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Research article

Mitigating geographical basis risk of weather derivatives using spatial-temporal regime-switching temperature model

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Abstract: In this paper, geographical basis risk in weather derivative design and pricing is mitigated by using spatial-temporal pricing models. A two-state regime-switching temperature model is constructed and extended to multi-dimensional locations that are highly correlated in temperature. The "normal" and "shifted" regime of this model are characterized by a heteroscedastic Ornstein-Uhlenbeck process and a Brownian motion with mean different from zero respectively. The correlation between the driving noise in each regime is assumed to be a function of the space between the locations and increases with decreasing space. A weight is assigned to each location in the temperature basket. However, a location with a higher risk is assigned a larger weight and vice versa. The weightings in the temperature basket gave considerable importance to farming locations having greater exposure to temperature risk. The further the farming location from the weather station, the larger the weight. With this spatial-temporal weather derivatives pricing model, the holder of a weather derivative contract will have the opportunity to select the most appropriate composite of weather stations with their desired weight that can reduce geographical basis risks.

Keywords: agriculture risk management; weather derivatives; regime-switching model; geographical basis risk; spatial-temporal model
Mathematics Subject Classification: 91G10, 91G20, 60G20, 60G15

1. Introduction

Agriculture is the main source of livelihood in Africa [1]. However, extreme changes in weather patterns, unpredictable temperature changes, frequent heat wave, and increasing temperature as a result of climate change make agriculture look like a costly gamble in Africa. The report of the Intergovernmental Panel on Climate Change (IPCC) 2007 [2] gives a comprehensive evaluation of the impacts of climate change on agriculture in Africa. As stated by the IPCC report, the estimated prediction of agriculture losses by 2100 is between 2% to 7% of gross domestic product (GDP) in parts of the Sahara. The report revealed that Northern and Southern Africa are anticipated to record GDP losses of between 0.4-1.3%, and between 2-4% for the Western and Central Africa. According to [3], Africa is extremely affected by climate change from two causes: a limited social, economic, and human abilities needed to adjust to the impact of climate change, and its geographical features of having a predominantly vast land lying across the warming tropics. From planting to harvest, extreme changes in weather can severely affect the quality and the complete production levels of crop yields.

The effects of weather do differ considerably in the agricultural supply and demand chain. Most farmers in Africa have used traditional ways to improve the negative effect of extreme weather conditions on their farmlands. However, most of these farmlands are in the same geographical locations and as a result crop losses are correlated across farmers due to the covariant nature of weather risks. This implies that farmers in the same geographical locations are vulnerable by the same weather event and are probably going to endure extensive losses concurrently. Covariant risk controls the success of traditional risk management techniques of smallholder farmers. Consequently, there is the need for a suitable and efficient risk management tool for farmers to control weather extremes and uncertainties. An emerging shift has been the success of a weather risk management tool-weather derivative, which is use to reduce the financial effects of weather extremes and uncertainties. The uptake of this tool has been lower than expected in the agricultural sector [4] due to basis risks. As defined by [5], "Basis risk arises when the production pattern of the individual operation is not perfectly correlated with the aggregated pattern of the area for which the derivative has been designed". Product-design and geographical basis risk are forms of basis risks. Product-design basis risks can easily be mitigated if the appropriate weather variable is used as the underlying index for designing the weather derivative. Geographical basis risk is difficult to mitigate especially in most developing countries due to the unavailability of weather stations at most farming locations under interest. For this reason, the need to develop spatial-temporal mathematical model that considers different farming locations and assigning weights to weather stations relative to the distance from the reference farming locations. This model is also beneficial to farmers with farmlands in the same geographical locations.

Unlike traditional agricultural insurance that is used to hedge against risks from idiosyncratic occurrences, weather derivatives allow agricultural stakeholders (farmers, input suppliers and other stakeholders at the farm level) to hedge against covariant risks. Weather derivative is seen as an efficient tool for mitigating risk that affects most farmers, input suppliers and other stakeholders at the farm level. In a survey report conducted in 2008-2009 and prepared for the weather risk management association, the agricultural sector contributed about 11%* of the total weather derivatives purchase in the weather market. Unlike traditional insurance, the payoff of a weather derivative depends on an index (specially designed measure) that is linked to the risk being hedged against.

The price of a weather derivative is usually dependent on different weather indices (heating degree days (HDD), cooling degree days (CDD), Pacific Rim (PRIM), cumulative average temperature (CAT), growing degree days (GDD)) that help in pricing weather derivatives. Different authors [6–8] have used HDD and CDD indices as the major indices for pricing weather derivatives in the energy industry. Similar to the CDD and HDD indices in the energy sector, the GDD and CAT indices are powerful

*the second largest percentage after the energy sector indices that can be use to price weather derivatives for the agricultural sector in Africa. For this reason, CAT and GDD are used as the major indices for the temperature based weather derivative pricing. GDD measures the growth and development of crops, weeds, and insects during a growing season.

The contributions made in this study are: (1) Motivated by [9], we develop a temperature dynamics model for spatial-temporal locations in Ghana. This model captures the stylized facts of temperature at different locations. (2) We develop an analytical weather derivative pricing model for basket futures written on CAT and GDD indices. With the proposed basket futures pricing model, it will be cost efficient and pragmatic for farmers to buy weather derivatives contracts for different but correlated farming locations than a single farming location. Geographical basis risk will also be reduced when using this spatial-temporal pricing model.

To the best our knowledge, the proposed analytical weather derivatives basket futures pricing formulas using multi-dimensional regime-switching model is the first of its kind in literature.

2. Theoretical concepts

Assume D_B represent the random payoff at expiry for the owner of a futures contract. At time $t \leq t_1 < t_2$ for a measurement period $[t_1, t_2]$, the holder of the contract enters into the contract. Let $F(t, t_1, t_2)$ represent the price against receiving the random payment D_B at time t_2. For a constant continuously compounded interest rate $r > 0$ and a risk-neutral probability measure $\mathbb{Q}$, the arbitrage-free future price for a measurement period $[t_1, t_2]$ on the CAT and GDD can be define as $\mathcal{F}_t$-adapted stochastic process satisfying

$$0 = e^{-r(t_2-t)}\mathbb{E}_{\mathbb{Q}}\left[D_B - F(t, t_1, t_2)|\mathcal{F}_t\right] \tag{2.1}$$

Assume that the futures price $F(t, t_1, t_2)$ is $\mathcal{F}_t$ adapted, then the futures price is define as

$$F(t, t_1, t_2) = \mathbb{E}_{\mathbb{Q}}[D_B|\mathcal{F}_t] \tag{2.2}$$

where t, t_1, t_2 are the current, starting and maturity time of the futures contract respectively and $\mathcal{F}_t$ is σ-algebra up to a specified time t. From equation 2.2, it is important to calculate the risk-neutral measure $\mathbb{Q}$ (also referred to as equivalent martingale measure) in order to determine futures prices.

