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Research article

Lie symmetry analysis of conformable differential equations

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Abstract: In this paper, we construct a proper extension of the classical prolongation formula of point transformations for conformable derivative. This technique is illustrated and employed to construct a symmetry group admitted by a conformable ordinary and partial differential equations. Using Lie symmetry analysis, we obtain an exact solution of the conformable heat equation.

Keywords: conformable derivative; fractional differential equations; Lie symmetry; conformable heat equation
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1. Introduction

In recent years, fractional differential equations have received considerable attention owing to their applicability in different fields of sciences such as chemistry, biology, diffusion, control theory, rheology, viscoelasticity and so on [1–4]. Many authors has been shown that the fractional derivatives used in this theory lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. Recently, a conformable derivative is introduced and it satisfies the basic properties of usual derivatives [5–7].

To construct exact solutions for fractional differential equations is not an easy task. Therefore, several methods such as homotopy perturbation method [8], sub-ODE method [9, 10], generalized tanh method [11], residual power series method [9, 12] and so on [13–18] are developed to obtain solutions of some nonlinear fractional differential equations.

Lie symmetry analysis or Lie group method has become of great interest in many aspects of the exact sciences [19–22], and has been extensively applied to construct exact solutions of ordinary and partial differential equations. However, this method has not been used much to investigate invariance properties of fractional partial differential equations. Gazizov et al. [23] have established the prolongation formula in the case of fractional differential equations in which the fractional derivative is considered in the sense of Caputo and Riemann-Liouville. Their fundamental work has become a key to investigate symmetry properties of some fractional differential equations and a crucial tool for other works, see for example [9, 12, 20–27].

In this work, we develop Lie symmetry method to establish the prolongation formula admitted by a conformable differential equations. The obtained prolongation formula is given in case of the dependent variables that are supposed to be differentiable functions. This method is used to obtain some solutions of conformable heat equation.

This paper is organized as follows: In Section 2, we recall some basic properties of conformable derivatives. Section 3 is devoted to determine the prolongation formula in the case of a conformable ordinary differential equation. While in Section 4, we establish the prolongation formula of a conformable partial differential equation, the method is illustrated by conformable heat equation. We concluded the paper in the last section by some comments.

2. Preliminaries and notations

Definition 2.1. *We consider* $f : \mathbb{R} \to \mathbb{R}$ *is a function, and* $\alpha \in (0, 1]$.

The left conformable derivative of order α *is defined by*

$$ {}_aT_t^{\alpha} f(t) = \lim_{h\to 0} \frac{f(t + h(t-a)^{1-\alpha}) - f(t)}{h}. \tag{2.1}$$

The right conformable derivative of order α *is*

$$ {}_tT_b^{\alpha} f(t) = \lim_{h\to 0} \frac{f(t - h(b-t)^{1-\alpha}) - f(t)}{h}. \tag{2.2}$$

If $a = 0$ and f is α-differentiable in some $(0, x)$, i.e., $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, so we define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

We can write $f^{(\alpha)}(t)$ for ${}_0T_t^{\alpha} f(t)$ to denote the conformable derivative of f of order α. In addition, if the conformable derivative of f of order α exists, then we simply say f is α-differentiable. Some useful formulas and properties of conformable derivative are summarized in [5, 6] and here we present some of its basic properties.

Proposition 1. *Let be* f *and* g *two* α*-differentiable functions at any point* $t > a$ *such that* $g \neq 0$*, and* $\lambda, \mu \in \mathbb{R}$*, we have*

$$ {}_aT_t^{\alpha}(\lambda f + \mu g) = \lambda\, {}_aT_t^{\alpha} f + \mu\, {}_aT_t^{\alpha} g, \tag{2.3}$$

$$ {}_aT_t^{\alpha}(fg) = f\, {}_aT_t^{\alpha} g + g\, {}_aT_t^{\alpha} f, \tag{2.4}$$

$$ {}_aT_t^{\alpha}\left(\frac{f}{g}\right) = \frac{g\, {}_aT_t^{\alpha} f - f\, {}_aT_t^{\alpha} g}{g^2}. \tag{2.5}$$

