l=0 \\ (-1)^l\frac{\alpha_l(\alpha_l-1)\dots(\alpha_l-l+1)}{l!} & \textit{for } l=1,2,3,\dots \end{cases} \tag{3} $$

where the term α_l means the value of a bounded order function.

Now, we move to the next definitions of recursive fractional derivatives.

Definition 2.3. *The $\mathcal{B}$-type fractional variable order derivative is defined as follows*

$$ {}_{a}^{\mathcal{B}}D_{x}^{\alpha(x)}f(x) = \lim_{h\to 0}\sum_{j=0}^{[\frac{x-a}{h}]}\frac{(-1)^{j}}{h^{\alpha(x-jh)}}\binom{\alpha(x-jh)}{j}f(x-jh) $$

The discrete form, of the above B-type fractional variable order definition, is the following:

$$ {}^{\mathcal{B}}\Delta^{\alpha_l}f_l = \sum_{j=0}^{l}\frac{(-1)^j}{h^{\alpha_{l-j}}}\binom{\alpha_{l-j}}{j}f_{l-j} $$

Where: $l = 0, 1, ..., N$, and f_l is the l-line of the discrete function f with the length N.
The $\mathcal{B}$-type of definition assumes that those coefficients for past samples are obtained for order that was present for these samples.

An other alternative definition [13], which owns a recursive nature, was obtained from the definition above, has the following form:

Definition 2.4. *The $\mathcal{E}$-type fractional variable order derivative is defined as follows*

$$ {}^{\mathcal{E}}_{a}D^{\alpha(x)}_{x} f(x) = \lim_{h\to 0}\left(\frac{f(x)}{h^{\alpha(x)}} - \sum_{j=1}^{[\frac{x-a}{h}]}(-1)^j \binom{-\alpha(x-jh)}{j}\frac{h^{\alpha(x-jh)}}{h^{\alpha(x)}}\, {}^{\mathcal{E}}_{a}D^{\alpha(x)}_{x-jh} f(x)\right) $$

Like the $\mathcal{B}$-type definition, we present the discrete form of $\mathcal{E}$-type fractional variable order derivative, which has the following expression:

$$ {}^{\mathcal{E}}\Delta^{\alpha_l} f_l = \frac{f_l}{h^{\alpha_l}} - \sum_{j=1}^{l}(-1)^j\binom{-\alpha_{l-j}}{j}\frac{h^{\alpha_{l-j}}}{h^{\alpha_l}}\ {}^{\mathcal{E}}\Delta^{\alpha_{l-j}} f_{l-j} \tag{4} $$

Where: $l = 1, ..., N$.

Remark 2.5. *For a constant order $\alpha(x) = const$, we get the same results as for constant order derivative and difference definitions, that is [12]:*

$$ {}^{G}_{a}D^{\alpha}_{x} f(x) \ = \ {}^{\mathcal{B}}_{a}D^{\alpha}_{x} f(x) \ = \ {}^{\mathcal{E}}_{a}D^{\alpha}_{x} f(x) $$

For the purpose of numerical calculations, we will attempt to present some interesting properties of the precedent definitions starting with the theorem 2.6, before that the $\mathcal{E}$-type derivative is equivalent to the switching scheme given in Figure 2.

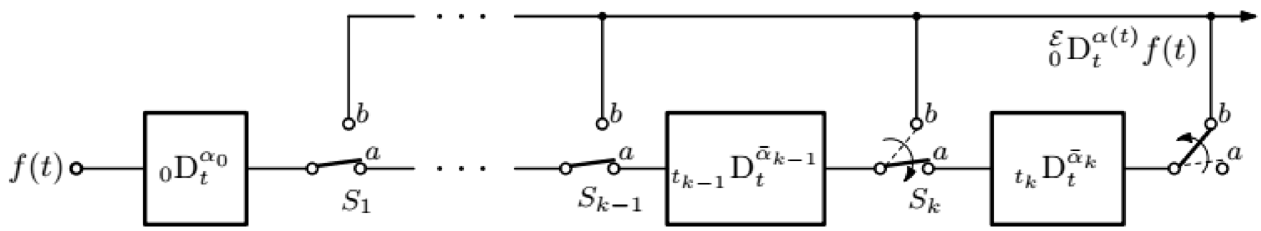


Figure 2. Realization of $\mathcal{E}$-type derivative in the form of switching scheme, where $\bar{\alpha}_j = \alpha_j - \alpha_{j-1}$, $j = 1, ..., N$, (configuration at $t = t_k$)

