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Research article

On the symmetric block design with parameters (280,63,14) admitting a Frobenius group of order 93

Menderes Gashi*

Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Prishtina, Avenue "Mother Teresa" 5, 10000 Prishtina, Kosovo

* **Correspondence:** Email: menderes_gashi@yahoo.com.

Abstract: In this paper we have proved that for a putative symmetric block design $\mathcal{D}$ with parameters (280,63,14), admitting a Frobenius group $G = \langle \rho, \mu | \rho^{31} = \mu^3 = 1, \rho^\mu = \rho^5 \rangle$ of order 93, there are exactly thirteen possible orbit structure up to isomorphism; two with the orbit distribution $[1; 31; 31; 31; 93; 93]$, eight with the orbit distribution $[1; 31; 31; 31; 31; 31; 31; 93]$ and three with the orbit distribution $[1; 31; 31; 31; 31; 31; 31; 31; 31; , 31]$.

Keywords: symmetric block design; orbit structure; automorphism group
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1. Introduction and preliminaries

A $2-(v,k,\lambda)$ design $(\mathcal{P}, \mathcal{B}, I)$ is said to be *symmetric* if the relation $|\mathcal{P}| = |\mathcal{B}| = v$ holds and in that case we often speak of a symmetric design with parameters (v,k,λ). The collection of the parameter sets (v,k,λ) for which a symmetric $2-(v,k,\lambda)$ design exists is often called the "spectrum". The determination of the spectrum for symmetric designs is a widely open problem. For example, a finite projective plane of order n is a symmetric design with parameters $(n^2+n+1, n+1, 1)$ and it is still unknown whether finite projective planes of non-prime-power order may exist at all.

The existence/non-existence of a symmetric design has often required "ad hoc" treatments even for a single parameter set (v,k,λ). The most famous instance of this circumstance is perhaps the non-existence of the projective plane of order 10, see [11].

It is of interest to study symmetric designs with additional properties, which often involve the assumption that a non-trivial automorphism group acts on the design under consideration, see for instance [4].

Among symmetric block designs of square order, a study of symmetric block designs of order 49 is of a particular interest. There are 15 possible parameters (v,k,λ) for symmetric designs of order 49, but until now only a few results are known (see [5, 8]). Due to the fact that symmetric designs of order 49 have a big number of points (blocks), the study of sporadic cases is very difficult, except, possibly, when the existence of a collineation group is assumed.

A few methods for the construction of symmetric designs are known and all of them have shown to be effective in certain situations. Here, we shall use the method of tactical decompositions, assuming that a certain automorphism group acts on the design we want to construct, used by Z. Janko in [9]; see also [4, 10] and [6]. The present paper is concerned with a symmetric design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ with parameters $(280, 63, 14)$: the existence/non-existence of such a design is still in doubt as far as we know. We shall further assume that the given design admits a certain automorphism group of order 93. We assume the reader is familiar with the basic facts of design theory, see for instance [2], [3] and [12]. If g is an automorphism of a symmetric design $\mathcal{D}$ with parameters (v, k, λ), then g fixes an equal number of points and blocks, see [12, Theorem 3.1, p.78]. We denote the sets of these fixed elements by $F_{\mathcal{P}}(g)$ and $F_{\mathcal{B}}(g)$ respectively, and their cardinality simply by $|F(g)|$. We shall make use of the following upper bound for the number of fixed points, see [12, Corollary 3.7, p. 82]:

$$|F(g)| \leq k + \sqrt{k - \lambda}. \tag{1}$$

It is also known that an automorphism group G of a symmetric design has the same number of orbits on the set of points $\mathcal{P}$ as on the set of blocks $\mathcal{B}$: [12, Theorem 3.3, p.79]. Denote that number by t.

2. Point- and block-orbits

We adopt the notation and terminology of Section 1 in [4]. In the following, for the sake of completeness, some fundamental relations are explicitly provided. Let $\mathcal{D}$ be a symmetric design with parameters (v, k, λ) and let G be a subgroup of the automorphism group $Aut(\mathcal{D})$ of $\mathcal{D}$. Denote the point orbits of G on $\mathcal{P}$ by $\mathcal{P}_1, \mathcal{P}_2, \ldots \mathcal{P}_t$ and the line orbits of G on $\mathcal{B}$ by $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_t$. Put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Obviously,

$$\sum_{r=1}^{t} \omega_r = \sum_{i=1}^{t} \Omega_i = v. \tag{2}$$

Let γ_{ir} be the number of points from $\mathcal{P}_r$, which lie on a line from $\mathcal{B}_i$; clearly this number does not depend on the chosen line. Similarly, let Γ_{js} be the number of lines from $\mathcal{B}_j$ which pass through a point from $\mathcal{P}_s$. Then, obviously,

$$\sum_{r=1}^{t} \gamma_{ir} = k \text{ and } \sum_{j=1}^{t} \Gamma_{js} = k. \tag{3}$$

By [3, Lemma 5.3.1. p.221], the partition of the point set $\mathcal{P}$ and of the block set $\mathcal{B}$ forms a tactical decomposition of the design $\mathcal{D}$ in the sense of [3, p.210]. Thus, the following equations hold:

$$\Omega_i \cdot \gamma_{ir} = \omega_r \cdot \Gamma_{ir} \tag{4}$$

$$\sum_{r=1}^{t} \gamma_{ir} \Gamma_{jr} = \lambda \Omega_j + \delta_{ij}(k - \lambda) \tag{5}$$

$$\sum_{i=1}^{t} \Gamma_{ir}\gamma_{is} = \lambda\omega_s + \delta_{rs}(k - \lambda) \tag{6}$$

where δ_{ij}, δ_{rs} are the Kronecker symbols.