To derive an explicit formula for the future price, we specify $\mathbb{Q}$ to help in calculating the expectation. Following the analysis of [11], we use the Girsanov theorem to find a sub-family of probability measures for the normal and shifted regimes. By using the option pricing technique under independent regime-switching model of [12], the CAT and GDD futures are priced by splitting the futures price into a normal and shifted price component. The two pricing component are joined up using the idea of weighted mixture of probability in each regime from t_1 to t_2.

2.1. CAT futures

Suppose that for a contract period $[t_1, t_2]$, the temperature dynamics follow the temperature model in Lemma 3.1. Then, there is a price dynamic of futures written on a CAT index with $t \leq t_1 < t_2$. From equation 2.1, the futures price of CAT is

$$0 = e^{-r(t_2-t)}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_1}^{t_2} T_x dx - F_{CAT}(t, t_1, t_2) \mid \mathcal{F}_t\right]. \tag{2.3}$$

Since the futures price $F(t, t_1, t_2)$ is $\mathcal{F}_t$ adapted under the measure $\mathbb{Q}$, F_{CAT} is $\mathcal{F}_t$-adapted. We can therefore define the CAT futures price $F_{CAT}(t, t_1, t_2)$ for a weather derivative contract as

$$F_{CAT}(t, t_1, t_2) = \mathbb{E}_{\mathbb{Q}}\left[\int_{t_1}^{t_2} T_x dx \mid \mathcal{F}_t\right] \tag{2.4}$$

2.2. GDD futures

Similar to the definition of the CAT futures price, the GDD futures price is given as

$$0 = e^{-r(t_2-t)}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_1}^{t_2} \max\left(T_x - K, 0\right) dx - F_{GDD}(t, t_1, t_2)\middle|\mathcal{F}_t\right] \tag{2.5}$$

Using the same idea in deriving the CAT futures price, the price of the GDD futures can be derived as

$$F_{GDD}(t, t_1, t_2) = \mathbb{E}_{\mathbb{Q}}\left[\int_{t_1}^{t_2} \max\left(T_x - K, 0\right) dx\middle|\mathcal{F}_t\right]. \tag{2.6}$$

Where K is the optimal normal temperature at which a crop will develop. The rate of development of most plants depends on the daily air temperature [13].
Because the market price of temperature risk remains unchanged for all derivatives that depends on temperature, we use the same risk-neutral measure $\mathbb{Q}$ used in pricing the CAT futures to price the GDD futures.

2.3. Change of measure

To find a mathematical expression for the futures price of the chosen indices, an arbitrage-free and explicit dynamics for future price of the indices are constructed. The real-world measure $\mathbb{P}$ is changed to a risk-neutral measure $\mathbb{Q}$, in a way that the discounted price process of the underlying is a martingale under $\mathbb{Q}$. To transform the $\mathbb{P}$ to $\mathbb{Q}$, the Girsanov theorem is employed. The Girsanov theorem provides techniques for transforming $\mathbb{P}$ to $\mathbb{Q}$ under the setting of a Brownian motion, where $\mathbb{Q}$ is a second probability measure.

Theorem 2.1 (Girsanov Theorem)**.** *Let W_t be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\lambda = \{\lambda_t : 0 \leq t \leq T\}$ is an adaptive process satisfying the Novikov condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \lambda_u^2 du\right)\right] < \infty. \tag{2.7}$$

$$\textit{Let} \qquad Z(t) = \exp\left(\int_0^t \lambda_u dW_u - \frac{1}{2}\int_0^t \lambda_u^2 du\right), \tag{2.8}$$

then $\mathbb{Q} \sim \mathbb{P}$ can be determined by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t = Z(t), \tag{2.9}$$

Then we can define the random process

$$\textit{and} \qquad V_t = W_t - \int_0^t \lambda_s ds. \tag{2.10}$$

Equivalently,

$$dV_t = dW_t - \lambda_t dt \tag{2.11}$$

The process V_t is a Brownian motion under the measure $\mathbb{Q}^\lambda$

Proof. See [14] for the proof of this theorem. □

2.4. Girsanov's Theorem in $\mathbb{R}^N$

Let $\boldsymbol{B}(t) = (B_1(t), B_2(t), B_3(t), \cdots, B_N(t))$ be an N-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\boldsymbol{\lambda} = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \cdots, \lambda_N(t))$ be an N-dimensional adapted process on $[0, T]$. Define

$$Z_\lambda(t) := \exp\Big(\int_0^t \boldsymbol{\lambda}(s) d\boldsymbol{B}(s) - \frac{1}{2}\int_0^t \parallel \boldsymbol{\lambda}(s) \parallel^2 ds \Big), \tag{2.12}$$

where $\parallel \boldsymbol{\lambda}(s) \parallel^2 = \sum_{i=1}^N \lambda_i(s)^2$.
Let

$$\tilde{\boldsymbol{B}}(t) = \boldsymbol{B}(t) + \int_0^t \boldsymbol{\lambda}(s) ds \tag{2.13}$$

and suppose that

$$\mathbb{E}\int_0^t \parallel \lambda(s) \parallel^2 Z(s)^2 ds < \infty,$$

then $\tilde{\boldsymbol{B}}(t)$ is a N-dimensional standard Brownian motion under the measure $\mathbb{Q}$ defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{F}_t = Z(T) \tag{2.14}$$

Observe that for each j,

$$\tilde{B}_j(t) = B_j(t) + \int_0^t \lambda_j(s) ds \tag{2.15}$$

The component process of $\tilde{\boldsymbol{B}}(t)$ are independent under the measure $\mathbb{Q}$.

Remark 1. The Novikov condition in the Girsanov theorem makes sure Z is positive martingale and whenever $\mathbb{E}(Z) = 1$. This is referred to as the Radon-Nikodym derivative.