Note that the above properties (2.4) and (2.5) are not preserved with other known fractional derivatives [4]. Furthermore, if α,β in $(0,1)$, and f is a 2-times differentiable, we have non commutative and non stability properties of conformable derivatives.

$$ {}_aT_t^{\alpha}({}_aT_t^{\beta}f) \neq {}_a T_t^{\beta}({}_aT_t^{\alpha}f), \qquad \text{and} \qquad {}_aT_t^{\alpha}({}_aT_t^{\beta}f) \neq {}_a T_t^{\alpha+\beta}f. \tag{2.6} $$

Proposition 2. *If f and g two functions such that g is differentiable in all t, and f is differentiable in all $g(t)$. we have*

$$ {}_aT_t^{\alpha}(f\circ g)(t) = f^{'}(g(t))_aT_t^{\alpha}g(t). \tag{2.7} $$

3. Prolongation formula for conformable ordinary differential equations

In this section, we consider the one parameter group of point transformations with $y = y(t)$ is a differentiable function,

$$ \bar{t} = \varphi(t,y,\epsilon) = t + \epsilon\xi(t,y) + o(\epsilon^2), \tag{3.1} $$

$$ \bar{y} = \phi(t,y,\epsilon) = y + \epsilon\eta(t,y) + o(\epsilon^2). \tag{3.2} $$

Where

$$ \xi(t,y) = \frac{\partial\varphi}{\partial\epsilon}\bigg|_{\epsilon=0} \qquad \text{and} \qquad \eta(t,y) = \frac{\partial\phi}{\partial\epsilon}\bigg|_{\epsilon=0}. \tag{3.3} $$

Along with (3.1) and (3.2), we consider the corresponding infinitesimal operator

$$ X = \xi(t,y)\frac{\partial}{\partial t} + \eta(t,y)\frac{\partial}{\partial y}. \tag{3.4} $$

With the condition of invariance of the equation $t = 0$, we obtain

$$ \xi(t,y(t))|_{t=0} = 0. \tag{3.5} $$

Theorem 3.1. *The infinitesimal transformation of the conformable derivative definition, we have*

$$ {}_0T_{\bar{t}}^{\alpha}\bar{y}(\bar{t}) = {}_0T_t^{\alpha}y(t) + \epsilon\eta^{(\alpha)} + o(\epsilon^2), \tag{3.6} $$

where

$$ \eta^{(\alpha)} = t^{1-\alpha}\eta_t + \left(\eta_y + \frac{1-\alpha}{t}\xi - \xi_t\right)\left({}_0T_t^{\alpha}y(t)\right) - t^{\alpha-1}\xi_y\left({}_0T_t^{\alpha}y(t)\right)^2. \tag{3.7} $$

Proof. According to the conformable derivative definition, we have

$$ {}_0T_{\bar{t}}^{\alpha}\bar{y}(\bar{t}) = \lim_{h\to 0}\frac{\bar{y}(\bar{t}+h\bar{t}^{1-\alpha}) - \bar{y}(\bar{t})}{h} \tag{3.8} $$

$$ = \bar{t}^{1-\alpha}\lim_{h\to 0}\frac{\bar{y}(\bar{t}+h\bar{t}^{1-\alpha}) - \bar{y}(\bar{t})}{h\bar{t}^{1-\alpha}} \tag{3.9} $$

$$ = \bar{t}^{1-\alpha}\frac{d\bar{y}}{d\bar{t}}. \tag{3.10} $$

In addition, from Eqs. (3.1) and (3.2), we get

$$\frac{d\bar{y}}{d\bar{t}} = \frac{y' + \epsilon(\eta_t + \eta_y y') + o(\epsilon^2)}{1 + \epsilon(\xi_t + \xi_y y') + o(\epsilon^2)}, \tag{3.11}$$

and,

$$\bar{t}^{1-\alpha} = \Big(t + \epsilon\xi(t,y) + o(\epsilon^2)\Big)^{1-\alpha} \tag{3.12}$$

$$= t^{1-\alpha}\Big(1 + \frac{\epsilon}{t}\xi\Big)^{1-\alpha} + o(\epsilon^2) \tag{3.13}$$

$$= t^{1-\alpha}\Big(1 + \frac{(1-\alpha)\epsilon}{t}\xi\Big) + o(\epsilon^2). \tag{3.14}$$