For a proof of these equations, the reader is referred to [3] and [4]. Equation (5), together with (4) yields

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k - \lambda). \tag{7}$$

Definition 1. *We denote*

$$[L_i, L_j] = \sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr}, 1 \leq i, j \leq t$$

and call these expressions the orbit products. The $(t \times t)$*-matrix* (γ_{ir}) *is called the orbit structure of the design* $\mathcal{D}$.

The first step in the construction of a design is to find all possible orbit structures. The second step of the construction is usually called indexing. In fact for each coefficient γ_{ir} of the orbit matrix one has to specify which γ_{ir} points of the point orbit $\mathcal{P}_r$ lie on the lines of the block orbit $\mathcal{B}_i$. Of course, it is enough to do this for a representative of each block orbit, as the other lines of that orbit can be obtained by producing all G-images of the given representative.

3. Action of the Frobenius group of order 93

In our construction of symmetric $2-(280, 63, 14)$ designs we assume the existence of an automorphism group $G = \langle \rho, \mu | \rho^{31} = \mu^3 = 1, \rho^{\mu} = \rho^5 \rangle$, which is a so called Frobenius group of order 93 with Frobenius kernel of order 31 (see [7]).

Lemma 3.1. *Let* ρ *be an element of G with* $o(\rho) = 31$. *Then* $\langle \rho \rangle$ *fixes precisely one point and one block.*

Proof. By [12, Theorem 3.1] the group $\langle \rho \rangle$ fixes the same number of points and blocks. Denote that number by f. Obviously $f \equiv 280 (\mathrm{mod}\ 31)$, i.e. $f \equiv 1 (\mathrm{mod}\ 31)$. The upper bound (1) for the number of fixed points yields $f \in \{1, 32, 63\}$. As $o(\rho) > \lambda$, an application of a result of M. Aschbacher [1, Lemma 2.6, p.274] forces the fixed structure to be a subdesign of $\mathcal{D}$. But there is no symmetric design with $v = 32$ or $v = 63$ and $\lambda = 14$ (there is no $k \in \mathbb{N}$ which satisfy $14 \cdot (v - 1) = k \cdot (k - 1)$). Hence, f is equal to 1. □

Our next task is to determine the lengths of the orbits of G on the sets of points and blocks of the symmetric block design $\mathcal{D}$. The possible orbit lengths are $1, 3, 31, 93$.

Lemma 3.2. *There is no orbit of length 3 of G.*

Proof. If false, then ρ would have at least three fixed points or three fixed blocks, which is not possible. □

Up to reordering, there are precisely four possibilities for the arrays expressing the lengths of the G-orbits on points and blocks, namely: $O_1 = [1; 93; 93; 93]$; $O_2 = [1; 31; 31; 31; 93; 93]$; $O_3 = [1; 31; 31; 31; 31; 31; 31; 93]; O_4 = [1; 31; 31; 31; 31; 31; 31; 31; 31; 31]$:

Lemma 3.3. *The case* $O_1 = [1; 93; 93; 93]$ *of the orbit distribution does not occur.*

Since for the case $O_1 = [1; 93; 93; 93]$ it is not possible to be constructed the fixed block, the following lemmas follow.

Lemma 3.4. *Up to isomorphism there are exactly two orbit structures for symmetric (280,63,14) designs and the automorphism group* $F_{31\cdot 3}$ *acting with the orbit distribution* $O_2 = [1; 31; 31; 31; 93; 93]$.

Proof. We put $\mathcal{P}_I = \{I_0, I_1, \cdots, I_{30}\}, I = 1, 2, 3$, $\mathcal{P}_4 = \{4_0, 4_1, \cdots, 4_{30}, 5_0, 5_1, \cdots, 5_{30}, 6_0, 6_1, \cdots, 6_{30}\}$, $\mathcal{P}_5 = \{7_0, 7_1, \cdots, 7_{30}, 8_0, 8_1, \cdots, 8_{30}, 9_0, 9_1, \cdots, 9_{30}\}$, for the non–trivial orbits of the group G. Thus, G acts on these point orbits as a permutation group in a unique way. Hence, for the two generators of G we may put

$$\rho = (\infty)(I_0, I_1, \cdots, I_{30}), I = 1, 2, \cdots, 9,$$

where ∞ is the fixed point of collineation, whereas non-trivial $\langle\rho\rangle$−orbits are numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ and $\infty, 1_0, 1_1, \cdots, 9_{30}$ all points of the symmetric block design $\mathcal{D}$, and the collineation μ of order 3 acts in the symmetric block design as permutation $(1)(2)(3)(4, 5, 6)(7, 8, 9)$ on orbit numbers, whereas on indices acts $\mu : x \to 5x \pmod{31}$ or

$$\mu = (\infty)(K_0)(K_1, K_5, K_{25})(K_2, K_{10}, K_{19})(K_3, K_{15}, K_{13})(K_4, K_{20}, K_7)$$

$$(K_6, K_{30}, K_{26})(K_8, K_9, K_{14})(K_{11}, K_{24}, K_{27})(K_{12}, K_{29}, K_{21})(K_{16}, K_{18}, K_{28})$$

$$(K_{17}, K_{23}, K_{22})(4_i, 5_{5i}, 6_{25i})(7_i, 8_{5i}, 9_{25i}), \quad K = 1, 2, 3, i = 0, \cdots, 30.$$

We immediately obtain the following.

