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Research article

Infinitesimal and tangent to polylogarithmic complexes for higher weight

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Abstract: Motivic and polylogarithmic complexes have deep connections with K-theory. This article gives morphisms (different from Goncharov's generalized maps) between $\mathbb{k}$-vector spaces of Cathelineau's infinitesimal complex for weight n. Our morphisms guarantee that the sequence of infinitesimal polylogs is a complex. We are also introducing a variant of Cathelineau's complex with the derivation map for higher weight n and suggesting the definition of tangent group $T\mathcal{B}_n(\mathbb{k})$. These tangent groups develop the *tangent to Goncharov's complex* for weight n.

Keywords: polylogarithm; infinitesimal complex; five term relation; tangent complex
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1. Introduction

The classical polylogarithms represented by Li_n are one valued functions on a complex plane (see [11]). They are called generalization of natural logarithms, which can be represented by an infinite series (power series):

$$\begin{aligned} Li_1(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1-z) \\ Li_2(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k^2} \\ &\vdots \\ Li_n(z) &= \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad \text{for } z \in \mathbb{C}, |z| < 1 \end{aligned}$$

The other versions of polylogarithms are Infinitesimal (see [8]) and Tangential (see [9]). We will discuss group theoretic form of infinitesimal and tangential polylogarithms in § 2.3, 2.4 and 2.5 below.

Dupont and Sah describe the connection between scissors congruence group and classical dilogarithm (polylogarithm for $n = 2$) (see [10]). Suslin (see [1]) defines the Bloch group that makes the famous Bloch-Suslin complex which is described in section 2.1 below. Zagier and Goncharov generalize the groups on which polylogarithmic functions are defined. This initiates a new era in the field of polylogarithms, arousing interest of algebraist and geometers. One of the milestones is the proof of Zagier's conjecture for weight $n = 2, 3$ (see [2, 3]).

On the basis of this study Goncharov introduces a motivic complex (2.1) below, which is called Goncharov's complex (see [2]). On the other hand Cathelineau ([7, 8]) uses a differential process to introduce infinitesimal form of motivic (Bloch-Suslin's and Goncharov's) complexes that consists of $\mathbb{k}$-vector spaces. These $\mathbb{k}$-vector spaces are algebraic representation of infinitesimal versions of the Bloch-Suslin and Goncharov's complexes for higher weight n (see [8]), which satisfies functional equations of infinitesimal polylogarithms. Cathelineau also uses a tangent functor to get the tangential analogue of the Bloch-Suslin complex, that allowing a new approach to view additive dialogalithms (regulator on $T\mathcal{B}_2(F)$) (see [9]). The tangent group $T\mathcal{B}_2(F)$ has two parts; first part comes from $\mathcal{B}_2(F)$ and the second part is the derivative of first part. He also suggests a framework for defining the additive trilogarithms.

Our work proposes an improved map (morphism), with the alternate signs between the $\mathbb{k}$-vector spaces that converts the sequence (2.3) into a complex. Further, we introduce a variant of infinitesimal $\mathbb{k}$-vector spaces which is structurally infinitesimal but has functional equations similar to classical polylogarithmic groups.

In §3.1, we are also giving an inductive definition of group $T\mathcal{B}_n(\mathbb{k})$ for higher weight n and putting this in a complex with suitable maps that make a tangent complex (3.1) to Goncharov's (motivic) complex.

2. Materials and method

2.1. Bloch-Suslin complex

Let $\mathbb{Z}[\mathbb{k}]$ be a free abelian group generated by $[a]$ for $a \in F$. Suslin defines the following map

$$\delta_2 : \mathbb{Z}[\mathbb{k}] \to \wedge^2\mathbb{k}^\times, [\theta] \mapsto \theta \wedge (1-\theta)$$

where $\wedge^2\mathbb{k}^\times = \mathbb{k}^\times \otimes \mathbb{k}^\times / \langle \theta\otimes\theta, \theta\otimes\phi + \phi\otimes\theta | \quad \theta,\phi \in \mathbb{k}^\times\rangle$. The Bloch-Suslin complex is defined as

$$\delta : \mathcal{B}_2(\mathbb{k}) \to \wedge^2\mathbb{k}^\times; [\theta]_2 \mapsto \theta \wedge (1-\theta)$$

where $\mathcal{B}_2(\mathbb{k})$ is the quotient of $\mathbb{Z}[\mathbb{k}]$ by the subgroup generated by Abel's five term relation

$$[\theta] - [\phi] + \left[\frac{\phi}{\theta}\right] - \left[\frac{1-\phi}{1-\theta}\right] + \left[\frac{1-\phi^{-1}}{1-\theta^{-1}}\right]$$

and δ is induced by δ_2. When $\mathbb{k}$ is algebraically closed with characteristic zero, the above complex can be inserted into the algebraic K-theory variant of the Bloch-Wigner sequence [9]