Furthermore,

$$\frac{1}{1 + \epsilon(\xi_t + \xi_y y') + o(\epsilon^2)} = 1 - \epsilon(\xi_t + \xi_y y') + o(\epsilon^2). \tag{3.15}$$

Substituting expressions (3.11), (3.14) and (3.15) in Eq. (3.10), we get

$$\begin{aligned} {}_0T^{\alpha}_{\bar{t}}\bar{y}(\bar{t}) &= \Big(t^{1-\alpha}\Big(1 + \frac{(1-\alpha)\epsilon}{t}\xi\Big) + o(\epsilon^2)\Big)\Big(y' + \epsilon(\eta_t + \eta_y y') + o(\epsilon^2)\Big) \\ &\times\Big(1 - \epsilon(\xi_t + \xi_y y') + o(\epsilon^2)\Big) & (3.16) \\ &= \Big(t^{1-\alpha}y' + \epsilon t^{1-\alpha}\Big(\eta_t + \Big(\eta_y + \frac{1-\alpha}{t}\xi\Big)y'\Big) + o(\epsilon^2)\Big) \\ &\times\Big(1 - \epsilon(\xi_t + \xi_y y') + o(\epsilon^2)\Big) & (3.17) \\ &= t^{1-\alpha}y' + \epsilon\Big[t^{1-\alpha}\Big(\eta_t + \Big(\eta_y + \frac{1-\alpha}{t}\xi\Big)y'\Big) - t^{1-\alpha}y'(\xi_t + \xi_y y')\Big] + o(\epsilon^2). \end{aligned}$$

Consequently,

$${}_0T^{\alpha}_{\bar{t}}\bar{y}(\bar{t}) = {}_0T^{\alpha}_{t}y(t) + \epsilon\eta^{(\alpha)} + o(\epsilon^2), \tag{3.18}$$

where

$$\eta^{(\alpha)} = t^{1-\alpha}\eta_t + \Big(\eta_y + \frac{1-\alpha}{t}\xi - \xi_t\Big)\big({}_0T^{\alpha}_{t}y(t)\big) - t^{\alpha-1}\xi_y\big({}_0T^{\alpha}_{t}y(t)\big)^2. \tag{3.19}$$

□

Example 1. *Consider the conformable Riccati equation*

$$y^{(\alpha)}(t) + y^2(t) = 0, \tag{3.20}$$

where $y^{(\alpha)}(t)$ *denotes the conformable derivative of* y*. In this example, we will compute the Lie symmetries of the above equation. Its corresponding infinitesimal generator is given by*

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}, \tag{3.21}$$

and the α-th order prolongation of X is

$$X^{(\alpha)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y} + \eta^{(\alpha)}\frac{\partial}{\partial y^{(\alpha)}}, \tag{3.22}$$

where $\eta^{(\alpha)}$ is given by the formula (3.7). If we put $\Delta = y^{(\alpha)}(t) + y^2(t)$, so the invariance criterion is given

$$X^{(\alpha)}[\Delta]|_{\Delta=0} = 0. \tag{3.23}$$

Then,

$$2\eta y + \eta^{(\alpha)}\Big|_{\Delta=0} = 0. \tag{3.24}$$

Substituting for $\eta^{(\alpha)}$ its expression, the above criterion leads to

$$t^{1-\alpha}\eta_t + 2\eta y - \left(\eta_y + \frac{1-\alpha}{t}\xi - \xi_t\right)y^2 - t^{\alpha-1}\xi_y y^4 = 0. \tag{3.25}$$