Corollary 1. *The element* μ *of* G *of order* 3 *fixes precisely* 4 *points and* 4 *blocks of* $\mathcal{D}$. *Each block orbit contains a unique block stabilized by* μ.

In what follows, we are going to construct a representative block for each block orbit. A representative block for the block orbit of length 31 will be the block fixed by μ. Hence the multiplicities of orbit numbers in orbit blocks, corresponding to point and block orbit of length 31, will be $\equiv 0, 1 \pmod 3$.

The $\langle\rho\rangle$−fixed block can be written in the form:

$$L_1 = \infty(1_0 1_1 \cdots 1_{30})(2_0 2_1 \cdots 2_{30})$$

or

$$L_1 = \infty 1_{31} 2_{31}.$$

Let L_2, L_3, L_4, L_5, L_6 be the representative blocks for the five non–trivial block orbits. There are exactly two non–fixed orbit blocks passing through the fixed point ∞. Let them be L_2, L_3. We write

$L_2 = \infty 1_{a_1} 2_{a_2} 3_{a_3} 4_{a_4} 5_{a_5}$

$$L_3 = \infty 1_{b_1} 2_{b_2} 3_{b_3} 4_{b_4} 5_{b_5}$$

where a_i, b_i denote the multiplicities of the appearance of orbit numbers in the orbit blocks L_2, L_3, respectively.

The multiplicities of the appearance of orbit numbers satisfy the following conditions:

$$a_1 + a_2 + a_3 + a_4 + a_5 = 62,$$

$$b_1 + b_2 + b_3 + b_4 + b_5 = 62.$$

Because $|L_i \cap L_1| = 14, i = 2,3$ and $\infty \in L_i, i = 1,2,3$ we have $a_1 + a_2 = 13$, $b_1 + b_2 = 13$, and consequently $a_3 + a_4 + a_5 = 49, b_3 + b_4 + b_5 = 49$. From (7) we have

$$[L_2, L_2] = 31/1 \cdot 1 \cdot 1 + 31/31 \cdot a_1^2 + 31/31 \cdot a_2^2 + 31/31 \cdot a_3^2 + 31/93 \cdot a_4^2 + 31/93 \cdot a_5^2$$

$$= 14 \cdot 31 + 63 - 14 = 483$$

$$[L_3, L_3] = 31/1 \cdot 1 \cdot 1 + 31/31 \cdot b_1^2 + 31/31 \cdot b_2^2 + 31/31 \cdot b_3^2 + 31/93 \cdot b_4^2 + 31/93 \cdot b_5^2$$

$$= 14 \cdot 31 + 63 - 14 = 483$$

$$[L_3, L_2] = 31 \cdot 1 \cdot 1 + 31/31 \cdot a_1 \cdot b_1 + 31/31 \cdot a_2 \cdot b_2 + 31/31 \cdot a_3 \cdot b_3 + 31/93 \cdot a_4 b_4 + 31/93 \cdot a_5 b_5$$

$$= 14 \cdot 31 = 434$$

where $0 \le a_i \le 13, i = 1,2, 0 \le a_3 \le 21, 0 \le a_i \le 38, i = 4,5$.

In order to reduce isomorphic cases that may appear in the orbit structures at the last stage, without loss of generality, for block L_2, we can use the reduction $a_1 \ge a_2, a_4 \ge a_5$.

Using the computer we have proved that there exist only six different orbit types for the block L_2 satisfying the above mentioned conditions:

	a_1	a_2	a_3	a_4	a_5
1.	10	3	7	21	21
2.	9	4	10	21	18
3.	9	4	4	24	21
4.	7	6	10	24	15
5.	7	6	7	27	15
6.	7	6	4	27	18

Further on, acting with the Frobenius group $G = F_{31\cdot 3}$, for orbit block L_3 we have:

Table 1. Triples $\{L_1, L_2, L_3\}$.

Block L_2	Number of orbit types for L_3	Number of triples $\{L_1, L_2, L_3\}$
Type 1.	1	1
Type 2.	1	1
Type 3.	1	1
Type 4.	1	1
Type 5.	1	1
Type 6.	1	1

The fourth orbit block L_4 has the form:

$$L_4 = 1_{c_1} 2_{c_2} 3_{c_3} 4_{c_4} 5_{c_5}$$

where $c_i, i = 1, 2, \cdots, 5$ are multiplicities of the appearance of orbit numbers 1,2,3, 4 and 5 in orbit block L_4.

We have: $c_1 + c_2 + c_3 + c_4 + c_5 = 63$,

$$[L_4, L_4] = c_1^2 + c_2^2 + c_3^2 + 1/3c_4^2 + 1/3c_5^2 = 14 \cdot 31 + 63 - 14 = 483,$$

$$[L_4, L_i] = 14 \cdot 31 = 434, (i = 2, 3).$$

$[L_4 \cap L_1] = 14$ implies $c_1 + c_2 = 14$, therefore $c_3 + c_4 + c_5 = 63 - 14 = 49$.

$[L_4, L_4] = 483$ implies $0 \leq c_3 \leq 21$, and $0 \leq c_i \leq 38, i = 4, 5$, whereas $c_1 + c_2 = 14$ implies $0 \leq c_i \leq 14, i = 1, 2$.

Further on, acting with the Frobenius group $G = F_{31 \cdot 3}$, for the orbit block L_4, for the number of triples L_1, L_2, L_3 given in Table 1, we have:

Table 2. Triples $\{L_2, L_3, L_4\}$.