$$0 \to \mu(\mathbb{k}) \to K_3^{ind}(\mathbb{k}) \to \mathcal{B}_2(\mathbb{k}) \to \wedge^2\mathbb{k}^\times \to K_2(\mathbb{k}) \to 0$$

if this sequence is tensored by $\mathbb{Q}$ then

$$0 \to K_3^{(2)}(\Bbbk) \to \mathcal{B}_2(\Bbbk) \otimes \mathbb{Q} \to \wedge^2 \Bbbk^\times \to K_2^{(2)}(\Bbbk) \to 0$$

where K-groups $K_n^{(i)}$ are the pieces of the Adams decomposition of $K_n(\Bbbk) \otimes \mathbb{Q}$ (see [6]). The homology of Bloch-Suslin complex is the K_n-groups for $n = 2, 3$ i.e. $\wedge^2 \Bbbk^\times / \mathrm{Im}\delta \cong \Bbbk^\times \otimes \Bbbk^\times / \langle \theta \wedge (1 - \theta) | \theta \in \Bbbk^\times) \rangle$ and $B(F) := \ker \delta$ is called Bloch group, which is isomorphic to K_3 group (see [11]).

2.2. Goncharov's (motivic) complexes

The free abelian group $\mathcal{B}_n(\Bbbk)$ is defined by Goncharov (see[2]) as

$$\mathcal{B}_n(\Bbbk) = \frac{\mathbb{Z}[\Bbbk]}{\mathcal{R}_n(\Bbbk)}$$

with the morphisms for $n = 2$

$$\begin{aligned} \delta_2 : &\mathbb{Z}[\Bbbk] \to \frac{\bigwedge_{\mathbb{Z}}^2 \Bbbk^\times}{(2 - torsion)} \\ &[x] \mapsto \begin{cases} 0 & \text{where } x = 0, 1 \\ x \wedge (1 - x) & \text{for all other } x, \end{cases} \end{aligned}$$

for $n \geq 3$

$$\begin{aligned} \delta_n : &\mathbb{Z}[\Bbbk] \to \mathcal{B}_{n-1}(\Bbbk) \otimes \Bbbk^\times \\ &[x] \mapsto \begin{cases} 0 & \text{if } x = 0, 1, \\ [x]_{n-1} \otimes x & \text{for all other } x, \end{cases} \end{aligned}$$

where $[x]_n$ is the class of x in $\mathcal{B}_n(\Bbbk)$. The subgroup $\mathcal{R}_1(\Bbbk)$ of $\mathbb{Z}[\Bbbk]$ is generated by $[x + y - xy] - [x] - [y]$ and $\mathbb{Z}[\Bbbk]$ is a free abelian group generated by the symbol $[x]$ for $0, 1 \neq x \in \Bbbk$, where $x, y \in \Bbbk \setminus \{1\}$ then $\mathcal{B}_1(\Bbbk) \cong \Bbbk^\times$. For $n = 2$, $\mathcal{R}_2(\Bbbk)$ is defined

$$\mathcal{R}_2(\Bbbk) = \left\langle [\theta] - [\phi] + \left[\frac{\phi}{\theta}\right] - \left[\frac{1 - \phi}{1 - \theta}\right] + \left[\frac{1 - \phi^{-1}}{1 - \theta^{-1}}\right]; 0, 1 \neq \theta, \phi \in \Bbbk \right\rangle$$

The above relation is the Suslin's form of Abel's relations([11]). For $n \geq 2$, $\mathcal{A}_n(\Bbbk)$ is defined as the kernel of δ_n and $\mathcal{R}_n(\Bbbk)$ is the subgroup of $\mathbb{Z}[\Bbbk]$ spanned by [0] and the elements $\sum n_i ([f_i(0)] - [f_i(1)])$, where f_i are rational fractions for indeterminate T, such that $\sum n_i [f_i] \in \mathcal{A}_n(\Bbbk(T))$.

Lemma 2.1. *(Goncharov [2, 3]) The following is the (cochain) complex*

$$\mathcal{B}_n(\Bbbk) \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes \Bbbk^\times \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \bigwedge{}^2 \Bbbk^\times \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{B}_2(\Bbbk) \bigwedge{}^{n-2} \Bbbk^\times \xrightarrow{\delta} \frac{\bigwedge{}^n \Bbbk^\times}{2 - torsion} \tag{2.1}$$

Proof. Proof requires direct calculation (we work here with modulo 2-torsion means $a \wedge a = 0$ and $a \wedge b = -b \wedge a$). □