Remark 2. λ is referred to as the market price of temperature risk (MPR). Since there is no real weather derivative market in Africa from which the prices can be obtained, λ is assumed to be a constant. For a constant λ, equation 2.11 can be redefined as

$$dV_t = dW_t - \lambda dt \tag{2.16}$$

3. Regime-switching temperature model

3.1. One-dimensional regime-switching model

Different temperature models have been proposed to capture the dynamics of temperature [6, 8–10, 15]. The usual assumptions in these models are: volatility of temperature is lower in summer than in winter, temperature is autoregressive, temperature follows a predicted pattern and it goes around a seasonal mean. Early models used autoregressive moving average (ARMA) processes, autoregressive process of order one (AR(1)) processes, mean reverting stochastic differential equations (SDE) [9, 15, 16]. All these models assumed no changes in the state of the dynamics of temperature. That is, they used single-regime model to describe the dynamics of temperature. However, temperature can go through different latent states in a particular period of time and a single SDE can not accurately capture all this states. More complex models have been proposed in recent literatures, an example is the model of [7, 17, 18], which uses a two-state Markov regime-switching model. [17] extended the model of [7] by replacing the constant volatility in the base regime with heteroscedastic volatility. This was necessary because volatility of temperature changes at different states. [18] further extended this model by replacing the Brownian process in the shifted regime with a Lévy process. However, none of these authors proposed a spatial-temporal pricing model for weather derivatives. Motivated by [7, 17, 18], a regime-switching temperature dynamics model is proposed and later extended to a multi-dimensional model. The switching dynamics between the regimes are assumed to be controlled by an unobservable latent variable R_t. The model is governed by a two-state regime-switching model $R_t = \{1, 2\}$ in which a two-state Markov chain controls the characterization of the probability law of switching between $R_t = 1$ and $R_t = 2$ with transition probabilities

$$p_{ik} = Pr(R_t = k \mid R_{t-1} = i) \qquad \forall i, k = 1, 2$$

$$0 \leq p_{ik} \leq 1 \qquad \text{and} \qquad \sum_{k=1}^{2} p_{ik} = 1$$

The daily average temperature on day t, T_t is modelled as the sum of a deseasonalized temperature $\tilde{T}_t$ and a deterministic seasonal component S_t,

$$T_t = \tilde{T}_t + S_t$$

The deterministic seasonality component at time t, S_t is given by

$$S_t = a_1 \sin\left(\frac{2\pi}{365}(t - \varphi)\right) + a_2 t + a_3$$

where a_1, a_2, a_3 and φ are constants.
The proposed model is distinctly appropriate to capture the dynamics of temperature through time. The proposed daily temperature model is given as

$$T_t = \begin{cases} T_{t,1} : dT_{t,1} = dS_{t,1} + \beta(T_{t,1} - S_{t,1})dt + \sigma_1 T_{t,1} dW_t, & \text{if } T_t \text{ is in the normal regime,} \\ T_{t,2} : dT_{t,2} = \mu dt + \sigma_2 dW_t, & \text{if } T_t \text{ is in the shifted regime,} \end{cases} \tag{3.1}$$

where $\sigma_1 T_{t,1}$ is the daily local volatility of the normal regime through time, σ_2 is the volatility of the shifted regime, and β is the mean-reversion rate of the daily temperature in the normal regime which reverses to the long term equilibrium level after the daily temperature has drifted from its equilibrium. The probabilities for the process to be in the normal and shifted regimes are p_1 and p_2 respectively and $p_1 + p_2 = 1$. $W_t \sim N(0, t)$ is the standard Brownian motion. $T_t(T(t))$ is the daily temperature at time t. The regimes are assumed to be independent to each other and the futures contract is calculated for each regime model. The final futures price is calculated using the weighted sum of the individual regimes.

Lemma 3.1. *If the daily average temperature $T_{t,1}$ follows the proposed model 3.1, then the explicit solution is given by*

$$T(t) = \begin{cases} T_{t,1} : T_{t,1} = S_{t,1} + (T_{0,1} - S_{0,1})e^{-\beta t} + \int_0^t \sigma T_u e^{\beta(t-s)} dW_s \\ T_{t,2} : T_{t,2} = T_{0,2} + \mu t + \int_0^t \sigma_2 dW_s \end{cases} \tag{3.2}$$

Proof. For the normal regime,

$$dT_t = dS_t + \beta(T_t - S_t)dt + \sigma T_t dB_t$$

$$d\tilde{T}_t = \beta \tilde{T}_t + \sigma T_t dB_t, \tag{3.3}$$

where $\tilde{T}_{t,1} = T_{t,1} - S_{t,1}$. Using the transformation below, $d\tilde{T}_{t,1}$ will be evaluated,

$$F[\tilde{T}_{t,1}, t] = \tilde{T}_{t,1} e^{-\beta t}$$

$$\frac{\partial F}{\partial \tilde{T}_{t,1}} = e^{-\beta t}; \qquad \frac{\partial^2 F}{\partial \tilde{T}_{t,1}^2} = 0; \qquad \frac{\partial F}{\partial t} = -\beta \tilde{T}_t e^{-\beta t}$$

By Itô's Lemma and from equation (3.15),

$$dF_{t,1} = \sigma T_{t,1} e^{-\beta t} dW_t \tag{3.4}$$

Integrating equation (3.17) over the interval $[0, t]$,

$$\begin{aligned} F_{t,1} &= F_{0,1} + \int_0^t \sigma T_{s,1} e^{-\beta s} dW_s \\ \tilde{T}_t e^{-\beta t} &= \tilde{T}_{0,1} + \int_0^t \sigma T_{s,1} e^{-\beta s} dW_s \\ \tilde{T}_t &= \tilde{T}_{0,1} e^{\beta t} + \int_0^t \sigma T_u e^{\beta(t-s)} dW_s \\ T_t &= S_{t,1} + (T_{0,1} - S_{0,1})e^{\beta t} + \int_0^t \sigma T_{s,1} e^{\beta(t-s)} dW_s \end{aligned}$$

For the shifted regime

$$\begin{aligned} dT_{t,2} &= \mu dt + \sigma_2 dW_t \\ \int_0^t dT_{t,2} &= \int_0^t \mu ds + \int_0^t \sigma_2 dW_s \\ T_{t,2} &= T_{0,2} + \mu t + \int_0^t \sigma_2 dW_s \end{aligned}$$

□

3.2. Spatial-temporal regime switching model

Let N be the spatial locations in the basket, $\left(\omega^i\right)_{i=1}^{N}$ be the collection of weights for spatial locations $\left(y^i\right)_{i=1}^{N}$. At time t, the basket of the deseasonalized average temperature at the N spatial locations is given as

$$D(t) := \sum_{i=1}^{N} \omega^i T_t^i \tag{3.5}$$

Where $\sum_{i=1}^{N} \omega^i = 1$.