Block L_2	Number of doubles $\{L_2, L_3\}$	Number of triples $\{L_2, L_3, L_4\}$
Type 1.	1	2
Type 2.	1	2
Type 3.	1	2
Type 4.	1	2
Type 5.	1	2
Type 6.	1	2

Therefore, we have found twelve compatible triples L_2, L_3, L_4, respectively twelve compatibile quadruples L_1, L_2, L_3, L_4.

Let $L_5 = 1_{d_1} 2_{d_2} 3_{d_3} 4_{d_4} 5_{d_5}$ be the fifth orbit block, where $d_i, i = 1, 2, \cdots, 5$ denote the multiplicities of the appearance of orbit numbers in the block L_5.

We have:

$$d_1 + d_2 + d_3 + d_4 + d_5 = 63,$$

$$[L_5, L_5] = 93/31d_1^2 + 93/31d_2^2 + 93/31d_3^2 + 93/93d_4^2 + 93/93d_5^2 = 14 \cdot 93 + 63 - 14 = 1351,$$

$$[L_5, L_i] = 14 \cdot 93 = 1302, (i = 2, 3, 4).$$

$|L_5 \cap L_1| = 14$ implies $d_1 + d_2 = 14$, therefore $d_3 + d_4 + d_5 = 49$.

$[L_5, L_5] = 1351$ implies $0 \leq d_3 \leq 21$, and $0 \leq d_i \leq 36, i = 4, 5$, whereas $d_1 + d_2 = 14$ implies $0 \leq d_i \leq 14, i = 1, 2$.

Further on, acting with the Frobenius group $G = F_{31 \cdot 3}$, for the orbit block L_5, for the number of triples L_2, L_3, L_4 given in Table 2, we have:

Table 3. Orbit types for L_5.

Quadruples $\{L_1, L_2, L_3, L_4\}$	Number of orbit types for L_5
Case 1.	1
Case 2.	1
Case 3.	1
Case 4.	1
Case 5.	1
Case 6.	1
Case 7.	2
Case 8.	1
Case 9.	1
Case 10.	2
Case 11.	1
Case 12.	1

Obviously, among blocks L_5 are also blocks L_6. Because of that, we choose doubles among candidates for the block L_5, such that every couple of them satisfies the intersection in 14 points. Based on this fact we have found that, from Case 7. and Case 10. in Table 3. for the number of orbit types for L_5, up to isomorphism, there are exactly two orbit structures:

Table 4. Orbit structures.

OS1.	1	31	31	31	93	93
	1	31	31	0	0	0
	1	7	6	10	24	15
	1	6	7	4	18	27
	0	10	4	10	15	24
	0	8	6	5	24	20
	0	5	9	8	20	21

OS2.	1	31	31	31	93	93
	1	31	31	0	0	0
	1	7	6	7	27	15
	1	6	7	7	15	27
	0	7	7	1	24	24
	0	9	5	8	20	21
	0	5	9	8	21	20

□

Lemma 3.5. *Up to isomorphism there are exactly eight orbit structures for symmetric (280,63,14) designs and the automorphism group* $F_{31\cdot 3}$ *acting with the orbit distribution* $O_3 = [1; 31; 31; 31; 31; 31; 31; 93]$.

Proof. We put $\mathcal{P}_I = \{I_0, I_1, \cdots, I_{30}\}, I = 1, 2, 3, 4, 5, 6,$ $\mathcal{P}_7 = \{7_0, 7_1, \cdots, 7_{30}, 8_0, 8_1, \cdots, 8_{30}, 9_0, 9_1, \cdots, 9_{30}\}$, for the non–trivial orbits of the group G. Thus, G acts on these point orbits as a permutation group in a unique way. Hence, for the two generators of G we may put

$$\rho = (\infty)(I_0, I_1, \cdots, I_{30}), I = 1, 2, \cdots, 9,$$

where ∞ is the fixed point of collineation, whereas non–trivial $\langle\rho\rangle$−orbits are numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ and $\infty, 1_0, 1_1, \cdots, 9_{30}$ all points of the symmetric block design $\mathcal{D}$, and the collineation μ of order 3 acts in the symmetric block design as permutation $(1)(2)(3)(4)(5)(6)(7, 8, 9)$ on orbit numbers, whereas on indices acts $\mu : x \to 5x \pmod{31}$ or

$$\mu = (\infty)(K_0)(K_1, K_5, K_{25})(K_2, K_{10}, K_{19})(K_3, K_{15}, K_{13})(K_4, K_{20}, K_7)$$
$$(K_6, K_{30}, K_{26})(K_8, K_9, K_{14})(K_{11}, K_{24}, K_{27})(K_{12}, K_{29}, K_{21})(K_{16}, K_{18}, K_{28})$$
$$(K_{17}, K_{23}, K_{22})(7_i, 8_{5i}, 9_{25i}), \quad K = 1, 2, 3, 4, 5, 6, i = 0, \cdots, 30.$$

We immediately obtain the following.

Corollary 2. *The element μ of G of order 3 fixes precisely 7 points and 7 blocks of $\mathcal{D}$. Each block orbit contains a unique block stabilized by μ.*

In what follows, we are going to construct a representative block for each block orbit. A representative block for the block orbit of length 31 will be the block fixed by μ. Hence the multiplicities of orbit numbers in orbit blocks, corresponding to point and block orbit of length 31, will be $\equiv 0, 1 \pmod 3$.