Theorem 2.6. *The $\mathcal{E}$-type fractional difference given by* (4) *can be expressed in the following matrix form:*

$$ \begin{pmatrix} {}^{\mathcal{E}}\Delta^{\alpha_0} f_0 \\ {}^{\mathcal{E}}\Delta^{\alpha_1} f_1 \\ \vdots \\ \vdots \\ {}^{\mathcal{E}}\Delta^{\alpha_N} f_N \end{pmatrix} = \mathcal{D}_0^N \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_N \end{pmatrix} $$

where:

$$ \mathcal{D}_0^N = \left(\begin{array}{c|c|c|c|c|c} h^{-\alpha_0} & 0 & 0 & \dots & 0 & 0 \\ \hline \mathfrak{q}_{2,1} & h^{-\alpha_1} & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline \mathfrak{q}_{N,1} & \mathfrak{q}_{N,2} & \mathfrak{q}_{N,3} & \dots & h^{-\alpha_{N-1}} & 0 \\ \hline \mathfrak{q}_{N+1,1} & \mathfrak{q}_{N+1,2} & \mathfrak{q}_{N+1,3} & \dots & \mathfrak{q}_{N+1,N} & h^{-\alpha_N} \end{array}\right) \tag{5} $$

where,

$$\mathfrak{q}_{i,j} = \begin{cases} q_{i-1}(\mathfrak{q}_{1,j}, \ldots, \mathfrak{q}_{i-1,j})^T & \text{if } i > j \\ h^{-\alpha_i} & \text{if } i = j \\ 0 & \text{if } i < j \end{cases} \tag{6}$$

for $i, j = 1, \ldots, N+1$.
and for $r = 1, \ldots, N$,

$$q_r = -(v_{-\alpha_0,r}\ ,\ v_{-\alpha_1,(r-1)}, \ldots,\ v_{-\alpha_{r-1},1}) \in \mathbb{R}^{1\times r}, \tag{7}$$

$$v_{-\alpha_{r-p},p} = (-1)^p \binom{-\alpha_{r-p}}{p} \frac{h^{\alpha_{r-p}}}{h^{-\alpha_r}},\ p = 1, \ldots, r. \tag{8}$$

To simplify more, the m-element of q_r, $m = 1, \ldots, r$.*:*

$$(q_r)_m = -v_{-\alpha_{m-1},r-m+1} = (-1)^{r-m+1} \binom{-\alpha_{m-1}}{r-m+1}$$

Proof. For the detailed proof, see [12] . □

To a more explicit form of (5), we have the following lemma:

Lemma 2.7. *The following holds*

$$\mathcal{D}_0^N = \mathcal{D}(\alpha_N, N) \ldots \mathcal{D}(\alpha_1, 1)\mathcal{D}(\alpha_0, 0), \tag{9}$$

Where, for $r = 0, 1, \ldots, N$

$$\mathcal{D}(\alpha_r, r) = \left(\begin{array}{c|c|c} I_{r,r} & 0_{r,1} & 0_{r,N-r} \\ \hline q_r & h^{-\alpha_r} & 0_{1,N-r} \\ \hline 0_{N-r,r} & 0_{N-r,1} & I_{N-r,N-r} \end{array}\right)$$

where q_r *is given by* (7), $0_{m,n}$ *and* $I_{m,n} \in \mathbb{R}^{n\times m}$ *stand, respectively, for zero and identity matrices.*

Proof. See [12] for the proof. □

Remark 2.8. *Generally, we have following relations*

$$^{\mathcal{B}}\Delta^{\pm\bar{\alpha}}(^{\mathcal{B}}\Delta^{\mp\bar{\alpha}} x_k) \neq x_k, \quad ^{\mathcal{B}}\Delta^{\pm\alpha(x)}(^{\mathcal{B}}\Delta^{\mp\alpha(x)} f(x)) \neq f(x)$$

and

$$^{\mathcal{E}}\Delta^{\pm\bar{\alpha}}(^{\mathcal{E}}\Delta^{\mp\bar{\alpha}} x_k) \neq x_k, \quad ^{\mathcal{E}}\Delta^{\pm\alpha(x)}(^{\mathcal{E}}\Delta^{\mp\alpha(x)} f(x)) \neq f(x)$$

where $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_N)$*, which means that, for a variable order derivative, the semi-group property does not hold. Otherwise, for a constant-order where, respectively,* $\alpha_0 = \cdots = \alpha_N = cons$ *and* $\alpha(x)$*=const, this property holds.*