Assume temperature is spatially correlated across the random noise term. To allow analytical pricing of the basket temperature derivatives contract, assume the risk-neutral distribution of temperature for each location is normally distributed in the temperature model. That is, the basket been a weighted sum of normally distributed temperature is also normally distributed. Consequently, we are able to outwit the principal difficulty associated with pricing basket options of assets when determining the distribution of the sum or average of the underlying assets. From these settings, a spatial-temporal temperature model at each spatial location y^i is proposed,

$$T_t^i = \begin{cases} T_{t,1}^i : dT_{t,1}^i = dS_{t,1}^i + \beta^i(T_{t,1}^i - S_{t,1}^i)dt + \sigma_1^i T_{t,1}^i dW_t^i, \\ T_{t,2}^i : dT_{t,2}^i = \mu^i dt + \sigma_2^i dW_t^i \end{cases} \tag{3.6}$$

Model 3.6 can be expressed for locations $i = 1, 2, \cdots, N$ as an N-dimensional system,

$$\boldsymbol{T}_t = \begin{cases} \boldsymbol{T}_{t,1} : d\boldsymbol{T}_{t,1} = d\boldsymbol{S}_{t,1} + \boldsymbol{\beta}(\boldsymbol{T}_{t,1} - \boldsymbol{S}_{t,1})dt + \boldsymbol{\sigma}_1 \boldsymbol{T}_{t,1} d\boldsymbol{W}_t, \\ \boldsymbol{T}_{t,2} : d\boldsymbol{T}_{t,2} = \boldsymbol{\mu} dt + \boldsymbol{\sigma}_2 d\boldsymbol{W}_t \end{cases} \tag{3.7}$$

where $\boldsymbol{W}_t \sim N(0, \boldsymbol{\Omega} t)$. From the property of linear transformation of multivariate normal distribution,

$$Y \sim N(\mu, \Sigma) \Rightarrow XY \sim N(X\mu, X\Sigma X^T)$$

Suppose $Z \sim N(0, \boldsymbol{I} t)$ and $Y = XZ$, then it follows that $Y \sim N(0, XX^T t)$. By applying Cholesky factorization to Σ, a lower triangular form for X is derived. Using this theory, $\boldsymbol{W}_t$ can be expressed as an N-dimensional Brownian motion $\boldsymbol{B}_t$,

$$\boldsymbol{W}_t = \boldsymbol{L}\boldsymbol{B}_t, \tag{3.8}$$

$\boldsymbol{L}\boldsymbol{L}^{\boldsymbol{T}} = \boldsymbol{\Omega}$, $\boldsymbol{L}$ is a lower triangular matrix with non-negative diagonal entries, $\boldsymbol{L}^{\boldsymbol{T}}$ is an upper traingular matrix, and $\boldsymbol{B}_t = (B_t^1, B_t^2, B_t^3, \cdots, B_t^N)^T$ with $dB_t^i dB_t^j = \delta_{ij} dt$. From equation 3.8, equation 3.7 can be reformulated as

$$\boldsymbol{T}_t = \begin{cases} \boldsymbol{T}_{t,1} : d\boldsymbol{T}_{t,1} = d\boldsymbol{S}_{t,1} + \boldsymbol{\beta}(\boldsymbol{T}_{t,1} - \boldsymbol{S}_{t,1})dt + \boldsymbol{\sigma}_1 \boldsymbol{T}_{t,1} \boldsymbol{L} d\boldsymbol{B}_t, \\ \boldsymbol{T}_{t,2} : d\boldsymbol{T}_{t,2} = \boldsymbol{\mu} dt + \boldsymbol{\sigma}_2 \boldsymbol{L} d\boldsymbol{B}_t \end{cases} \tag{3.9}$$

Equation 2.8 can be transformed for the normal and shifted regime,

$$Z_t^{\lambda} = \begin{cases} \exp\Big(\int_0^t (\boldsymbol{\sigma}_1 \boldsymbol{T}_{s,1} \boldsymbol{L})^{-1} \boldsymbol{\lambda}_s d\boldsymbol{B}_s - \frac{1}{2} \int_0^t \| \boldsymbol{\sigma}_1 \boldsymbol{T}_{s,1} \boldsymbol{L} \|^{-2} \| \boldsymbol{\lambda}_s \|^2 \, ds \Big) \\ \exp\Big(\int_0^t (\boldsymbol{\sigma}_2 \boldsymbol{L})^{-1} \boldsymbol{\lambda}(s) d\boldsymbol{B}_s - \frac{1}{2} \int_0^t \| \boldsymbol{\sigma}_2 \boldsymbol{L} \|^{-2} \| \boldsymbol{\lambda}(\boldsymbol{s}) \|^2 \, ds \Big) \end{cases} \tag{3.10}$$

From equation 3.10, and assuming λ is a constant for each reference measurement station, it can be inferred that;

$$\boldsymbol{V}_t^{\lambda} = \begin{cases} \boldsymbol{B}_t - \int_0^t (\boldsymbol{\sigma}_1 \boldsymbol{T}_{s,1} \boldsymbol{L})^{-1} \lambda ds \\ \boldsymbol{B}_t - \int_0^t (\boldsymbol{\sigma}_2 \boldsymbol{L})^{-1} \lambda ds \end{cases} \tag{3.11}$$

Equivalently

$$d\boldsymbol{V}_t^{\lambda} = \begin{cases} d\boldsymbol{B}_t - (\boldsymbol{\sigma}_1 \boldsymbol{T}_{t,1} \boldsymbol{L})^{-1} \lambda dt \\ d\boldsymbol{B}_t - (\boldsymbol{\sigma}_2 \boldsymbol{L})^{-1} \lambda dt \end{cases} \tag{3.12}$$

Where $\boldsymbol{V}_t^{\lambda}$ is a Brownian motion under the measure $\mathbb{Q}_{\lambda}$
Consequently, the TML model under the equivalent martingale measure $\mathbb{Q}^{\lambda}$ is

$$\begin{cases} d\mathbf{T}_{t,1} = d\mathbf{S}_{t,1} + (\boldsymbol{\lambda} + \boldsymbol{\beta}(\boldsymbol{T}_{t,1} - \boldsymbol{S}_{t,1}))\, dt + \boldsymbol{\sigma}_1 \mathbf{T}_{t,1} \boldsymbol{L} d\boldsymbol{V}_{\lambda}(t), \\ d\boldsymbol{T}_{t,2} = (\boldsymbol{\mu} + \boldsymbol{\lambda})\, dt + \boldsymbol{\sigma}_2 \boldsymbol{L} d\boldsymbol{V}_{\lambda}(t), \end{cases} \tag{3.13}$$