The $\langle\rho\rangle$−fixed block can be writen in the form:

$$L_1 = \infty(1_0 1_1 \cdots 1_{30})(2_0 2_1 \cdots 2_{30})$$

or

$$L_1 = \infty 1_{31} 2_{31}.$$

Let $L_2, L_3, L_4, L_5, L_6, L_7, L_8$ be the representative blocks for the seven non–trivial block orbits. There are exactly two non–fixed orbit blocks passing through the fixed point ∞. Let them be L_2, L_3. We write

$$L_2 = \infty 1_{a_1} 2_{a_2} 3_{a_3} 4_{a_4} 5_{a_5} 6_{a_6} 7_{a_7}$$
$$L_3 = \infty 1_{b_1} 2_{b_2} 3_{b_3} 4_{b_4} 5_{b_5} 6_{b_6} 7_{b_7}$$

where a_i, b_i denote the multiplicities of the appearance of orbit numbers in the orbit blocks L_2, L_3, respectively.

The multiplicities of the appearance of orbit numbers satisfy the following conditions:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 62,$$
$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = 62.$$

Because $|L_i \cap L_1| = 14, i = 2, 3$ and $\infty \in L_i, i = 1, 2, 3$ we have $a_1 + a_2 = 13$, $b_1 + b_2 = 13$, and consequently $a_3 + a_4 + a_5 + a_6 + a_7 = 49, b_3 + b_4 + b_5 + b_6 + b_7 = 49$. From (7) we have

$$[L_2, L_2] = 31/1 \cdot 1 \cdot 1 + 31/31 \cdot a_1^2 + 31/31 \cdot a_2^2 + 31/31 \cdot a_3^2 + 31/31 \cdot a_4^2 + 31/31 \cdot a_5^2$$

$$+31/31 \cdot a_6^2 + 31/93 \cdot a_7^2 = 14 \cdot 31 + 63 - 14 = 483$$

$$[L_3, L_3] = 31/1 \cdot 1 \cdot 1 + 31/31 \cdot b_1^2 + 31/31 \cdot b_2^2 + 31/31 \cdot b_3^2 + 31/31 \cdot b_4^2 + 31/31 \cdot b_5^2$$

$$+31/31 \cdot b_6^2 + 31/93 \cdot b_7^2 = 14 \cdot 31 + 63 - 14 = 483$$

$$[L_3, L_2] = 31 \cdot 1 \cdot 1 + 31/31 \cdot a_1 \cdot b_1 + 31/31 \cdot a_2 \cdot b_2 + 31/31 \cdot a_3 \cdot b_3 + 31/31 \cdot a_4 b_4 + 31/31 \cdot a_5 \cdot b_5$$

$$+31/31 \cdot a_6 b_6 + 31/93 \cdot a_7 b_7 = 14 \cdot 31 = 434$$

where $0 \leq a_i \leq 13, i = 1, 2, 0 \leq a_i \leq 21, i = 3, 4, 5, 6, 0 \leq a_7 \leq 38$.

In order to reduce isomorphic cases that may appear in the orbit structures at the last stage, without loss of generality, for block L_2, we can use the reduction $a_1 \geq a_2, a_3 \geq a_4 \geq a_5 \geq a_6$.

Using the computer we have proved that there exist only ten different orbit types for the block L_2 satisfying the above mentioned conditions:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
1.	10	3	7	7	7	7	21
2.	9	4	10	7	7	7	18
3.	9	4	10	6	6	6	21
4.	9	4	9	9	7	6	18
5.	9	4	7	7	7	4	24
6.	7	6	9	9	9	7	15
7.	7	6	9	9	9	4	18
8.	7	6	9	9	7	3	21
9.	7	6	9	7	6	3	24
10.	7	6	6	6	6	4	27

Further on, acting with the Frobenius group $G = F_{31 \cdot 3}$, for orbit block L_3 we have:

Table 5. Triples $\{L_1, L_2, L_3\}$.

Block L_2	Number of orbit types for L_3	Number of triples $\{L_1, L_2, L_3\}$
Type 1.	1	1
Type 2.	1	1
Type 3.	0	0
Type 4.	0	0
Type 5.	1	1
Type 6.	0	0
Type 7.	0	0
Type 8.	0	0
Type 9.	0	0
Type 10.	0	0

The fourth orbit block L_4 has the form:

$$L_4 = 1_{c_1} 2_{c_2} 3_{c_3} 4_{c_4} 5_{c_5} 6_{c_6} 7_{c_7}$$

where $c_i, i = 1, 2, \cdots, 7$ are multiplicities of the appearance of orbit numbers 1,2,3, 4,5,6 and 7 in orbit block L_4.

We have: $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = 63$,

$$[L_4, L_4] = c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2 + 1/3c_7^2 = 14 \cdot 31 + 63 - 14 = 483,$$

$$[L_4, L_i] = 14 \cdot 31 = 434, (i = 2, 3).$$

$[L_4 \cap L_1] = 14$ implies $c_1 + c_2 = 14$, therefore $c_3 + c_4 + c_5 + c_6 + c_7 = 63 - 14 = 49$.

$[L_4, L_4] = 483$ implies $0 \le c_i \le 21, i = 3, 4, 5, 6$, and $0 \le c_7 \le 38$, whereas $c_1 + c_2 = 14$ implies $0 \le c_i \le 14, i = 1, 2$.

Further on, acting with the Frobenius group $G = F_{31 \cdot 3}$, for the orbit block L_4, for the number of triples L_1, L_2, L_3 given in Table 5, we have:

Table 6. Quadruples $\{L_1, L_2, L_3, L_4\}$.