Example 2.2. *For weight $n = 3$ the following is a complex*

$$\mathcal{B}_3(\mathbb{k}) \xrightarrow{\delta} \mathcal{B}_2(\mathbb{k}) \otimes \mathbb{k}^\times \xrightarrow{\delta} \wedge^3 \mathbb{k}^\times$$

$$\begin{aligned}\delta\delta([\theta]_3) =& \delta\left([\theta]_2 \otimes \theta\right)\\ =&(1-\theta) \wedge \underbrace{\theta \wedge \theta}_{0}\\ =&0\end{aligned}$$

2.3. Cathelineau's infinitesimal complexes

Let $\mathbb{k}$ be a field with a zero characteristic and $\mathbb{k}^{\bullet\bullet} = K - \{0, 1\}$, subspace $\beta_n(\mathbb{k})$ is defined in [3, 9] as

$$\beta_n(\mathbb{k}) = \frac{\mathbb{k}[\mathbb{k}^{\bullet\bullet}]}{\rho_n(\mathbb{k})}$$

where $\rho_n(K)$ is the kernel of the following map

$$\partial_n : \mathbb{k}[\mathbb{k}^{\bullet\bullet}] \to (\beta_{n-1} \otimes \mathbb{k}^\times) \oplus (\mathbb{k} \otimes \mathcal{B}_{n-1}(\mathbb{k}))$$

$$\partial_n : [\theta] \mapsto \langle\theta\rangle_{n-1} \otimes \theta + (1-\theta) \otimes [\theta]_{n-1} \tag{2.2}$$

where $\langle\theta\rangle_n$ is the coset-class of θ in $\beta_n(\mathbb{k})$ and $\rho_2(\mathbb{k})$ generated by Cathelineau's relation,

$$[\theta] - [\phi] + \theta\left[\frac{\phi}{\theta}\right] + (1-\theta)\left[\frac{1-\phi}{1-\theta}\right]$$

For $n = 1$ we have $\beta_1(\mathbb{k}) \cong \mathbb{k}$.

Vector space $\beta_n(\mathbb{k})$ has some non-trivial elements from the functional relations of Li_n for $n \leq 7$ while one can find only inversion and distribution relations in $\beta_n(\mathbb{k})$ for $n > 7$(see [11]).

The following is the Cathelineau's infinitesimal complex to the Goncharov's complex for weight n (see §2 of [4] and [9]):

$$\beta_n(\mathbb{k}) \xrightarrow{\partial} \begin{smallmatrix}\beta_{n-1}(\mathbb{k})\otimes\mathbb{k}^\times\\ \oplus\\ \mathbb{k}\otimes\mathcal{B}_{n-1}(\mathbb{k})\end{smallmatrix} \xrightarrow{\partial} \begin{smallmatrix}\beta_{n-2}(\mathbb{k})\otimes\wedge^2\mathbb{k}^\times\\ \oplus\\ \mathbb{k}\otimes\mathcal{B}_{n-2}(\mathbb{k})\otimes\mathbb{k}^\times\end{smallmatrix} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \begin{smallmatrix}\beta_2(\mathbb{k})\otimes\wedge^{n-2}\mathbb{k}^\times\\ \oplus\\ \mathbb{k}\otimes\mathcal{B}_2(\mathbb{k})\otimes\wedge^{n-3}\mathbb{k}^\times\end{smallmatrix} \xrightarrow{\partial} \mathbb{k} \otimes \wedge^{n-1}\mathbb{k}^\times \tag{2.3}$$

Example 2.3. *For weight $n = 3$, the following infinitesimal version satisfying the definition of a complex:*

$$\beta_3(\mathbb{k}) \xrightarrow{\partial} \beta_2(\mathbb{k}) \otimes \mathbb{k}^\times \bigoplus \mathbb{k} \otimes \mathcal{B}_2(\mathbb{k}) \xrightarrow{\partial} \mathbb{k} \otimes \wedge^2 \mathbb{k}^\times \tag{2.4}$$

$$\begin{aligned}\partial\partial(\langle\theta\rangle_3) &= \partial\left(\langle\theta\rangle_2 \otimes \theta + (1-\theta) \otimes [\theta]_2\right)\\ &= -\theta \otimes \underbrace{\theta \wedge \theta}_{0} - (1-\theta) \otimes (1-\theta) \wedge \theta + (1-\theta) \otimes (1-\theta) \wedge \theta\\ &= 0\end{aligned}$$