Lemma 3.2. *For a spatial location i, if the dynamics of the daily average temperature follows equation 3.13, then the explicit solution for the i^{th} location y_i is given as*

$$\begin{cases} T_{t,1}^i = S_{t,1}^i + (T_{0,1}^i - S_{0,1}^i) e^{\beta^i t} + \left(\frac{\lambda}{\beta}\right)^i (e^{\beta^i t} - 1) + \int_0^t \sigma_1^i T_{s,1}^i e^{\beta^i (t-s)} \sum_{j=1}^i L^{ij} dV_{\lambda}^j(s), \\ T_{t,2}^i = T_{0,2}^i + (\mu + \lambda)^i t + \int_0^t \sigma_2^i \sum_{j=1}^i L^{ij} dV_{\lambda}^j(s) \end{cases} \tag{3.14}$$

Proof. From the normal regime of model 3.13,

$$d\mathbf{T}_{t,1} = d\mathbf{S}_{t,1} + (\boldsymbol{\lambda} + \boldsymbol{\beta}(\boldsymbol{T}_{t,1} - \boldsymbol{S}_{t,1}))\, dt + \boldsymbol{\sigma}_1 \mathbf{T}_{t,1} \boldsymbol{L} d\boldsymbol{V}_{\lambda}(t),$$

$$d\tilde{\boldsymbol{T}}_{t,1} = \beta \tilde{\boldsymbol{T}}_{t,1} + \lambda dt + \boldsymbol{\sigma}_1 \mathbf{T}_{t,1} \boldsymbol{L} d\boldsymbol{V}_{\lambda}(t), \tag{3.15}$$

where $\tilde{\boldsymbol{T}}_{t,1} = \boldsymbol{T}_{t,1} - \boldsymbol{S}_{t,1}$. $d\tilde{\boldsymbol{T}}_{t,1}$ will be evaluated using the transformation in equation 3.16,

$$F[\tilde{\boldsymbol{T}}_{t,1}, t] = \tilde{\boldsymbol{T}}_{t,1} e^{-\boldsymbol{\beta} t} \tag{3.16}$$

By Itô's Lemma and from equation (3.15),

$$dF = \boldsymbol{\lambda} e^{-\boldsymbol{\beta} t} dt + \boldsymbol{\sigma}_1 \boldsymbol{T}_{t,1} e^{-\boldsymbol{\beta} t} \boldsymbol{L} d\boldsymbol{V}_t \tag{3.17}$$

Integrating equation (3.17) over the interval $[0, t]$ gives

$$\boldsymbol{T}_{t,1} = \boldsymbol{S}_{t,1} + (\boldsymbol{T}_{0,1} - \boldsymbol{S}_{0,1}) e^{\boldsymbol{\beta} t} + \frac{\boldsymbol{\lambda}}{\boldsymbol{\beta}} (e^{\boldsymbol{\beta} t} - 1) + \int_0^t \boldsymbol{\sigma}_1 \boldsymbol{T}_{s,1} e^{\boldsymbol{\beta}(t-s)} \boldsymbol{L} d\boldsymbol{V}_s$$

At location y^i,

$$T_{t,1}^i = S_{t,1}^i + (T_{0,1}^i - S_{0,1}^i) e^{\beta^i t} + \left(\frac{\lambda}{\beta}\right)^i (e^{\beta^i t} - 1) + \int_0^t \sigma_1^i T_{s,1}^i e^{\beta^i (t-s)} \sum_{j=1}^i L^{ij} dV_{\lambda}^j(s)$$

From the shifted regime of model 3.13,

$$dT_{t,2} = \mu dt + \lambda dt + \sigma_2 L dB_\lambda(t)$$
$$T_{t,2} = T_{0,2} + (\mu+\lambda)t + \int_0^t \sigma_2 L dV_\lambda(s)$$

At location y^i,

$$T_{t,2} = T^i_{0,2} + (\mu+\lambda)^i t + \int_0^t \sigma^i_2 \sum_{j=1}^{i} L^{ij} dV^j_\lambda(s),$$

hence the lemma. □

4. Pricing futures on temperature basket

The CAT and GDD futures will be priced by splitting the futures price into normal and shifted price components and added up with probabilities in each regime for the contract period.

$$F(t,t_1,t_2) = Pr(R_{[t_1,t_2]} = N)F^N(t,t_1,t_2) + Pr(R_{[t_1,t_2]} = S)F^S(t,t_1,t_2) \tag{4.1}$$

Where $F^N(t,t_1,t_2)$ and $F^S(t,t_1,t_2)$ are the futures price of the normal and shifted regimes respectively. $Pr(R_{[t_1,t_2]} = N)$ and $Pr(R_{[t_1,t_2]} = S)$ are the probability of the observed daily average temperature data under the normal and shifted regimes throughout the contract period $[t_1,t_2]$. The splitting is possible since it is assumed that the futures price under the normal regime is independent of the futures price under the shifted regime.

4.1. CAT and GDD futures on temperature basket

By the same reasoning as in deriving the futures price of a single CAT, the futures price of a basket CAT is given by

$$F_{CAT}(t,t_1,t_2;D) = \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{N}\omega^i\Big(\int_{t_1}^{t_2} T^i_x dx\Big) \mid \mathcal{F}_t\right] \tag{4.2}$$

Definition 4.1. At a spatial location y^i and a specificied contract period, $t \leq t_1 < t_2$, the GDD futures price is define as

$$\begin{aligned} GDD(t_1,t_2) &:= \int_{t_1}^{t_2} max\Big\{D(t) - K\,,\,0\Big\}dt \\ &= \int_{t_1}^{t_2} max\Big\{\sum_{i=1}^{N}\omega^i T^i_t - K\,,\,0\Big\}dt \end{aligned} \tag{4.3}$$

Analogously, the futures price of a basket GDD is defined as

$$\begin{aligned} F_{GDD}(t,t_1,t_2;D) &= \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_1}^{t_2} max\Big\{\sum_{i=1}^{N}\omega^i T^i_x - K\,,\,0\Big\}dx\Big|\mathcal{F}_t\Big) \\ &= \int_{t_1}^{t_2}\mathbb{E}_{\mathbb{Q}}\Big(max\Big\{\sum_{i=1}^{N}\omega^i T^i_x - K\,,\,0\Big\}\Big|\mathcal{F}_t\Big)dx \end{aligned} \tag{4.4}$$

Proposition 4.2. *At a spatial location y^i, the futures contract price on basket of CAT index following the normal regime in equation 3.6 is calculated as*

$$F^N_{CAT}(t,t_1,t_2;D) = \sum_{i=1}^{N}\omega^i\Big[\int_{t_1}^{t_2} S^{\,i}_{x,1}dx + \int_{t_1}^{t_2}(T^i_{t,1} - S^{\,i}_{t,1})e^{\beta^i(x-t)}dx + \int_{t_1}^{t_2}\left(\frac{\lambda}{\beta}\right)^i(e^{\beta^i(x-t)} - 1)dx\Big]$$