Triple L_1, L_2, L_3	Number of orbit types for L_4	Number of quadruples $\{L_1, L_2, L_3, L_4\}$
Case 1. (Type 1 for L_2)	100	100
Case 2. (Type 2 for L_2)	28	28
Case 3. (Type 5 for L_2)	28	28

Note that in set of possible candidates for the orbit block L_4 are also orbit blocks L_5, L_6, L_7, because they satisfy the same conditions. Therefore, we investigate quadruples of blocks $\{L_4, L_5, L_6, L_7\}$ which are pairwise compatible. In this way, by computer, for all three cases for the number of orbit types for L_4 given in Table 6, we find quadruples $\{L_4, L_5, L_6, L_7\}$, respectively septuples $\{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ and have:

Table 7. Septuples $\{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$.

Triple L_1, L_2, L_3	Number of septuples $\{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$
Case 1.	57
Case 2.	15
Case 3.	15

The eighth orbit block L_8 has the form:

$$L_8 = 1_{d_1} 2_{d_2} 3_{d_3} 4_{d_4} 5_{d_5} 6_{d_6} 7_{d_7}$$

where $d_i, i = 1, 2, \cdots, 7$ are multiplicity of the appearance of orbit numbers 1,2,3, 4,5,6 and 7 in orbit block L_8.

We have: $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 = 63$,

$$[L_8, L_8] = 3 \cdot d_1^2 + 3 \cdot d_2^2 + 3 \cdot d_3^2 + 3 \cdot d_4^2 + 3 \cdot d_5^2 + 3 \cdot d_6^2 + d_7^2 = 14 \cdot 93 + 63 - 14 = 1351,$$

$$[L_8, L_i] = 14 \cdot 93 = 1302, (i = 2, 3, 4, 5, 6, 7).$$

$[L_8 \cap L_1] = 14$ implies $d_1 + d_2 = 14$, therefore $d_3 + d_4 + d_5 + d_6 + d_7 = 63 - 14 = 49$.

$[L_8, L_8] = 1351$ implies $0 \leq d_i \leq 21, i = 3, 4, 5, 6$, and $0 \leq d_7 \leq 36$, whereas $d_1 + d_2 = 14$ implies $0 \leq d_i \leq 14, i = 1, 2$.

Further on, acting with the Frobenius group $G = F_{31\cdot 3}$, for the number of septuples $\{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$ given in Table 7. we find orbit block L_8. By computer we found, up to isomorphism, exactly eight orbit structure:

Table 8. Orbit structures.

OS3.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	10	3	7	7	7	7	21
	1	3	10	7	7	7	7	21
	0	7	7	13	6	6	6	18
	0	7	7	6	13	6	6	18
	0	7	7	6	6	13	6	18
	0	7	7	6	6	6	13	18
	0	7	7	6	6	6	6	25

OS4.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	10	3	7	7	7	7	21
	1	3	10	7	7	7	7	21
	0	7	7	13	6	6	6	18
	0	7	7	6	12	9	7	15
	0	7	7	6	9	4	3	27
	0	7	7	6	7	3	12	21
	0	7	7	6	5	9	7	22

OS5.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	10	3	7	7	7	7	21
	1	3	10	7	7	7	7	21
	0	7	7	13	6	6	6	18
	0	7	7	6	12	9	4	18
	0	7	7	6	9	4	12	18
	0	7	7	6	4	12	9	18
	0	7	7	6	6	6	6	25

OS6.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	10	3	7	7	7	7	21
	1	3	10	7	7	7	7	21
	0	7	7	12	9	7	6	15
	0	7	7	9	3	6	4	27
	0	7	7	7	6	3	12	21
	0	7	7	6	4	12	9	18
	0	7	7	5	9	7	6	22

OS7.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	9	4	10	7	7	7	18
	1	4	9	4	7	7	7	24
	0	10	4	4	6	6	6	27
	0	7	7	6	13	6	6	18
	0	7	7	6	6	13	6	18
	0	7	7	6	6	6	13	18
	0	6	8	9	6	6	6	22

OS8.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	9	4	10	7	7	7	18
	1	4	9	4	7	7	7	24
	0	10	4	4	6	6	6	27
	0	7	7	6	12	9	4	18
	0	7	7	6	9	4	12	18
	0	7	7	6	4	12	9	18
	0	6	8	9	6	6	6	22

OS9.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	9	4	10	7	7	7	18
	1	4	9	4	7	7	7	24
	0	10	4	3	9	7	6	24
	0	7	7	9	4	3	6	27
	0	7	7	7	3	12	6	21
	0	7	7	6	6	6	13	18
	0	6	8	8	9	7	6	19

OS10.	1	31	31	31	31	31	31	93
	1	31	31	0	0	0	0	0
	1	9	4	10	7	7	7	18
	1	4	9	4	7	7	7	24
	0	10	4	3	9	7	6	24
	0	7	7	9	3	6	4	27
	0	7	7	7	6	3	12	21
	0	7	7	6	4	12	9	18
	0	6	8	8	9	7	6	19

□

Lemma 3.6. *Up to isomorphism there are exactly three orbit structures for symmetric (280,63,14) designs and the automorphism group* $F_{31\cdot 3}$ *acting with the orbit distribution* $\mathcal{O}_4 = [1; 31; 31; 31; 31; 31; 31; 31, 31, 31]$.