As $\xi(t, y) = \xi(t)$ and $\eta(t, y) = 0$, we obtain the equation

$$\xi'(t) - \frac{1-\alpha}{t}\xi(t) = 0. \tag{3.26}$$

Hence,

$$\xi(t) = Ct^{1-\alpha}, \quad C = cte. \tag{3.27}$$

Consequently, the conformable Riccati equation (3.20) is invariant under the one-parameter group of transformations generated by the infinitesimal operator

$$X = t^{1-\alpha}\frac{\partial}{\partial t}. \tag{3.28}$$

4. Prolongation formula of conformable partial differential equations

Let us assume that a conformable partial differential equation of independent variables (x, t) and dependent variable u is invariant under a one-parameter (ϵ) continuous point transformations

$$\bar{t} = t + \epsilon\xi(t, x, y) + o(\epsilon^2), \tag{4.1}$$

$$\bar{x} = x + \epsilon\tau(t, x, y) + o(\epsilon^2), \tag{4.2}$$

$$\bar{y} = y + \epsilon\eta(t, x, y) + o(\epsilon^2). \tag{4.3}$$

Where

$$\frac{d\bar{t}}{d\epsilon}\Big|_{\epsilon=0} = \xi(t, x, y), \quad \frac{d\bar{x}}{d\epsilon}\Big|_{\epsilon=0} = \tau(t, x, y), \quad \frac{d\bar{y}}{d\epsilon}\Big|_{a=0} = \eta(t, x, y). \tag{4.4}$$

Then, we have the following result.

Theorem 4.1. *The infinitesimal transformation of the conformable derivative is given by*

$${}_0T^{\alpha}_{\bar{t}}\bar{y} = {}_0T^{\alpha}_{t}y + \epsilon\eta^{(\alpha)}_t + o(\epsilon^2), \tag{4.5}$$

where

$$\eta^{(\alpha)}_t = {}_0T^{\alpha}_{t}\eta - \left(t^{\alpha-1}({}_0T^{\alpha}_{t}\xi) - \frac{1-\alpha}{t}\xi\right)({}_0T^{\alpha}_{t}y) - y_x.({}_0T^{\alpha}_{t}\tau), \tag{4.6}$$

Proof. Equations (4.1) and (4.2) may be inverted (locally) to give t and x in terms of $\bar{t}$ and $\bar{x}$, provided that the Jacobian is nonzero, that is,

$$\mathcal{J} = \begin{vmatrix} D_t[\bar{t}] & D_t[\bar{x}] \\ D_x[\bar{t}] & D_x[\bar{x}] \end{vmatrix} \neq 0, \tag{4.7}$$

where $y = y(t, x)$ and

$$D_t = \frac{\partial}{\partial t} + y_t \frac{\partial}{\partial y} + y_{tt} \frac{\partial}{\partial y_t} + y_{tx} \frac{\partial}{\partial y_x} + ... \tag{4.8}$$

$$D_x = \frac{\partial}{\partial x} + y_x \frac{\partial}{\partial y} + y_{xx} \frac{\partial}{\partial y_x} + y_{xt} \frac{\partial}{\partial y_t} + ..., \tag{4.9}$$

denote the total derivatives that treat the dependent variable y and its derivatives according to the independent variables. if Eq. (4.7) is satisfied, then the equation (4.3) can be rewritten as:

$$\bar{y} = \bar{y}(\bar{t}, \bar{x}). \tag{4.10}$$

Applying the chain rule to (4.10), we obtain

$$\begin{bmatrix} D_t[\bar{y}] \\ D_x[\bar{y}] \end{bmatrix} = \begin{bmatrix} D_t[\bar{t}] & D_t[\bar{x}] \\ D_x[\bar{t}] & D_x[\bar{x}] \end{bmatrix} \begin{bmatrix} \bar{y}_{\bar{t}} \\ \bar{y}_{\bar{x}} \end{bmatrix}, \tag{4.11}$$

and therefore, by Cramer's rule, we obtain

$$\bar{y}_{\bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t[\bar{y}] & D_t[\bar{x}] \\ D_x[\bar{y}] & D_x[\bar{x}] \end{vmatrix}, \qquad \bar{y}_{\bar{x}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t[\bar{t}] & D_t[\bar{y}] \\ D_x[\bar{t}] & D_x[\bar{y}] \end{vmatrix}. \tag{4.12}$$

Dropping all terms with of order ϵ^2, we get

$$\mathcal{J} = 1 + \epsilon\Big((\xi_t + y_t\xi_y) + (\tau_x + y_x\tau_y)\Big) + o(\epsilon^2). \tag{4.13}$$

Thus,

$$\frac{1}{\mathcal{J}} = 1 - \epsilon\Big((\xi_t + y_t\xi_y) + (\tau_x + y_x\tau_y)\Big) + o(\epsilon^2). \tag{4.14}$$