Between the two above types of definitions, there exists a property, called duality [19, 20], which is given as follows:

$$^{\mathcal{B}}\Delta^{\pm\bar{\alpha}}(^{\mathcal{E}}\Delta^{\mp\bar{\alpha}}x_k) = x_k, \quad ^{\mathcal{B}}\Delta^{\pm\alpha(x)}(^{\mathcal{E}}\Delta^{\mp\alpha(x)}f(x)) = f(x)$$

and

$$^{\mathcal{E}}\Delta^{\pm\bar{\alpha}}(^{\mathcal{B}}\Delta^{\mp\bar{\alpha}}x_k) = x_k, \quad ^{\mathcal{E}}\Delta^{\pm\alpha(x)}(^{\mathcal{B}}\Delta^{\mp\alpha(x)}f(x)) = f(x)$$

2.2. The proposed model

We propose a variational model based on the variable-order derivative, similar to (2). our proposed model has the following form:

$$\min_u \left\{ E(u) = TV^{\hat{\alpha}}(u) + \frac{\lambda}{2}\|u - u_0\|_2^2 \right\} \tag{10}$$

where, $u : \Omega \longrightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, $u(x)$ is the value of the grayscale at the pixel $x \in \Omega$, the regularization parameter $\lambda > 0$ and $1 < \hat{\alpha} < 2$.

The function $\hat{\alpha}$ above depends, in our model, on the norm of the pixel x, i.e. $\hat{\alpha} = \alpha(|x|)$.

Where:

$$\begin{aligned} \alpha : \mathbb{R}^+ &\longrightarrow [1, 2] \\ s &\longmapsto \alpha(s) \end{aligned}$$

The value of $\hat{\alpha}$ has been chosen numerically, in our simulation, we took $\alpha(|x|) = 1 + e^{-\left(\frac{|x|}{k}\right)^2}$, where k a positive constant, which is also chosen numerically by using the PSNR and the SSIM.

The idea behind this choice is that:

(i) In homogeneous regions, the derivation order $\hat{\alpha}$ will be close to 2, which reduces the staircasing effect.
(ii) In nonhomogeneous regions, $\hat{\alpha}$ will be closer to 1, which is exactly the famous model TV, proposed by Rudin, Osher and Fatemi in [18], it is known that this particular model preserve edges, features and corners.

In order to define the semi-norm $TV^{\hat{\alpha}}$, we first need to consider the following definition of space of test functions:

Definition 2.9. *Let $C^1(\Omega, \mathbb{R}^2)$ denote the space of continuously differentiable functions of order one. Furthermore, for any $C^1(\Omega, \mathbb{R}^2) \ni v : \Omega \longmapsto \mathbb{R}^2$ if the 2^{nd} order derivative $v^{(2)}$ is integrable and $\frac{\partial^i v(x)}{\partial n^i}|_{\partial\Omega} = 0$ for $i = 0, 1$, then v is a compactly supported continuous-integrable function in Ω. Therefore, the one-compactly supported continuous-integrable function space is denoted by $\mathscr{C}_0^1(\Omega, \mathbb{R}^2)$.*

Now, we can introduce $TV^{\bar{\alpha}}$ in the next definition.

Definition 2.10. *(total α-order variation)*
Let K denote the space of special test functions :

$$K = \left\{ \phi \in \mathscr{C}_0^1(\Omega, \mathbb{R}^2), \ where \ |\phi(x)| \leq 1, \ for \ all \ x \in \Omega \right\},$$

where $|\phi(x)| = \sqrt{\sum_{i=1}^{2} \phi_i^2(x)}$, *Then the total* α*-order variation of* u *is defined by:*

$$TV^{\hat{\alpha}} = \int_{\Omega} \left|D^{\hat{\alpha}} u\right| = \sup_{\phi \in K} \int_{\Omega} \Big(- u \, div^{\hat{\alpha}} \phi \Big) dx,$$

where $div^{\hat{\alpha}}\phi = \sum_{i=1}^{2} \frac{\partial^{\hat{\alpha}} \phi_i}{\partial x_i^{\hat{\alpha}}}$ *and* $\frac{\partial^{\alpha} \phi_i}{\partial x_i^{\alpha}}$ *denotes a fractional* α*-order derivative of* ϕ_i *along the* x_i *direction.*

Remark 2.11. *The existence and uniqueness of the problem* (2)*, where* $\hat{\alpha}$ *is constant, can be found in [21].*

3. Discretization and the proposed algorithm

3.1. Discretization of the fractional derivative

We present a finite difference discretization of the fractional derivative. Of course, we start with a spatial partition of image domain Ω (x_k, y_l), for all k=0,1,...,N+1 and l=0,...,M+1. We discretized the α-order fractional derivative on Ω along the x-direction at the inner point (x_k, y_l), for $k = 1, ..., N$ and $l = 1, ..., M$; because of the boundary condition.