Proof. We put $\mathcal{P}_I = \{I_0, I_1, \cdots, I_{30}\}, I = 1,2,3,4,5,6,7,8,9$, for the non–trivial orbits of the group G. Thus, G acts on these point orbits as a permutation group in a unique way. Hence, for the two generators of G we may put

$$\rho = (\infty)(I_0, I_1, \cdots, I_{30}), I = 1,2,\cdots,9,$$

where ∞ is the fixed point of collineation, whereas non–trivial $\langle\rho\rangle$−orbits are numbers $1,2,3,4,5,6,7,\ 8,9$ and $\infty, 1_0, 1_1, \cdots, 9_{30}$ all points of the symmetric block design $\mathcal{D}$, and the collineation μ of order 3 acts in the symmetric block design as permutation (1)(2)(3)(4)(5)(6)(7)(8)(9) on orbit numbers, whereas on indices acts $\mu : x \to 5x \pmod{31}$ or

$$\mu = (\infty)(K_0)(K_1, K_5, K_{25})(K_2, K_{10}, K_{19})(K_3, K_{15}, K_{13})(K_4, K_{20}, K_7)$$
$$(K_6, K_{30}, K_{26})(K_8, K_9, K_{14})(K_{11}, K_{24}, K_{27})(K_{12}, K_{29}, K_{21})$$
$$(K_{16}, K_{18}, K_{28})(K_{17}, K_{23}, K_{22}), \quad K = 1,2,3,4,5,6,7,8,9.$$

We immediately obtain the following.

Corollary 3. *The element* μ *of* G *of order* 3 *fixes precisely* 10 *points and* 10 *blocks of* $\mathcal{D}$*. Each block orbit contains a unique block stabilized by* μ*.*

In what follows, we are going to construct a representative block for each block orbit. A representative block for the block orbit of length 31 will be the block fixed by μ. Hence the multiplicities of orbit numbers in orbit blocks, will be $\equiv 0, 1 \pmod 3$.

The $\langle\rho\rangle$−fixed block can be writen in the form:

$$L_1 = \infty(1_0 1_1 \cdots 1_{30})(2_0 2_1 \cdots 2_{30})$$

or

$$L_1 = \infty 1_{31} 2_{31}.$$

Let $L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}$ be the representative blocks for the nine non–trivial block orbits. There are exactly two non–fixed orbit blocks passing through the fixed point ∞. Let them be L_2, L_3. We write

$$L_2 = \infty 1_{a_1} 2_{a_2} 3_{a_3} 4_{a_4} 5_{a_5} 6_{a_6} 7_{a_7} 8_{a_8} 9_{a_9}$$
$$L_3 = \infty 1_{b_1} 2_{b_2} 3_{b_3} 4_{b_4} 5_{b_5} 6_{b_6} 7_{b_7} 8_{b_8} 9_{b_9}$$

where a_i, b_i denote the multiplicities of the appearance of orbit numbers in the orbit blocks L_2, L_3, respectively.

The multiplicities of the appearances of orbit numbers satisfy the following conditions:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 62.$$
$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 = 62.$$

Because $|L_i \cap L_1| = 14, i = 2,3$ and $\infty \in L_i, i = 1,2,3$ we have $a_1 + a_2 = 13$, $b_1 + b_2 = 13$, and consequently $a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 49, b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 = 49$. From (7) we have

$$[L_2, L_2] = 31/1 \cdot 1 \cdot 1 + 31/31 \cdot a_1^2 + 31/31 \cdot a_2^2 + 31/31 \cdot a_3^2 + 31/31 \cdot a_4^2 + 31/31 \cdot a_5^2 + 31/31 \cdot a_6^2$$
$$+31/31 \cdot a_7^2 + 31/31 \cdot a_8^2 + 31/31 \cdot a_9^2 = 14 \cdot 31 + 63 - 14 = 483$$
$$[L_3, L_3] = 31/1 \cdot 1 \cdot 1 + 31/31 \cdot b_1^2 + 31/31 \cdot b_2^2 + 31/31 \cdot b_3^2 + 31/31 \cdot b_4^2 + 31/31 \cdot b_5^2 + 31/31 \cdot b_6^2$$
$$+31/31 \cdot b_7^2 + 31/31 \cdot b_8^2 + 31/31 \cdot b_9^2 = 14 \cdot 31 + 63 - 14 = 483$$
$$[L_3, L_2] = 31 \cdot 1 \cdot 1 + 31/31 \cdot a_1 \cdot b_1 + 31/31 \cdot a_2 \cdot b_2 + 31/31 \cdot a_3 \cdot b_3 + 31/31 \cdot a_4 b_4 + 31/31 \cdot a_5 \cdot b_5$$
$$+31/31 \cdot a_6 b_6 + 31/31 \cdot a_7 b_7 + 31/31 \cdot a_8 b_8 + 31/31 \cdot a_9 b_9 = 14 \cdot 31 = 434$$

where $0 \le a_i \le 13, i = 1,2, 0 \le a_i \le 21, i = 3,4,\cdots,9$.

In order to reduce isomorphic cases that may appear in the orbit structures at the last stage, without loss of generality, for block L_2, we can use the reduction $a_1 \ge a_2, a_3 \ge a_4 \ge a_5 \ge a_6 \ge a_7 \ge a_8 \ge a_9$.

Using the computer we have proved that there exist only six different orbit types for the block L_2 satisfying the above mentioned conditions:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1.	10	3	7	7	7	7	7	7	7
2.	9	4	10	7	7	7	6	6	6
3.	9	4	9	9	7	6	6	6	6
4.	7	6	10	9	7	7	6	6	4
5.	7	6	9	9	9	6	6	6	4
6.	7	6	9	9	7	7	7	7	3

Further on, acting with the Frobenius group $G = F_{31\cdot 3}$, for orbit block L_3 we have:

Table 9. Triples $\{L_1, L_2, L_3\}$.