2.4. Variant of Cathelineau's complex

We put $[\![a]\!]^D = \frac{D(a)}{a(1-a)}[a]$ where $D(a) \in Der_{\mathbb{Z}}(\mathbb{k}, \mathbb{k})$ and is called general derivation, $\beta_n^D(\mathbb{k})$ is defined as

$$\beta_n^D(\mathbb{k}) = \frac{\mathbb{k}[\mathbb{k}^{\bullet\bullet}]}{\rho_n^D(\mathbb{k})}$$

where $\rho_n^D(\mathbb{k})$ is a kernel of the following map

$$\partial_n^D : \mathbb{k}[\mathbb{k}^{\bullet\bullet}] \to \left(\beta_{n-1}^D(\mathbb{k}) \otimes \mathbb{k}^\times\right) \oplus (\mathbb{k} \otimes \mathcal{B}_{n-1})$$

$$\partial_n^D : [\theta]^D \mapsto [\![\theta]\!]_{n-1}^D \otimes \theta + D\log(\theta) \otimes [a]_{n-1}$$

and $[\![\theta]\!]_n^D$ is a class of θ in $\beta_n^D(\mathbb{k})$ which is equal to $\frac{D(\theta)}{\theta(1-\theta)}\langle\theta\rangle_n$. The following is a subspace of $\mathbb{k}[\mathbb{k}^{\bullet\bullet}]$:

$$\rho_2^D(\mathbb{k}) = \left\langle [\![\theta]\!]^D - [\![\psi]\!]^D + \left[\!\!\left[\frac{\psi}{\theta}\right]\!\!\right]^D - \left[\!\!\left[\frac{1-\psi}{1-\theta}\right]\!\!\right]^D + \left[\!\!\left[\frac{1-\psi^{-1}}{1-\theta^{-1}}\right]\!\!\right]^D ; 0, 1 \neq \theta, \phi \in \mathbb{k} \right\rangle$$

For $n \geq 4$, one can write only inversion relations in $\beta_n^D(\mathbb{k})$ while for $n \leq 3$ we have other non-trivial relations as well. The following sequence is a complex. One can easily prove in a completely analogous way as Lemma 3.1

$$\beta_n^D(\mathbb{k}) \xrightarrow{\partial^D} \begin{array}{c}\beta_{n-1}^D(\mathbb{k})\otimes\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes B_{n-1}(\mathbb{k})\end{array} \xrightarrow{\partial^D} \cdots \xrightarrow{\partial^D} \begin{array}{c}\beta_2^D(\mathbb{k})\otimes\wedge^{n-2}\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes B_2(\mathbb{k})\otimes\wedge^{n-3}\mathbb{k}^\times\end{array} \xrightarrow{\partial^D} \mathbb{k} \otimes \wedge^{n-1}\mathbb{k}^\times \tag{2.5}$$

Example 2.4. *This $D\log$ version of Cathelineau's complex is also satisfying the definition of a complex when the above maps are used for weight $n = 3$.*

$$\beta_3^D(\mathbb{k}) \xrightarrow{\partial} \beta_2^D(\mathbb{k}) \otimes \mathbb{k}^\times \bigoplus \mathbb{k} \otimes \mathcal{B}_2(\mathbb{k}) \xrightarrow{\partial} \mathbb{k} \otimes \wedge^2\mathbb{k}^\times \tag{2.6}$$

$$\begin{aligned}
\partial\partial(\langle\theta\rangle_3^D) &= \partial\left(\langle\theta\rangle_2^D \otimes \theta + D\log\theta \otimes [\theta]_2\right) \\
&= -D\log(1-\theta) \otimes \underbrace{\theta \wedge \theta}_{0} + D\log\theta \otimes (1-\theta) \wedge \theta + D\log\theta \otimes \theta \wedge (1-\theta) \\
&= D\log\theta \otimes (1-\theta) \wedge \theta - D\log\theta \otimes (1-\theta) \wedge \theta \\
&= 0
\end{aligned}$$

2.5. Tangent to Bloch-Suslin complex

We represent a ring of dual numbers by $\mathbb{k}[\varepsilon]_2 = \mathbb{k}[\varepsilon]/\langle\varepsilon^2\rangle$ where $\mathbb{k}$ is algebraically closed field with zero characteristic. There is a $\mathbb{k}^\star$-action on $\mathbb{k}[\varepsilon]_2$ for $\lambda \in \mathbb{k}^\times$

$$\lambda : \mathbb{k}[\varepsilon]_2 \to \mathbb{k}[\varepsilon]_2$$

$$\lambda \star (\theta + \theta'\varepsilon) = \theta + \lambda\theta'\varepsilon$$

For dual numbers $\mathbb{k}[\varepsilon]_2$, we define a free abelian group $\mathbb{Z}[\mathbb{k}[\varepsilon]_2]$ generated by $[\theta + \phi\varepsilon]$ for $\theta + \phi\varepsilon \in \mathbb{k}[\varepsilon]_2$. Define a morphism

$$\partial : \mathbb{Z}[\mathbb{k}[\varepsilon]_2] \to \wedge^2\mathbb{k}[\varepsilon]_2^\times \tag{2.7}$$

$$\partial : [\mu] \mapsto \mu \wedge (1-\mu)$$

for all $\mu \in \Bbbk[\varepsilon]_2$. Similarly, if we replace $\Bbbk$ by $\Bbbk[\varepsilon]_2$ in the Bloch-Suslin complex, we get

$$\partial : \mathcal{B}_2(\Bbbk[\varepsilon]_2) \to \wedge^2 \Bbbk[\varepsilon]_2^\times \tag{2.8}$$