For comparability reasons, we first start with the discretization of fractional derivative, where $\hat{\alpha}$ is constant, by using the approach :

$$\begin{aligned} D^{\alpha}_{[a,x]} u(x_k, y_l) &= \frac{\delta^{\alpha} u(x_k, y_l)}{h^{\alpha}} + O(h) \\ &= \frac{1}{2}\Big(\frac{\delta^{\alpha}_{-} u(x_k, y_l)}{h^{\alpha}} + \frac{\delta^{\alpha}_{+} u(x_k, y_l)}{h^{\alpha}} + O(h) \Big) \\ &= \frac{1}{2}\Big(h^{-\alpha} \sum_{j=0}^{k+1} \omega_j^{\alpha} u^l_{k-j+1} \\ &\quad + h^{-\alpha} \sum_{j=0}^{N-k+2} \omega_j^{\alpha} u^l_{k+j-1} \Big) + O(h) \end{aligned} \tag{11}$$

where $u^l_s = u(x_s, y_l)$, $\omega_j^{\alpha} = (-1)^j \binom{\alpha}{j}$, $j = 0, 1, ..., N+1$, with the fact that ω_j^{α} is a recursive sequence, defined with the following form:

$$\omega_0^{\alpha} = 1, \;\; \omega_j^{\alpha} = (1 - \frac{1+\alpha}{j}) \omega_{j-1}^{\alpha}, \;\; for \;\; j > 0$$

Combining the zero boundary condition with the matrix approximation of fractional derivative, (11) can be written in the matrix approach along the x direction :

$$\begin{pmatrix} \delta^{\alpha} u(x_1, y_l) \\ \delta^{\alpha} u(x_2, y_l) \\ \vdots \\ \vdots \\ \delta^{\alpha} u(x_N, y_l) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\omega_1^{\alpha} & \omega & \omega_3^{\alpha} & \dots & \omega_N^{\alpha} \\ \omega & 2\omega_1^{\alpha} & \ddots & \ddots & \vdots \\ \omega_3^{\alpha} & \ddots & \ddots & \ddots & \omega_3^{\alpha} \\ \vdots & \ddots & \ddots & 2\omega_1^{\alpha} & \omega \\ \omega_N^{\alpha} & \dots & \omega_3^{\alpha} & \omega & 2\omega_1^{\alpha} \end{pmatrix} \begin{pmatrix} u^l_1 \\ u^l_2 \\ \vdots \\ \vdots \\ u^l_N \end{pmatrix}$$

where $\omega = \omega_0^\alpha + \omega_2^\alpha$
It is the same procedure on Ω along the y-directions.

Before we present the matrix approach of the variable-order derivative, we start to the discretization of $\alpha(|x|)$:
For $k = 1, ..., N$ and $l = 1, ..., M$

$$\alpha_k^l = \alpha(|x(k,l)|) = \alpha(|x_k| + |x_l|)$$

and finally, the matrix approach of the variable-order derivative along the x-direction has the following form:

$$\begin{pmatrix} {}^{\mathcal{E}}\Delta^{\alpha_1^l}u(x_1,y_l) \\ {}^{\mathcal{E}}\Delta^{\alpha_2^l}u(x_2,y_l) \\ \vdots \\ \vdots \\ {}^{\mathcal{E}}\Delta^{\alpha_N^l}u(x_N,y_l) \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c} h^{-\alpha_1^l} & 0 & 0 & \dots & 0 & 0 \\ \hline \mathrm{q}_{2,1} & h^{-\alpha_2^l} & 0 & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline \mathrm{q}_{N-1,1} & \mathrm{q}_{N-1,2} & \mathrm{q}_{N-1,3} & \dots & h^{-\alpha_{N-1}^l} & 0 \\ \hline \mathrm{q}_{N,1} & \mathrm{q}_{N,2} & \mathrm{q}_{N,3} & \dots & \mathrm{q}_{N,N-1} & h^{-\alpha_N^l} \end{array}\right) \begin{pmatrix} u_1^l \\ u_2^l \\ \vdots \\ \vdots \\ u_N^l \end{pmatrix}$$

where q_r is given by (6).
On the other hand, where the order of derivation is non-constant, using the lemma (2.7) in the previous section, and exactly the form (9), that simplifies the matrix approach of the desired derivative.

3.2. PDPG algorithm

The Primal Dual Projected Gradient (PDPG) algorithm is a method based on the Primal Dual approach that transforms the variational model into a saddle point problem, by using the Legendre-Fenchel transform. After that, many methods to resolve the saddle point problem have been proposed in the literature, but in our case we used the projected gradient (or subgradient and supergradient). For more details, we refer the interested reader to, e.g., [23].