Then,

$${}_0T^{\alpha}_{\bar{t}}\bar{y} = \bar{t}^{1-\alpha}\bar{y}_{\bar{t}} \tag{4.15}$$

$$\begin{aligned} &= \Big(t^{1-\alpha}\Big(1 + \frac{(1-\alpha)\epsilon}{t}\xi\Big) + o(\epsilon^2)\Big)\Big(1 - \epsilon\Big((\xi_t + y_t\xi_y) + (\tau_x + y_x\tau_y)\Big) \\ &+ o(\epsilon^2)\Big) \times \Big[\Big(y_t + \epsilon(\eta_t + y_t\eta_y) + o(\epsilon^2)\Big)\Big(1 + \epsilon(\tau_x + y_x\tau_y) + o(\epsilon^2)\Big) \\ &- \Big(y_x + \epsilon(\eta_x + y_x\eta_y) + o(\epsilon^2)\Big)\Big(\epsilon(\tau_t + y_t\tau_y) + o(\epsilon^2)\Big)\Big] \\ &= \Big(t^{1-\alpha}\Big(1 + \frac{(1-\alpha)\epsilon}{t}\xi\Big) + o(\epsilon^2)\Big)\Big[y_t + \epsilon\Big(\eta_t + (\eta_y - \xi_t - y_t\xi_y)y_t \\ &- (\tau_t + y_t\tau_y)y_x\Big) + o(\epsilon^2)\Big] \end{aligned} \tag{4.16}$$

$$= {}_0T_t^{\alpha}y + \epsilon\Big[t^{1-\alpha}\eta_t + t^{1-\alpha}(\eta_y + \frac{1-\alpha}{t}\xi - \xi_t)y_t - t^{1-\alpha}(\tau_t + y_t\tau_y)y_x$$
$$- t^{1-\alpha}\xi_y y_t^2\Big] + o(\epsilon^2). \tag{4.17}$$

Hence,

$$ {}_0T_{\bar{t}}^{\alpha}\bar{y} = {}_0T_t^{\alpha}y + \epsilon\eta_t^{(\alpha)} + o(\epsilon^2), \tag{4.18}$$

where

$$\eta_t^{(\alpha)} = {}_0T_t^{\alpha}\eta - \Big(t^{\alpha-1}({}_0T_t^{\alpha}\xi) - \frac{1-\alpha}{t}\xi\Big)({}_0T_t^{\alpha}y) - y_x.({}_0T_t^{\alpha}\tau), \tag{4.19}$$

with

$$ {}_0T_t^{\alpha}\tau = t^{1-\alpha}D_t\tau = t^{1-\alpha}(\tau_t + y_t\tau_y) \text{ and } {}_0T_t^{\alpha}\eta = t^{1-\alpha}D_t\eta = t^{1-\alpha}(\eta_t + y_t\eta_y).$$

□

Example 2. *We consider the time conformable heat equation defined by*

$$y_t^{(\alpha)} = y_{xx}. \tag{4.20}$$

In the particular case $\alpha = 1$, the above equation is invariant under the following group transformations

$$\bar{x} = x + \epsilon,\ \bar{t} = t,\ \bar{y} = y. \tag{4.21}$$
$$\bar{x} = x,\ \bar{t} = t + \epsilon,\ \bar{y} = y. \tag{4.22}$$
$$\bar{x} = x,\ \bar{t} = t,\ \bar{y} = e^{\epsilon}y. \tag{4.23}$$
$$\bar{x} = e^{\epsilon}x,\ \bar{t} = e^{2\epsilon}t,\ \bar{y} = y. \tag{4.24}$$
$$\bar{x} = x + 2\epsilon t,\ \bar{t} = t,\ \bar{y} = ye^{-(\epsilon x + \epsilon^2 t)}. \tag{4.25}$$
$$\bar{x} = \frac{x}{1-4\epsilon t},\ \bar{t} = \frac{t}{1-4\epsilon t},\ \bar{y} = y\sqrt{1-4\epsilon t}\exp\Big(\frac{-\epsilon x^2}{1-4\epsilon t}\Big). \tag{4.26}$$
$$\bar{x} = x, \bar{t} = t, \bar{y} = y + \epsilon\beta(x,t). \tag{4.27}$$