The right hand side of (2.8) is canonically isomorphic to $\bigwedge^2 \Bbbk^\times \bigoplus \Bbbk \otimes \Bbbk^\times \bigoplus \bigwedge^2 \Bbbk$ with

$$(\theta + \phi\varepsilon) \wedge (\theta' + \phi'\varepsilon) \mapsto \theta \wedge \theta' \oplus \left(\theta \otimes \frac{\phi'}{\theta'} - \theta' \otimes \frac{\phi}{\theta}\right) \oplus \frac{\phi}{\theta} \wedge \frac{\phi'}{\theta'}$$

while the left hand side is isomorphic to $\mathcal{B}_2(\Bbbk) \bigoplus \beta_2(\Bbbk) \bigoplus \bigwedge^2 \Bbbk \bigoplus \Bbbk$ (see [9])

Define a $\mathbb{Z}$-module $\mathbb{Z}'[\Bbbk[\varepsilon]_2]$ generated by $\langle \theta; \phi] = [\theta + \phi\varepsilon] - [\theta]$ for $\theta, \phi \in \Bbbk$ and define $\mathcal{R}_2^\varepsilon(\Bbbk[\varepsilon]_2)$ as a submodule of $\mathbb{Z}'[\Bbbk[\varepsilon]_2]$ generated by the five term relation (see [9] and [12])

$$\langle \theta; \theta'] - \langle \psi; \psi'] + \left\langle \frac{\psi}{\theta}; \left(\frac{\psi}{\theta}\right)' \right] - \left\langle \frac{1-\psi}{1-\theta}; \left(\frac{1-\psi}{1-\theta}\right)' \right] + \left\langle \frac{\theta(1-\psi)}{\psi(1-\theta)}; \left(\frac{\theta(1-\psi)}{\psi(1-\theta)}\right)' \right], \quad \theta, \psi \neq 0, 1, \theta \neq \psi \tag{2.9}$$

where

$$\left(\frac{\psi}{\theta}\right)' = \frac{\theta\psi' - \theta'\psi}{\theta^2},$$

$$\left(\frac{1-\psi}{1-\theta}\right)' = \frac{(1-\psi)\theta' - (1-\theta)\psi'}{(1-\theta)^2}$$

and

$$\left(\frac{\theta(1-\psi)}{\psi(1-\theta)}\right)' = \frac{\psi(1-\psi)\theta' - \theta(1-\theta)\psi'}{(\psi(1-\theta))^2}$$

Define

$$T\mathcal{B}_2(\Bbbk) = \frac{\mathbb{Z}'[\Bbbk[\varepsilon]_2]}{\mathcal{R}_2^\varepsilon(\Bbbk[\varepsilon]_2)}$$

Remark 2.5. *The tangent group $T\mathcal{B}_2(\Bbbk)$ is isomorphic to $\beta_2(\Bbbk) \bigoplus \bigwedge^2 \Bbbk \bigoplus \Bbbk$ (Theorem 1.1 of [9]) and $\mathbb{Z}'[\Bbbk[\varepsilon]_2]$ is isomorphic to $\mathcal{B}_2(\Bbbk[\varepsilon]_2)$*

3. Main results and discussion

Consider the sequence (2.3) above. Here we suggest a map (morphism) different from the one which is defined in §2 of [3] and the relation (2.2) above between the abelian groups of sequence (2.3), since the map without alternate sign does not follow the definition of a complex. Thus, the above sequence becomes a complex if we put alternate signs for ∂:

when $n = 2$, we put

$$\partial : \langle \theta \rangle_2 \mapsto -(\theta \otimes \theta + (1-\theta) \otimes (1-\theta))$$

and for $n \geq 3$, we suggest to use

$$\partial : \langle \theta \rangle_n \mapsto \langle \theta \rangle_{n-1} \otimes \theta + (-1)^{n-1}(1-\theta) \otimes [\theta]_{n-1}$$

Theorem 3.1. *The sequence* (2.3) *is a complex for the ∂ defined above.*

Proof. To prove that the sequence (2.3) is a complex we consider $2 \le k \le n-2$

$$\cdots \xrightarrow{\partial} \begin{array}{c}\beta_{n-k+1}(\mathbb{k})\otimes\wedge^{k-1}\mathbb{k}^\times\\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k+1}(\mathbb{k})\otimes\wedge^{k-2}\mathbb{k}^\times\end{array} \xrightarrow{\partial} \begin{array}{c}\beta_{n-k}(\mathbb{k})\otimes\wedge^{k}\mathbb{k}^\times\\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k}(\mathbb{k})\otimes\wedge^{k-1}\mathbb{k}^\times\end{array} \xrightarrow{\partial} \begin{array}{c}\beta_{n-k-1}(\mathbb{k})\otimes\wedge^{k+1}\mathbb{k}^\times\\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k-1}(\mathbb{k})\otimes\wedge^{k}\mathbb{k}^\times\end{array} \xrightarrow{\partial} \cdots$$