Block L_2	Number of orbit types for L_3	Number of triples $\{L_1, L_2, L_3\}$
Type 1.	1	1
Type 2.	0	0
Type 3.	0	0
Type 4.	0	0
Type 5.	0	0
Type 6.	0	0

Hence, we have only one double L_2, L_3, respectively only one triple L_1, L_2, L_3:

	1	31	31	31	31	31	31	31	31	31
$L1$	1	31	31	0	0	0	0	0	0	0
$L2$	1	10	3	7	7	7	7	7	7	7
$L3$	1	3	10	7	7	7	7	7	7	7

The fourth orbit block L_4 has the form:

$$L_4 = 1_{c_1}2_{c_2}3_{c_3}4_{c_4}5_{c_5}6_{c_6}7_{c_7}8_{c_8}9_{c_9}$$

where $c_i, i = 1, 2, \cdots, 9$ are multiplicities of the appearance of orbit numbers 1,2,3, 4,5,6,7,8 and 9 in orbit block L_4.

We have: $c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 = 63$,

$$[L_4, L_4] = c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2 + c_8^2 + c_9^2 = 14 \cdot 31 + 63 - 14 = 483,$$

$$[L_4, L_i] = 14 \cdot 31 = 434, (i = 2, 3).$$

$[L_4 \cap L_1] = 14$ implies $c_1 + c_2 = 14$, therefore $c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 = 63 - 14 = 49$.

$[L_4, L_4] = 483$ implies $0 \leq c_i \leq 21, i = 3, 4, 5, 6, 7, 8, 9$, whereas $c_1 + c_2 = 14$ implies $0 \leq c_i \leq 14, i = 1, 2$.

Using the computer we have proved that for the number of triples L_1, L_2, L_3 given in Table 9., there exist exactly 2527 dfferent orbit types for the block L_4 satisfying the above mentioned conditions:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
1.	7	7	13	6	6	6	6	6	6
2.	7	7	12	9	6	6	6	6	4
3.	7	7	12	9	6	6	6	4	6
...									
2525.	7	7	3	4	7	7	9	10	9
2526.	7	7	3	4	7	7	9	9	10
2527.	7	7	3	4	6	9	9	9	9

Note that in set of possible candidates for the orbit block L_4 are also orbit blocks L_5, L_6, L_7, L_8 and L_9, because they satisfy the same conditions. Therefore, we investigate sextuples of blocks

$\{L_4, L_5, L_6, L_7, L_8, L_9\}$ which are pairwise compatible. In this way, by computer, we find sextuples $\{L_4, L_5, L_6, L_7, L_8, L_9\}$, respectively all orbit structures. Thus, up to isomorphism, we have exactly three orbit structure:

Table 10. Orbit structures.

OS11.	1	31	31	31	31	31	31	31	31	31
	1	31	31	0	0	0	0	0	0	0
	1	10	3	7	7	7	7	7	7	7
	1	3	10	7	7	7	7	7	7	7
	0	7	7	13	6	6	6	6	6	6
	0	7	7	6	13	6	6	6	6	6
	0	7	7	6	6	13	6	6	6	6
	0	7	7	6	6	6	13	6	6	6
	0	7	7	6	6	6	6	13	6	6
	0	7	7	6	6	6	6	6	13	6
	0	7	7	6	6	6	6	6	6	13

OS12.	1	31	31	31	31	31	31	31	31	31
	1	31	31	0	0	0	0	0	0	0
	1	10	3	7	7	7	7	7	7	7
	1	3	10	7	7	7	7	7	7	7
	0	7	7	13	6	6	6	6	6	6
	0	7	7	6	13	6	6	6	6	6
	0	7	7	6	6	13	6	6	6	6
	0	7	7	6	6	6	13	6	6	6
	0	7	7	6	6	6	6	12	9	4
	0	7	7	6	6	6	6	9	4	12
	0	7	7	6	6	6	6	4	12	9

OS13.	1	31	31	31	31	31	31	31	31	31
	1	31	31	0	0	0	0	0	0	0
	1	10	3	7	7	7	7	7	7	7
	1	3	10	7	7	7	7	7	7	7
	0	7	7	13	6	6	6	6	6	6
	0	7	7	6	12	9	6	6	6	4
	0	7	7	6	9	4	6	6	6	12
	0	7	7	6	6	6	12	9	4	6
	0	7	7	6	6	6	9	4	12	6
	0	7	7	6	6	6	4	12	9	6
	0	7	7	6	4	12	6	6	6	9

□

Thus we have

Theorem 3.7. *Up to isomorphism, there are exactly thirteen orbit structures for a symmetric block design with parameters* $(280, 63, 14)$ *admitting the Frobenius Group* $G = \langle \rho, \mu | \rho^{31} = \mu^3 = 1, \rho^\mu = \rho^5 \rangle$ *of order* 93*; two with the orbit distribution* $[1; 31; 31; 31; 93; 93]$ *(Table 4.), eight with the orbit distribution* $[1; 31; 31; 31; 31; 31; 31; 93]$ *(Table 8.) and three with the orbit distribution* $[1; 31; 31; 31; 31; 31; 31; 31; 31; 31]$ *(Table 10.).*

Remark 1. The actual indexing of these thirteen orbit structures in order to produce an example is still an open problem.

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Conflict of interest

The author declares there is no conflicts of interest in this paper.

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