4. Numerical results and analysis

In this section we will test the validity of our model (10) in the visual context which is based on the conservation of edges and features of the initial image, and also the elimination of staircasing effect. Using the Primal Dual Projected Gradient (PDPG) algorithm for the numerical simulation. The best variable-order $\hat{\alpha}$ derivative expression has been chosen numerically. In Figure 3 we can see that the reliable choice lays where PSNR and SSIM values are bigger. Thus, the best choice is $\hat{\alpha}_1$.
We took in the Figure 3, $\alpha_1 = 1 + \exp(\frac{-|x|^2}{k})$, $\alpha_2 = 1 + \frac{1}{\sqrt{\frac{|x|^2}{k}+1}}$ and $\alpha_3 = 1 + \frac{1}{\frac{|x|^2}{k}+1}$

4.1. Denoising results

In this subsection, we delete the noise from four different images using the Primal Dual Projected Gradient (PDPG) Algorithm, with the usual TV and the fractional TV as the competitive models, with our approach $TV^{\hat{\alpha}}$ with the two discretization $\mathcal{E}$-type and VFOBD,

Algorithm 1 $TV^{\hat{\alpha}}$ fractional denoising model, done by PDPG algorithm :

Inputs:

λ (regularization parameter)

$\hat{\alpha}$ (fractional variable-order derivative)

r_1, r_2 (fixed-point parameters)

u_0 (initial image)

ϕ^0 (the initial dual variable)

Initialize:

$u^0 = u_0$

$\phi^0 = 0$

repeat: for $k > 0$ $\begin{cases} \bar{\phi}^{k+1} & \leftarrow \phi^k - r_1 \nabla^{\hat{\alpha}} u^k \\ \phi^{k+1} & \leftarrow \dfrac{\bar{\phi}^{k+1}}{max(|\bar{\phi}^{k+1}|, 1)} \\ u^{k+1} & \leftarrow (1 - \lambda r_2) u^k - r_2 (div^{\hat{\alpha}} \phi^{k+1} - \lambda u_0) \end{cases}$

until convergence

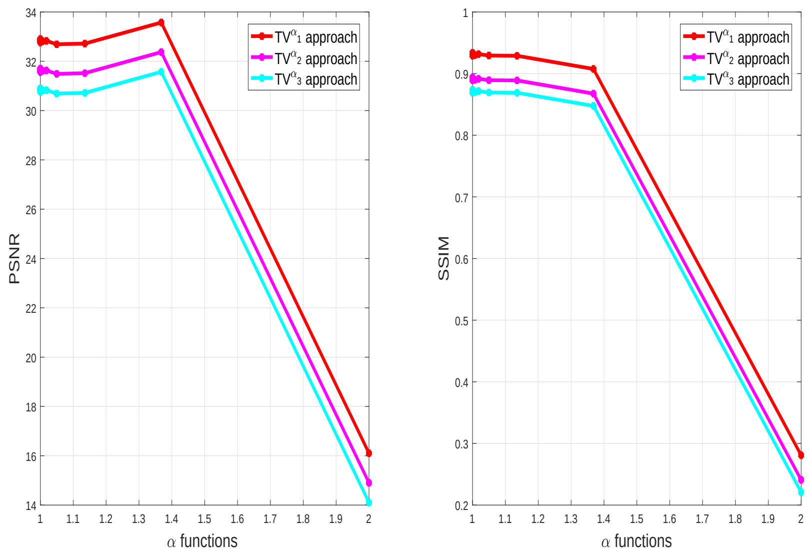
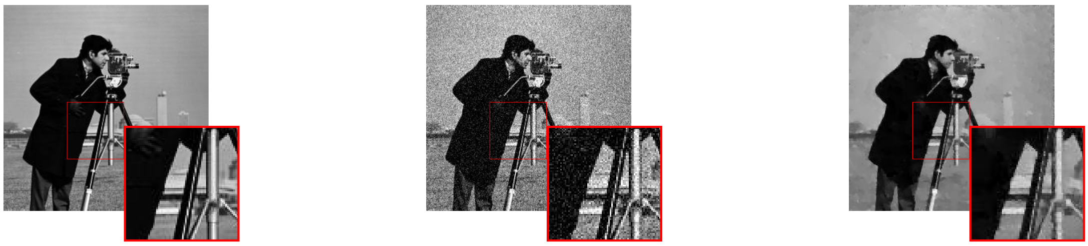


Figure 3. 3 different variable-order derivatives PSNR(α_i) and SSIM(α_i), i=1,2 and 3.