Let $\langle u\rangle_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} v_i + \theta \otimes [\phi]_{n-k+1} \otimes \bigwedge_{j=1}^{k-2} \psi_j \in \begin{array}{c}\beta_{n-k+1}(\mathbb{k})\otimes\wedge^{k-1}\mathbb{k}^\times\\ \oplus \\ K\otimes\mathcal{B}_{n-k+1}(\mathbb{k})\otimes\wedge^{k-2}\mathbb{k}^\times\end{array}$

Now compute $\partial\left(\partial\left(\langle u\rangle_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} v_i + \theta\otimes[\phi]_{n-k+1}\otimes \bigwedge_{j=1}^{k-2}\psi_j\right)\right)$.

To make calculation simple, first we compute

$$\begin{aligned}
&\partial\left(\partial\left(\langle u\rangle_{n-k+1} \otimes \bigwedge_{i=1}^{k-1} v_i\right)\right)\\
=&\partial\left(\langle u\rangle_{n-k} \otimes u \wedge \bigwedge_{i=1}^{k-1} v_i + (-1)^{n-k}(1-u)\otimes[u]_{n-k}\otimes \bigwedge_{i=1}^{k-1} v_i\right)\\
=&\langle u\rangle_{n-k-1} \otimes \underbrace{u\wedge u}_{0} \wedge \bigwedge_{i=1}^{k-1} v_i + (-1)^{n-k-1}(1-u)\otimes[u]_{n-k-1}\otimes u \wedge \bigwedge_{i=1}^{k-1} v_i\\
&+(-1)^{n-k}(1-u)\otimes[u]_{n-k-1}\otimes u\wedge \bigwedge_{i=1}^{k-1} v_i\\
=&0
\end{aligned}$$

then find

$$\begin{aligned}
\partial\left(\partial\left(\theta\otimes[\phi]_{n-k+1}\otimes\bigwedge_{j=1}^{k-2}\psi_j\right)\right) =&\partial\left(\theta\otimes[\phi]_{n-k}\otimes\phi\wedge\bigwedge_{j=1}^{k-2}\psi_j\right)\\
=&\theta\otimes[\phi]_{n-k-1}\otimes\underbrace{\phi\wedge\phi}_{0}\wedge\bigwedge_{j=1}^{k-2}\psi_j\\
=&0
\end{aligned}$$

Now the last case is for $k = 1$ with $\bigwedge_{i=0}^{0} v_i = 1 \in \mathbb{Z}$ and using $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ for any ring R. □

Similarly, for the variant of Cathelineau's complex (2.5) and tangential version of Goncharv's complex (3.1), we have similar results.

Theorem 3.2. *The above sequence (2.5) is a complex.*

Proof. There is not much effort required to prove the above sequence is a complex except to use $D\log$ maps. We just follow the steps of Theorem 3.1 and use $D\log$. □

3.1. Tangent to Goncharov's complex

Here, we suggest that how to define a tangent group $T\mathcal{B}_n(\mathbb{k})$ for any n in the same spirit as $\beta_n(\mathbb{k})$ is defined in [3] and give its appropriateness by relating them in a suitable complex.

Inductively, for any n,we define a tangent group $T\mathcal{B}_n(\mathbb{k})$ by defining the map

$$\partial : \mathbb{Z}'[\mathbb{k}[\varepsilon]_2] \to T\mathcal{B}_{n-1}(\mathbb{k}) \otimes \mathbb{k}^\times \oplus \mathbb{k} \otimes \mathcal{B}_{n-1}(\mathbb{k})$$

thus $T\mathcal{B}_n(\mathbb{k})$ is

$$T\mathcal{B}_n(\mathbb{k}) = \frac{\mathbb{Z}'[\mathbb{k}[\varepsilon]_2]}{\mathcal{R}_n^\varepsilon(\mathbb{k}[\varepsilon]_2)}$$

where $\mathcal{R}_n^\varepsilon(\mathbb{k}[\varepsilon]_2)$ is a kernel of the following map

$$\partial_{\varepsilon,n} : \mathbb{Z}'[\mathbb{k}[\varepsilon]_2] \to T\mathcal{B}_{n-1}(\mathbb{k}) \otimes \mathbb{k}^\times \oplus \mathbb{k} \otimes \mathcal{B}_{n-1}(\mathbb{k})$$