Where $\beta(t,x)$ is an arbitrary solution of the heat equation. Now, we show that transformations (4.21) to (4.27) do not all preserve the conformable heat equation (4.20) where $\alpha \neq 0$:
In the case of transformation (4.22), we have

$$\bar{y}_{\bar{t}}^{(\alpha)} = \bar{t}^{1-\alpha}\bar{y}_{\bar{t}} = (t+\epsilon)^{1-\alpha}y_{\bar{t}}$$
$$= (t+\epsilon)^{1-\alpha}y_t = \Big(1 + \frac{\epsilon}{t}\Big)^{1-\alpha}y_t^{(\alpha)} = \Big(1 + \frac{\epsilon}{t}\Big)^{1-\alpha}y_{xx}, \tag{4.28}$$

and $\bar{y}_{\bar{x}\bar{x}} = y_{xx}$, consequently,

$$\bar{y}_{\bar{t}}^{(\alpha)} \neq \bar{y}_{\bar{x}\bar{x}}. \tag{4.29}$$

In a similar way, we find also that transformations (4.24), (4.25) and (4.26) do not preserve the conformable heat equation. However, the transformations (4.21) satisfies

$$\bar{y}_{\bar{x}\bar{x}} = \frac{\partial}{\partial\bar{x}}\bar{y}_{\bar{x}} = \frac{\partial}{\partial\bar{x}}y_x = y_{xx} = y_t^{(\alpha)} = \bar{y}_{\bar{t}}^{(\alpha)}, \tag{4.30}$$

then, it preserves equation (4.20). Similarly, the transformations (4.23) and (4.27) will also leave the equation (4.20) invariant.

Now, Let us assume that the conformable heat equation (4.20) is invariant under a one-parameter (ϵ) continuous point transformations

$$\bar{t} = t + \epsilon\xi(t,x,y) + o(\epsilon^2), \tag{4.31}$$
$$\bar{x} = x + \epsilon\tau(t,x,y) + o(\epsilon^2), \tag{4.32}$$
$$\bar{y} = y + \epsilon\eta(t,x,y) + o(\epsilon^2), \tag{4.33}$$
$$\bar{y}_{\bar{t}}^{(\alpha)} = y_t^{(\alpha)} + \epsilon\eta_t^{(\alpha)} + o(\epsilon^2), \tag{4.34}$$
$$\bar{y}_{\bar{x}} = y_x + \epsilon\eta_x^{(1)} + o(\epsilon^2), \tag{4.35}$$
$$\bar{y}_{\bar{x}\bar{x}} = y_{xx} + \epsilon\eta_x^{(2)} + o(\epsilon^2). \tag{4.36}$$

Where $\eta_x^{(1)}$ and $\eta_x^{(2)}$ are given by

$$\eta_x^{(1)} = \eta_x + (\eta_y - \tau_x)y_x - \xi_x y_t - \tau_y y_x^2 - \xi_y y_x y_t, \tag{4.37}$$
$$\begin{aligned}\eta_x^{(2)} = {} & \eta_{xx} + (2\eta_{xy} - \tau_{xx})y_x - \xi_{xx}y_t + (\eta_{yy} - 2\tau_{xy})y_x^2 - 2\xi_{xy}y_x y_t - \tau_{yy}y_x^3 \\ & - \xi_{yy}y_x^2 y_t + (\eta_y - 2\tau_x)y_{xx} - 2\xi_x y_{xt} - 3\tau_y y_{xx}y_x - \xi_y y_{xx}y_t - 2\xi_y y_{xt}y_x.\end{aligned} \tag{4.38}$$