Proof. For $x \geq t$ in Lemma 3.2,

$$T^i_{x,1} = S^{\,i}_{x,1} + (T^i_{t,1} - S^{\,i}_{t,1})e^{\beta^i(x-t)} + \left(\frac{\lambda}{\beta}\right)^i(e^{\beta^i(x-t)} - 1) + \int_t^x \sigma^i_1 T^i_{s,1}e^{\beta^i(x-s)}\sum_{j=1}^{i}L^{ij}dV^j_\lambda(s) \tag{4.5}$$

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[\int_{t_1}^{t_2} T^i_{x,1}dx \mid \mathcal{F}_t\Big] &= \mathbb{E}_{\mathbb{Q}}\Bigg[\int_{t_1}^{t_2}\Big(S^{\,i}_{x,1} + (T^i_{t,1} - S^{\,i}_{t,1})e^{\beta^i(x-t)} + \left(\frac{\lambda}{\beta}\right)^i(e^{\beta^i(x-t)} - 1) + \\ &\quad \int_t^x \sigma^i_1 T^i_{s,1}e^{\beta^i(x-s)}\sum_{j=1}^{i}L^{ij}dV^j_\lambda(s)\Big)dx\Big|\mathcal{F}_t\Bigg] \\ &= \int_{t_1}^{t_2} S^{\,i}_{x,1}dx + \int_{t_1}^{t_2}(T^i_{t,1} - S^{\,i}_{t,1})e^{\beta^i(x-t)}dx + \\ &\quad \int_{t_1}^{t_2}\left(\frac{\lambda}{\beta}\right)^i(e^{\beta^i(x-t)} - 1)dx \\ F^N_{CAT}(t,t_1,t_2;D) &= \sum_{i=1}^{N}\omega^i\mathbb{E}_{\mathbb{Q}}\left[\int_{t_1}^{t_2} T^i_{x,1}dx \mid \mathcal{F}_t\right] \\ &= \sum_{i=1}^{N}\omega^i\Big[\int_{t_1}^{t_2} S^{\,i}_{x,1}dx + \int_{t_1}^{t_2}(T^i_{t,1} - S^{\,i}_{t,1})e^{\beta^i(x-t)}dx + \\ &\quad \int_{t_1}^{t_2}\left(\frac{\lambda}{\beta}\right)^i(e^{\beta^i(x-t)} - 1)dx\Big]\end{aligned}$$

□

Proposition 4.3. *For a specificied contract period, $t \leq t_1 < t_2$ at spatial location y^i, the futures contract price on basket of CAT index following the shifted regime in equation 3.6 is*

$$F^S_{CAT}(t,t_1,t_2;D) = \sum_{i=1}^{N}\omega^i\left[T^i_{t,2}(t_2 - t_1) + \frac{1}{2}(\mu+\lambda)^i\left((t_2 - t)^2 - (t_1 - t)^2\right)\right] \tag{4.6}$$

Proof. We first integrate the shifted regime in lemma 3.2 at a spatial location y_i over the time interval $[t_1, t_2]$.

$$\begin{aligned}\int_{t_1}^{t_2} T^i_{t,2}dx &= \int_{t_1}^{t_2} T^i_{0,2}dx + \int_{t_1}^{t_2}(\mu+\lambda)^i t dx + \int_{t_1}^{t_2}\int_0^t \sigma^i_2\sum_{j=1}^{i}L^{ij}dV^j_\lambda(s)dx \\ &= T^i_{0,2}(t_2 - t_1) + \frac{1}{2}(\mu+\lambda)^i(t_2^2 - t_1^2) + \int_{t_1}^{t_2}\int_0^t \sigma^i_2\sum_{j=1}^{i}L^{ij}dV^j_\lambda(s)dx\end{aligned} \tag{4.7}$$

For $x \geq t$ and a spatial location y^i

$$\int_{t_1}^{t_2} T^i_{x,2}dx = T^i_{t,2}(t_2 - t_1) + \frac{1}{2}(\mu + \lambda)^i\left((t_2 - t)^2 - (t_1 - t)^2\right) + \int_{t_1}^{t_2}\int_t^x \sigma^i_2 \sum_{j=1}^{i} L^{ij} dV^j_\lambda(s)dx$$

$$\begin{aligned} F^S_{CAT}(t, t_1, t_2; D) &= \sum_{i=1}^{N} \omega^i \mathbb{E}_{\mathbb{Q}}\left[\int_{t_1}^{t_2} T^i_{x,2}dx \mid \mathcal{F}_t\right] \\ &= \sum_{i=1}^{N} \omega^i \mathbb{E}_{\mathbb{Q}}\Big[\Big(T^i_{t,2}(t_2 - t_1) + \frac{1}{2}(\mu + \lambda)^i\left((t_2 - t)^2 - (t_1 - t)^2\right) + \\ &\quad \int_{t_1}^{t_2}\int_t^x \sigma^i_2 \sum_{j=1}^{i} L^{ij} dV^j_\lambda(s)dx\Big)\Big|\mathcal{F}_t\Big] \\ &= \sum_{i=1}^{N} \omega^i\Big(T^i_{t,2}(t_2 - t_1) + \frac{1}{2}(\mu + \lambda)^i\left((t_2 - t)^2 - (t_1 - t)^2\right)\Big) \end{aligned}$$

□

Generally, if we assume that the daily average temperature follows model 3.6, then the CAT futures price on temperature basket is computed using Equation 4.1, Proposition 4.2 and 4.3.

Proposition 4.4. *For a specificied contract period, $t \leq t_1 < t_2$ at spatial location y^i, the futures contract price on basket GDD index following the normal regime in equation 3.6 is given by*

$$F^N_{GDD}(t, t_1, t_2; D) = \int_{t_1}^{t_2}\left(\xi(t, x) + 2\varDelta(t, x)\right)^{\frac{1}{2}}\Big(\phi(\Lambda(t, x)) + \Lambda(t, x)\Phi(\Lambda(t, x))\Big)dx, \tag{4.8}$$

where Φ is the cumulative standard normal distribution function, ϕ is the standard normal density function,

$$\Lambda(t, x) = \frac{\Psi(t, x) - K}{\left(\xi(t, x) + 2\varDelta(t, x)\right)^{\frac{1}{2}}},$$

$$\Psi(t, x) = \sum_{i=1}^{N} \omega^i\Big(S^{\,i}_{t,1} + (T^i_{0,1} - S^{\,i}_{0,1})e^{\beta^i t} + \left(\frac{\lambda}{\beta}\right)^i (e^{\beta^i t} - 1)\Big),$$

$$\xi(t, x) = \sum_{i=1}^{N} \omega_i^2 \mathfrak{I}^2(t, x) = \sum_{i=1}^{N} \omega_i^2 \sum_{j=1}^{i} \int_t^x \sigma_i^2 T^2_{(t,1),i} L^2_{ij} e^{2\beta_i(x-s)}ds,$$

$$\begin{aligned} \varDelta(t, x) &= \sum_{i=1}^{N}\sum_{j=i+1}^{N} \omega^i\omega^j\Big(\sum_{q=1}^{i} L^{iq}L^{jq}\Big)\Upsilon^{ij}(t, x) \\ &= \sum_{i=1}^{N}\sum_{j=i+1}^{N} \omega^i\omega^j\Big(\sum_{q=1}^{i} L^{iq}L^{jq}\Big)\int_t^x \sigma^i_1\sigma^j_1 T^i_{u,1}T^j_{u,1} e^{(\beta^i+\beta^j)(x-s)}ds \end{aligned}$$