$$\partial_{\varepsilon,n} : \langle\theta;\psi] \mapsto \langle\theta;\psi]_{n-1} \otimes \theta + (-1)^{n-1}\frac{\psi}{\theta} \otimes [\theta]_{n-1}$$

where $\langle\theta;\psi] = [\theta + \psi\varepsilon] - [\theta]$ and $\langle\theta;\psi]_n$ is the class of $\langle\theta,\psi]$ in $T\mathcal{B}_n(\mathbb{k})$, by using the above definition, the following becomes a complex

$$T\mathcal{B}_n(\mathbb{k}) \xrightarrow{\partial_\varepsilon} \begin{smallmatrix} T\mathcal{B}_{n-1}(\mathbb{k})\otimes\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-1}(\mathbb{k}) \end{smallmatrix} \xrightarrow{\partial_\varepsilon} \cdots \xrightarrow{\partial_\varepsilon} \begin{smallmatrix} T\mathcal{B}_2(\mathbb{k})\otimes\wedge^{n-2}\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_2(\mathbb{k})\otimes\wedge^{n-3}\mathbb{k}^\times \end{smallmatrix} \xrightarrow{\partial_\varepsilon} \left(\mathbb{k} \otimes \bigwedge{}^{n-1}\mathbb{k}^\times\right) \oplus \left(\bigwedge{}^2\mathbb{k} \otimes \bigwedge{}^{n-2}\mathbb{k}^\times\right) \tag{3.1}$$

where ∂_ε is induced by $\partial_{\varepsilon,n}$ and when ∂_ε is applied to the group $\mathcal{B}_n(\mathbb{k})$ then it agrees with δ_n defined above and in [11].

Theorem 3.3. *For weight n = 3, the tangent to Goncharov's complex is also a complex.*

$$T\mathcal{B}_3(\mathbb{k}) \xrightarrow{\partial_\varepsilon} T\mathcal{B}_2(\mathbb{k}) \otimes \mathbb{k}^\times \bigoplus \mathbb{k} \otimes \mathcal{B}_2(\mathbb{k}) \xrightarrow{\partial_\varepsilon} \mathbb{k} \otimes \wedge^2\mathbb{k}^\times \bigoplus \wedge^2\mathbb{k} \otimes \mathbb{k}^\times$$

where $\partial_\varepsilon(\langle\theta;\phi]_3) = \langle\theta;\phi]_2 \otimes \theta + \frac{\phi}{\theta} \otimes [\theta]_2$ *and*

$$\partial_\varepsilon\left(\langle\theta;\phi]_2 \otimes \psi + x \otimes [y]_2\right) = -\tfrac{\phi}{1-\theta} \otimes \theta \wedge \psi - \tfrac{\phi}{\theta} \otimes (1-\theta) \wedge \psi + x \otimes (1-y) \wedge y + \tfrac{\phi}{1-\theta} \wedge \tfrac{\phi}{\theta} \otimes \psi + \tfrac{\phi}{\theta} \wedge \tfrac{\phi}{1-\theta} \otimes y$$

Proof. Here we will prove that how the above sequence is a complex for weight $n = 3$.

$$\begin{aligned} \partial_\varepsilon\partial_\varepsilon\left(\langle\theta;\phi]_3\right) &= \partial_\varepsilon\left(\langle\theta;\phi]_2 \otimes \theta + \frac{\phi}{\theta} \otimes [\theta]_2\right) \\ &= -\frac{\phi}{1-\theta} \otimes \underbrace{\theta \wedge \theta}_{0} - \frac{\phi}{\theta} \otimes (1-\theta) \wedge \theta + \frac{\phi}{\theta} \otimes (1-\theta) \wedge \theta + \frac{\phi}{1-\theta} \wedge \frac{\phi}{\theta} \otimes \theta + \frac{\phi}{\theta} \wedge \frac{\phi}{1-\theta} \otimes \theta \\ &= 0 \quad \text{(by invoking the antisymmetric relation in the last two terms)} \end{aligned}$$

□

Theorem 3.4. *The above sequence (3.1) is a complex.*

Proof. We can show that (3.1) is a complex, by considering two cases:

Case 1: Consider

$$T\mathcal{B}_n(\mathbb{k}) \xrightarrow{\partial_\varepsilon} \begin{matrix} T\mathcal{B}_{n-1}(\mathbb{k})\otimes\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-1}(\mathbb{k}) \end{matrix} \xrightarrow{\partial_\varepsilon} \begin{matrix} T\mathcal{B}_{n-2}(\mathbb{k})\otimes\wedge^2\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-2}(\mathbb{k})\otimes\mathbb{k}^\times \end{matrix} \xrightarrow{\partial_\varepsilon} \cdots$$