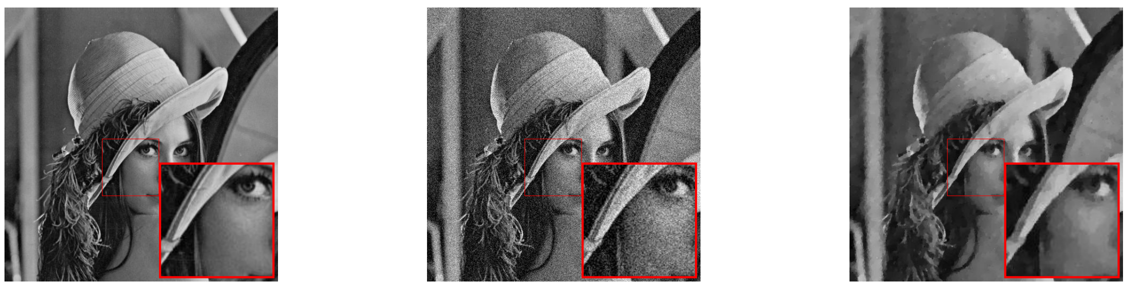
In Figures 4, 5, 6 and 7, we can see clearly the suppression of noise in all of the images. Fortunately for us, our model can delete noise while preserving corners and features like in camera's leg, Lena's cheek or in any other detail presented in the four images.



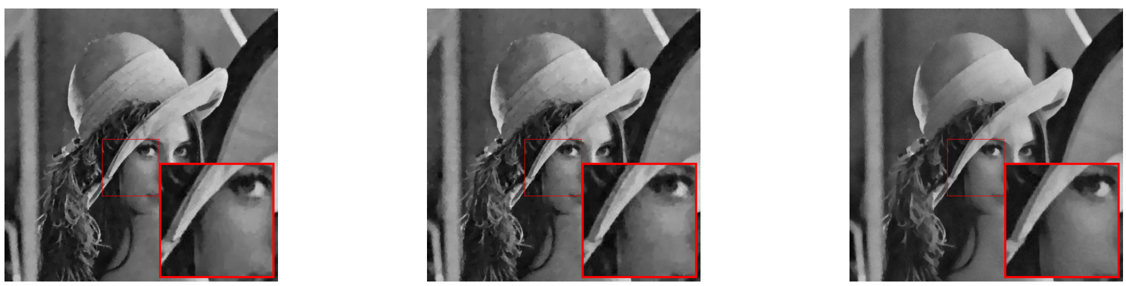
(a) Original image **(b)** Noisy image **(c)** TV model [18]

(d) TV^{α} model [21] **(e)** Our approach using VFOBD [14] **(f)** Our approach using the $\mathcal{E}$-type

Figure 4. Different total variation models applied to the test image denoising Cameraman, where the noise level $\sigma = 40$.

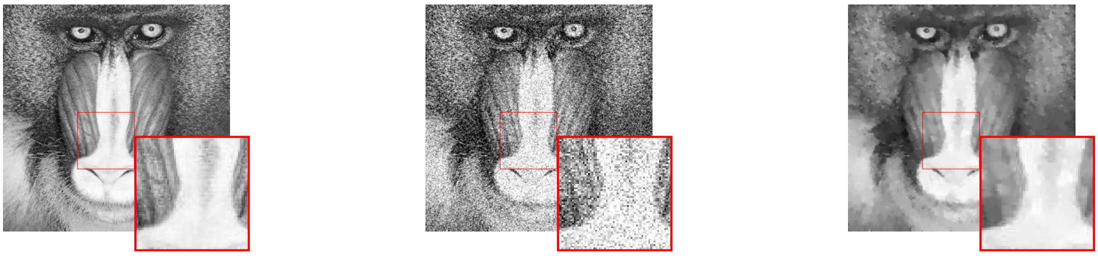


(a) Original image **(b)** Noisy image **(c)** TV model [18]

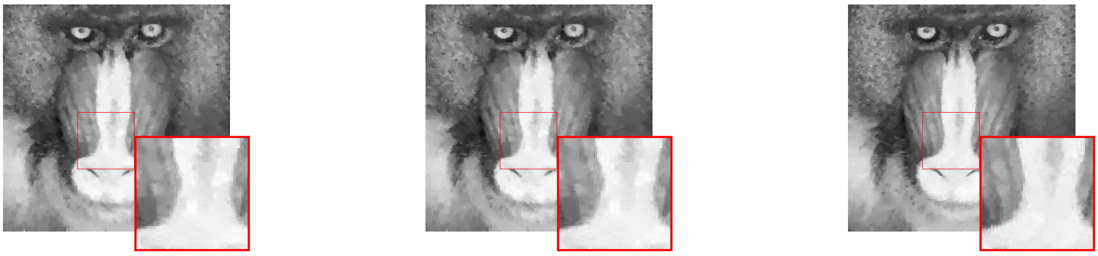


(d) TV^{α} model [21] **(e)** Our approach using VFOBD [14] **(f)** Our approach using the $\mathcal{E}$-type

Figure 5. Comparison of our approach with Total Variation models using *Lena* image where the noise variance is σ=40.