Proof. Let

$$D(x) = \sum_{i=1}^{N} \omega^i T_x^i \tag{4.9}$$

For $x \geq t$ in Lemma 3.2,

$$T_{x,1}^i = S_{x,1}^i + (T_{t,1}^i - S_{t,1}^i)e^{\beta^i(x-t)} + \left(\frac{\lambda}{\beta}\right)^i (e^{\beta^i(x-t)} - 1) + \int_t^x \sigma_1^i T_{s,1}^i e^{\beta^i(x-s)} \sum_{j=1}^{i} L^{ij} dV_\lambda^j(s) \tag{4.10}$$

For convenience, we shall denote the deterministic and random component of equation 4.10 as $G^i(t,x)$ and $H^i(t,x)$ respectively. That is,

$$G^i(t,x) = S_{x,1}^i + (T_{t,1}^i - S_{t,1}^i)e^{\beta^i(x-t)} + \left(\frac{\lambda}{\beta}\right)^i (e^{\beta^i(x-t)} - 1)$$

$$H^i(t,x) = \int_t^x \sigma_1^i T_{s,1}^i e^{\beta^i(x-s)} \sum_{j=1}^{i} L^{ij} dV_\lambda^j(s) = \sum_{j=1}^{i} \int_t^x \sigma_1^i T_{s,1}^i e^{\beta^i(x-s)} L^{ij} dV_\lambda^j(s)$$

Hence

$$D(x) = \sum_{i=1}^{N} \omega^i \Big(G^i(t,x) + H^i(t,x)\Big) \tag{4.11}$$

At time t, the distribution of the basket $D(x)$ can be computed. However, $G^i(t,x)$ is deterministic. Hence, at time t and by Itô isometry,

$$\int_t^x \sigma_1^i T_{s,1}^i e^{\beta^i(x-s)} L^{ij} dV_\lambda^j(s) \sim N\Big(0, \int_t^x \sigma_i^2 T_{(s,1),i}^2 L_{ij}^2 e^{2\beta_i(x-s)} ds\Big)$$

But $V_\lambda^j(s)$ are independent for each j. So the variances can be summed to obtain the variance of $H^i(t,x)$,

$$H^i(t,x) \sim N\Big(0, \sum_{j=1}^{i} \int_t^x \sigma_i^2 T_i^2 L_{ij}^2 e^{2\beta_i(x-u)} du\Big) = N\Big(0, \Im^2(t,x)\Big)$$

Since $\sum_{i=1}^{N} \omega^i H^i(t,x)$ is a sum of normally distributed random variables, it implies that it is normally distributed with the following respective mean and variance:

$$\mathbb{E}\Big(\sum_{i=1}^{N} \omega^i H^i(t,x)\Big) = \sum_{i=1}^{N} \omega^i \mathbb{E}\Big(H^i(t,x)\Big) = 0$$

and

$$\begin{aligned} Var\Big(\sum_{i=1}^{N} \omega^i H^i(t,x)\Big) &= \sum_{i=1}^{N} Var\Big(\omega^i H^i(t,x)\Big) + 2\sum_{i<j} Cov\Big(\omega^i H^i, \omega^j H^j\Big) \\ &= \sum_{i=1}^{N} \omega_i^2 Var(H^i(t,x)) + 2\sum_{i<j} \omega^i \omega^j Cov\Big(H^i, H^j\Big) \\ &= \sum_{i=1}^{N} \omega_i^2 \Im^2(t,x) + 2\sum_{i<j} \omega^i \omega^j Cov\Big(H^i, H^j\Big) \end{aligned}$$

Now, take into account $Cov(H^i, H^j)$ for $j > 1$. Both H^1 and H^j are in the same integral form with respect to the standard Brownian motion $V_\lambda^1(u)$. By the independence property of $V_\lambda^1(u)$ and $V_\lambda^j(u)$, the covariance only exist between these two integrals. Therefore,

$$\begin{aligned} Cov\Big(H^i, H^j\Big) &= \int_t^x \sigma_1^i \sigma_1^j T_{s,1}^i T_{s,1}^j e^{(\beta^i+\beta^j)(x-s)} \Big(\sum_{q=1}^{i} L^{iq} L^{jq} \Big) ds, \quad \forall j > 1 \\ &= \Big(\sum_{q=1}^{i} L^{iq} L^{jq} \Big) \int_t^x \sigma_1^i \sigma_1^j T_{s,1}^i T_{s,1}^j e^{(\beta^i+\beta^j)(x-s)} ds, \quad \forall j > 1 \end{aligned}$$

Define $\Upsilon^{ij}(t,x) := \int_t^x \sigma_1^i \sigma_1^j T_{s,1}^i T_{s,1}^j e^{(\beta^i+\beta^j)(x-s)} ds$.

$$\begin{aligned} Var\Big(\sum_{i=1}^{N} \omega^i H^i(t,x) \Big) &= \sum_{i=1}^{N} \omega_i^2 \Im^2(t,x) + 2 \sum_{i<j} \omega^i \omega^j \Big(\sum_{q=1}^{i} L^{iq} L^{jq} \Big) \Upsilon^{ij}(t,x) \\ &= \sum_{i=1}^{N} \omega_i^2 \Im^2(t,x) + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \omega^i \omega^j \Big(\sum_{q=1}^{i} L^{iq} L^{jq} \Big) \Upsilon^{ij}(t,x) \end{aligned}$$

From 4.11,

$$D(x) \sim N\Big(\sum_{i=1}^{N} \omega^i G^i(t,x), \sum_{i=1}^{N} \omega_i^2 \Im^2(t,x) + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} \omega^i \omega^j \Big(\sum_{q=1}^{i} L^{iq} L^{jq} \Big) \Upsilon^{ij}(t,x) \Big)$$