$$\begin{aligned}
\partial_\varepsilon\left(\partial_\varepsilon\left(\langle\theta;\phi]_n\right)\right) &= \partial_\varepsilon\left(\langle\theta;\phi]_{n-1}\otimes\theta + (-1)^{n-1}\frac{\phi}{\theta}\otimes[\theta]_{n-1}\right) \\
&= \langle\theta;\phi]_{n-2}\underbrace{\theta\wedge\theta}_{0} + (-1)^{n-2}\frac{\phi}{\theta}\otimes[\theta]_{n-2}\otimes\theta + (-1)^{n-1}\frac{\phi}{\theta}\otimes[\theta]_{n-2}\otimes\theta \\
&= -(-1)^{n-1}\frac{\phi}{\theta}\otimes[\theta]_{n-2}\otimes\theta + (-1)^{n-1}\frac{\phi}{\theta}\otimes[\theta]_{n-2}\otimes\theta \\
&= 0
\end{aligned}$$

Case 2: We consider

$$\cdots\xrightarrow{\partial} \begin{matrix} T\mathcal{B}_{n-k+1}(\mathbb{k})\otimes\wedge^{k-1}\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k+1}(\mathbb{k})\otimes\wedge^{k-2}\mathbb{k}^\times \end{matrix} \xrightarrow{\partial} \begin{matrix} T\mathcal{B}_{n-k}(\mathbb{k})\otimes\wedge^{k}\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k}(\mathbb{k})\otimes\wedge^{k-1}\mathbb{k}^\times \end{matrix} \xrightarrow{\partial} \begin{matrix} T\mathcal{B}_{n-k-1}(\mathbb{k})\otimes\wedge^{k+1}\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k-1}(\mathbb{k})\otimes\wedge^{k}\mathbb{k}^\times \end{matrix} \xrightarrow{\partial} \cdots$$

Let $\langle\theta;\phi]_{n-k+1}\otimes\wedge_{i=1}^{k-1}\phi_i + x\otimes[y]\otimes\wedge_{j=1}^{k-2}z_j \in \begin{matrix} T\mathcal{B}_{n-k+1}(\mathbb{k})\otimes\wedge^{k-1}\mathbb{k}^\times \\ \oplus \\ \mathbb{k}\otimes\mathcal{B}_{n-k+1}(\mathbb{k})\otimes\wedge^{k-2}\mathbb{k}^\times \end{matrix}$

Now applying maps

$$\begin{aligned}
&\partial_\varepsilon\left(\partial_\varepsilon\left(\langle\theta;\phi]_{n-k+1}\otimes\wedge_{i=1}^{k-1}\phi_i + x\otimes[y]\otimes\wedge_{j=1}^{k-2}z_j\right)\right) \\
&= \partial_\varepsilon\left(\langle\theta;\phi]_{n-k}\otimes\theta\otimes\bigwedge_{i=1}^{k-1}\phi_i + (-1)^{n-k}\frac{\phi}{\theta}\otimes[\theta]_{n-k}\otimes\bigwedge_{i=1}^{k-1}\phi_i + x\otimes[y]_{n-k}\otimes y\otimes\bigwedge_{j=1}^{k-2}z_j\right) \\
&= \langle\theta;\phi]_{n-k-1}\otimes\underbrace{\theta\wedge\theta}_{0}\otimes\bigwedge_{i=1}^{k-1}\phi_i + (-1)^{n-k-1}\frac{\phi}{\theta}\otimes[\theta]_{n-k-1}\otimes\theta\otimes\bigwedge_{i=1}^{k-1}\phi_i \\
&+ (-1)^{n-k}\frac{\phi}{\theta}\otimes[\theta]_{n-k-1}\otimes\theta\otimes\bigwedge_{i=1}^{k-1}\phi_i + x\otimes[y]_{n-k-1}\otimes\underbrace{y\wedge y}_{0}\wedge\bigwedge_{j=1}^{k-2}z_j \\
&= 0 \quad (\text{ two middle terms are opposite in sign})
\end{aligned}$$

□

4. Conclusion

We have shown that the sequences (2.3), (2.5) and (3.1) are complexes. Complexes (2.3) and (2.5) have only inversion and distribution relations (functional equations) for $n > 3$. However, there are some non-trivial but non-defining relations known for $n \leq 7$ (see [5, 11]). There is insufficient information for the complex (3.1) (kernels of ∂_ε and defining relations are unknown) to compute the homologies for $n \geq 3$, but it is expected to come out in a similar way as the homology of the complex (2.1).

The original construction of the tangent to Bloch-Suslin complex (see [9]) is described by the application of a tangent functor on the Bloch-Suslin, resulting in the first derivative on $\mathcal{B}_2(F)$ and $\wedge^2 F^\times$. One can find the higher order derivatives (tangent order) on Goncharov's complex or precisely on $T\mathcal{B}_n(F)$ in a similar way as done in [13] for Bloch-Suslin complex.

Conflict of interest

The author declares there is no conflicts of interest in this paper.

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