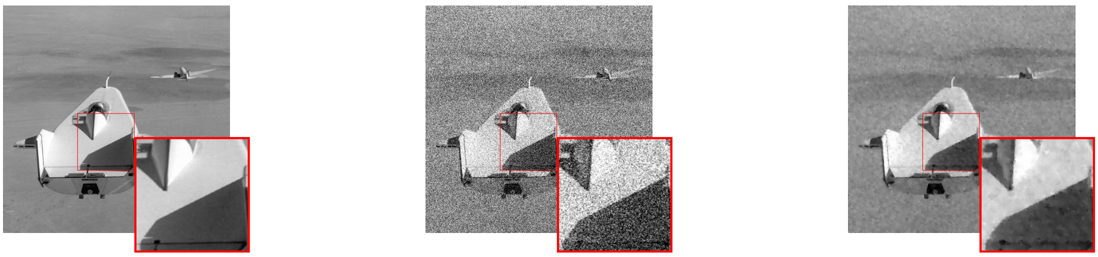


(a) Original image of Mandrill **(b)** Noisy image with $\sigma = 40$ **(c)** TV model [18]

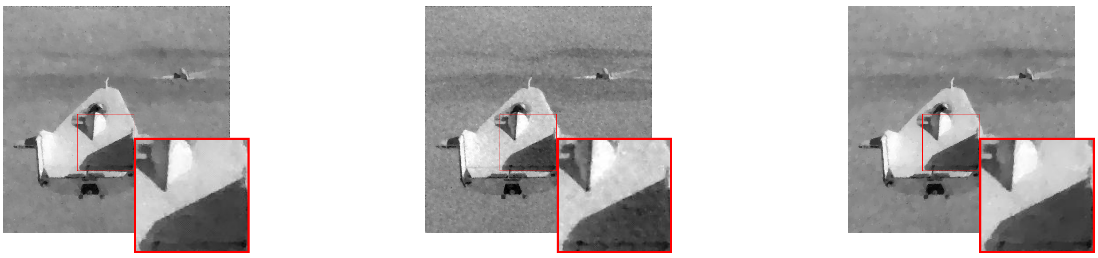


(d) TV^{α} model [21] **(e)** Our approach using VFOBD [14] **(f)** Our approach using the $\mathcal{E}$-type

Figure 6. Comparison of our approach with other models using *Mandrill* image where the noise variance is $\sigma = 40$.



(a) Original image of liftingbody **(b)** Noisy image with $\sigma = 50$ **(c)** TV model [18]



(d) TV^{α} model [21] **(e)** Our approach using VFOBD [14] **(f)** Our approach using the $\mathcal{E}$-type

Figure 7. Testing the robustness of our approach with other models using Liftingbody image at $\sigma = 40$.

Visually, it is clear that our approach is well adapted to the denoising. Furthermore, the Tables 1 and 2 show that our approach is better than the competitive models according to the PSNR and SSIM.

Table 1. The value of PSNR for the different model using Lena image.

Noise level σ	TV	TV^{α}	$TV^{\hat{\alpha}}$ using the $\mathcal{E}$-type
10	31.07	32.77	**33.02**
20	28.94	30.07	**30.21**
30	27.05	28.44	**28.65**

Table 2. The value of SSIM for the different model using cameraman image.

Noise level σ	TV	TV^{α}	$TV^{\hat{\alpha}}$ using the $\mathcal{E}$-type
10	0.905	0.920	**0.924**
20	0.803	0.874	**0.888**
30	0.715	**0.868**	0.855

4.2. The parameters effect

In all of the competitive models, the parameters play a role in the performance of PDPG algorithm, we mention λ a regularization parameter, and of course the order of the derivation α.

In this subsection, we will show the importance and the optimal values of the preceding parameters.

The Figure 8 shows the effect of fractional derivation, with α as a constant and $\sigma = 30$. An optimal value of α can be determined numerically but it depends on the initial image. Because for every image the value of α changes.

In Figure 9, the parameter λ is a coefficient that control the reduction of staicasing effect. We can sense the influence of λ and its optimal value is located at the neighborhood of $\frac{1}{\sigma}$.