Let

$$\Psi(t,x) = \sum_{i=1}^{N} \omega^i G^i(t,x); \quad \xi(t,x) = \sum_{i=1}^{N} \omega_i^2 \Im^2(t,x)$$

$$\varDelta(t,x) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \omega^i \omega^j \Big(\sum_{q=1}^{i} L^{iq} L^{jq} \Big) \Upsilon^{ij}(t,x)$$

$D(x)$ can be written in the form of a standard normal random variable $Z \sim N(0,1)$ as

$$F(x) = \Psi(t,x) + \Big(\xi(t,x) + 2\varDelta(t,x) \Big)^{\frac{1}{2}} Z \tag{4.12}$$

From equation 4.4, consider

$$\sum_{i=1}^{N} \omega^i T_t^i - K > 0 \tag{4.13}$$

This requires

$$\Big(\xi(t,x) + 2\varDelta(t,x) \Big)^{\frac{1}{2}} Z > K - \Psi(t,x)$$

$$Z > \frac{K - \Psi(t,x)}{\Big(\xi(t,x) + 2\varDelta(t,x) \Big)^{\frac{1}{2}}} := \Lambda'(t,x) \tag{4.14}$$

From Equation 4.14,

$$K = \Psi(t,x) + \Lambda'(t,x)\Big(\xi(t,x) + 2\mathit{\Delta}(t,x)\Big)^{\frac{1}{2}} \tag{4.15}$$

From equations 4.4 and 4.14,

$$\mathbb{E}_{\mathbb{Q}}\Bigg(max\Big\{\sum_{i=1}^{N}\omega^i T_t^i - C\,,\,0\Big\}dx\Big|\mathcal{F}_t\Bigg) = \int_{\Lambda'(t,x)}^{+\infty}\Big(D(x) - K\Big)\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz \tag{4.16}$$

Substituting 4.12 and 4.15,

$$\mathbb{E}_{\mathbb{Q}}\Bigg(max\Big\{\sum_{i=1}^{N}\omega^i T_t^i - K\,,\,0\Big\}dx\Big|\mathcal{F}_t\Bigg) = \int_{\Lambda'(t,x)}^{+\infty}\Bigg(\Psi(t,x) + (\xi(t,x) + 2\mathit{\Delta}(t,x))^{\frac{1}{2}}z -$$

$$\Psi(t,x) - \Lambda'(t,x)(\xi(t,x) + 2\mathit{\Delta}(t,x))^{\frac{1}{2}}\Bigg)\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz$$

$$= \int_{\Lambda^1(t,x)}^{+\infty}\Bigg((\xi(t,x) + 2\mathit{\Delta}(t,x))^{\frac{1}{2}}z - \Lambda'(t,x)\Big(\xi(t,x) + 2\mathit{\Delta}(t,x)\Big)^{\frac{1}{2}}\Bigg)\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz$$

$$=\Big(\xi(t,x) + 2\mathit{\Delta}(t,x)\Big)^{\frac{1}{2}}\Bigg(\int_{\Lambda'(t,x)}^{+\infty}\frac{ze^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz + \Lambda^1(t,x)\Phi(-\Lambda'(t,x)\Bigg)$$

$$=(\xi(t,x) + 2\mathit{\Delta}(t,x))^{\frac{1}{2}}\Bigg(\frac{e^{-\frac{1}{2}\Lambda(t,x)^2}}{\sqrt{2\pi}} + \Lambda(t,x)\Phi(\Lambda(t,x)\Bigg)$$

$$=(\xi(t,x) + 2\mathit{\Delta}(t,x))^{\frac{1}{2}}(\phi(\Lambda(t,x)) + \Lambda(t,x)\Phi(\Lambda(t,x)))$$

Therefore

$$F_{GDD}^{N}(t,t_1,t_2;D) = \int_{t_1}^{t_2}(\xi(t,x) + 2\mathit{\Delta}(t,x))^{\frac{1}{2}}(\phi(\Lambda(t,x)) + \Lambda(t,x)\Phi(\Lambda(t,x)))dx$$

where

$$\Lambda(t,x) = -\Lambda'(t,x) = \frac{\Psi(t,x) - K}{\Big(\xi(t,x) + 2\mathit{\Delta}(t,x)\Big)^{\frac{1}{2}}}$$

□

Proposition 4.5. *The price of a futures contract on basket GDD index following the shifted regime in equation 3.6 at time $t \le t_1 < t_2$ is given by*

$$F_{GDD}^{E}(t,t_1,t_2;D) = \int_{t_1}^{t_2}\Big(S(t,x) + 2Y(t,x)\Big)^{\frac{1}{2}}\Big(\phi\big(g(t,x)\big) + g(t,x)\Phi\big(g(t,x)\big)\Big)dx$$

where ϕ and Φ as their usual meaning as in Proposition 4.4.

$$g(t,x) = \frac{U(t,x) - K}{\Big(S(t,x) + 2Y(t,x)\Big)^{\frac{1}{2}}},\quad U(t,x) = \sum_{i=1}^{N}\omega^i\Big(T_{t,2}^i + (\mu+\lambda)^i\Big)$$

$$S(t,x) = \sum_{i=1}^{N} \omega_i^2 \Sigma^2(t,x) = \sum_{i=1}^{N} \omega_i^2 \sum_{j=1}^{i} \int_t^x \sigma_{2,i}^2 L_{ij}^2 du$$

$$Y(t,x) = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \omega^i \omega^j \sum_{q=1}^{i} L^{iq} L^{jq} \int_t^x \sigma_2^i \sigma_2^j du$$

Proof. The proof of Proposition 4.5 follows in the same way as the proof of Proposition 4.4 □

In summary, if the daily average temperature follows the regime-switching model in equation 3.6, equation 4.1, Proposition 4.4, and Proposition 4.5 are used to calculate the GDD futures price on the temperature basket.

5. Conclusion

In this paper, a regime-switching temperature dynamics model for spatial-temporal farming location was developed. To allow analytical tractability of the pricing models, the driving noise of the regimes were captured by a Brownian motion. Based on this model, pricing models for basket futures written on cumulative average temperature (CAT) and growing degree-days (GDD) indices were proposed. Pricing futures on temperature basket provides significant benefit as it mitigates geographical basis risks and changing of contracts relative to risk exposures of hedgers. With the proposed spatial-temporal regime-switching pricing model, investors in the weather derivative market have the opportunity to select the most appropriate composite of weather stations with their desired weight to optimize basis risk.

An extension of this research would be to use the multi-dimensional regime-switching temperature model to price basket options on futures at different locations, also called rainbow options.

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Conflict of interest

The authors declare that there is no conflict of interest.

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