Our approach defeats the competitive models, with a big difference in the values of the PSNR and SSIM as shown in Figure 10.

Our model is not complete because it still needs a theoretical part and a reasonable choice of $\hat{\alpha}$. All these perspectives will be discussed in future works.

5. Conclusion

In this work, we managed to present a variational model based on a variable-order derivative as a regularizer, we use the Primal-Dual Projected Gradient Algorithm for the simulation, comparing our approach with the well known TV model, and also a model based on the TV^{α} as a regularizer. Our model shows its potential in both vision and statistics, visually in deleting the noise, preserving the edges and features, statistically in PSNR and SSIM as referees.

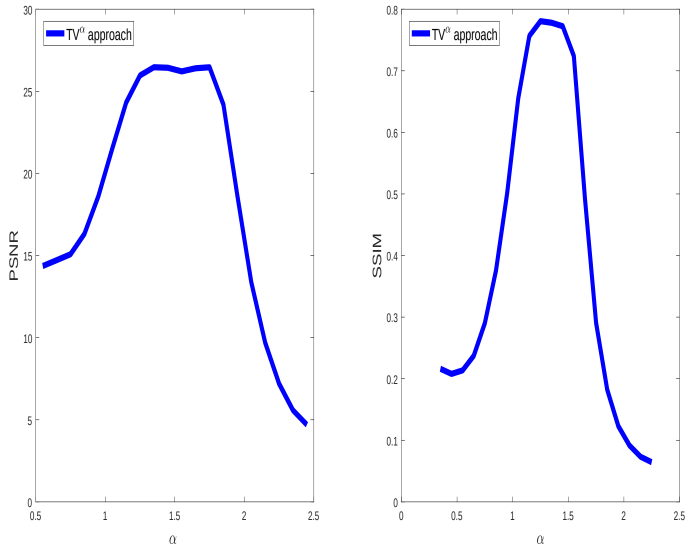


Figure 8. PSNR(α) and SSIM(α).

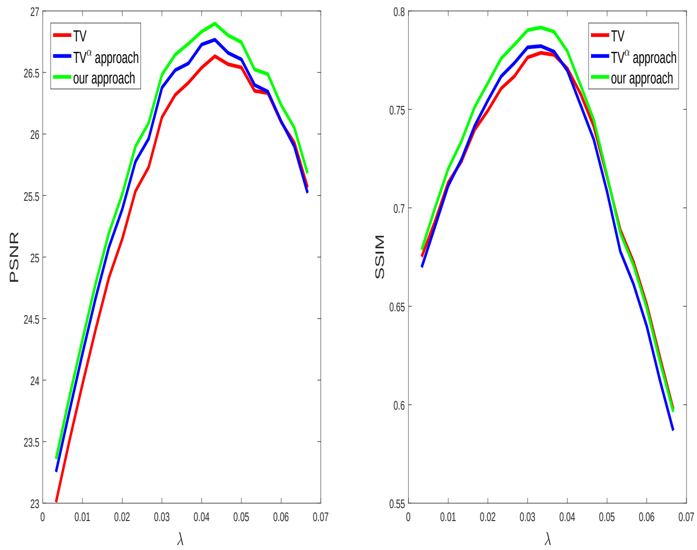


Figure 9. PSNR(λ) and SSIM(λ).

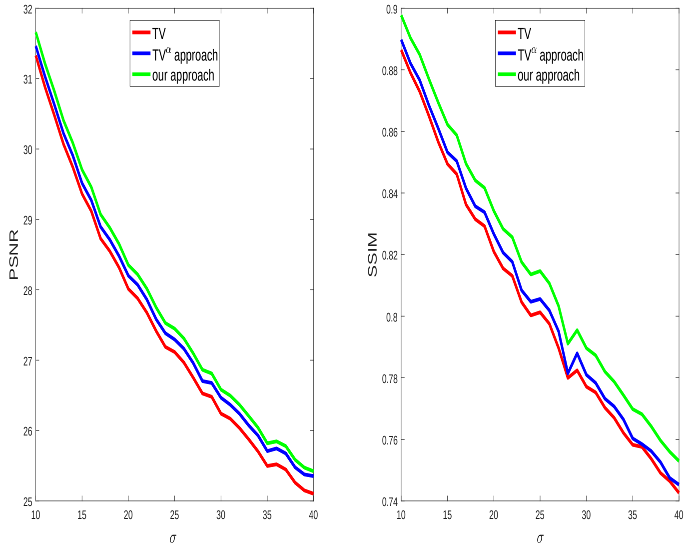


Figure 10. PSNR(σ) and SSIM(σ).

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Conflict of interest

The authors declares that no conflict of interest.

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