Suppose

$$X = \xi(t,x,y)\frac{\partial}{\partial t} + \tau(t,x,y)\frac{\partial}{\partial x} + \eta(t,x,y)\frac{\partial}{\partial y}, \tag{4.39}$$

be a symmetry operator for the Eq. (4.20). According to the invariance criterion, Eq. (4.20) admits the group transformations (4.31)–(4.33) if the prolonged $X^{(\alpha)}$ annihilates (4.20) on its solution, namely

$$X^{(\alpha)}(\Delta y)|_{(\Delta y=0)} = 0, \tag{4.40}$$

where $\Delta y = y_t^{(\alpha)} - y_{xx}$, and

$$X = X + \eta_x^{(1)}\frac{\partial}{\partial y_x} + \eta_x^{(2)}\frac{\partial}{\partial y_{xx}} + \eta_t^{(\alpha)}\frac{\partial}{\partial y_t^{(\alpha)}}. \tag{4.41}$$

Then, the invariance criterion is written as

$$t^{1-\alpha}\eta_t + \left(\eta_y + \frac{1-\alpha}{t}\xi - \xi_t\right)y_{xx} - t^{\alpha-1}\xi_y(y_{xx})^2 - t^{1-\alpha}(\tau_t + t^{\alpha-1}y_{xx}\tau_y)y_x - \eta_x^{(2)} = 0. \tag{4.42}$$

Solving the above equation, we derive the determining system

$$-t^{1-\alpha}\tau_t + \tau_{xx} - 2\eta_{xy} = 0, \tag{4.43}$$
$$2\tau_{xy} - \eta_{yy} = 0, \tag{4.44}$$
$$\tau_{yy} = 0, \tag{4.45}$$
$$-\xi_t + \frac{1-\alpha}{t}\xi + 2\tau_x + t^{\alpha-1}\xi_{xx} = 0, \tag{4.46}$$
$$-t^{\alpha-1}\xi_y + \xi_y t^{\alpha-1} = 0, \tag{4.47}$$

$$2\xi_{xy}t^{\alpha-1} + 2\tau_y = 0, \tag{4.48}$$
$$\xi_{yy}t^{\alpha-1} = 0, \tag{4.49}$$
$$2\xi_x = 0, \tag{4.50}$$
$$2\xi_y = 0, \tag{4.51}$$
$$t^{1-\alpha}\eta_t - \eta_{xx} = 0. \tag{4.52}$$

Solution of the determining system, gives

$$\xi(t,x,y) = c_6 t^{1-\alpha} + \frac{t}{\alpha}\Big(-\frac{4c_1}{\alpha}t^\alpha + c_3\Big), \tag{4.53}$$
$$\tau(t,x,y) = \Big(-\frac{4c_1}{\alpha}t^\alpha + c_3\Big)x + \Big(-\frac{2c_2}{\alpha}t^\alpha + c_4\Big), \tag{4.54}$$
$$\eta(t,x,y) = \Big(c_1x^2 + c_2x + \Big(\frac{2c_1}{\alpha}t^\alpha + c_5\Big)\Big)y + c_7\gamma(t,x). \tag{4.55}$$

where $c_1, ..., c_6$ are arbitrary constants and $\gamma(t,x)$ is an arbitrary solution of the conformable heat equation. Thus, the Lie symmetry algebra admitted by equation (4.20) is spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = t^{1-\alpha}\frac{\partial}{\partial t}, \qquad X_3 = y\frac{\partial}{\partial y}, \tag{4.56}$$
$$X_4 = \frac{2t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad X_5 = \frac{2t^\alpha}{\alpha}\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}, \tag{4.57}$$
$$X_6 = \frac{4t^{1+\alpha}}{\alpha^2}\frac{\partial}{\partial t} + \frac{4xt^\alpha}{\alpha}\frac{\partial}{\partial x} - \Big(x^2 + \frac{2t^\alpha}{\alpha}\Big)y\frac{\partial}{\partial y}. \tag{4.58}$$

and the infinite dimensional subalgebra

$$X_\infty = \gamma(t,x)\frac{\partial}{\partial y}, \tag{4.59}$$

where γ is an arbitrary solution of the conformable heat equation. To obtain the group transformation generated by each infinitesimal symmetry X_k, we solve the system of first order ordinary differential equations,

$$\frac{d\bar{t}}{da} = \xi(\bar{t},\bar{x},\bar{y}), \qquad \frac{d\bar{x}}{da} = \tau(\bar{t},\bar{x},\bar{y}), \qquad \frac{d\bar{y}}{da} = \eta(\bar{t},\bar{x},\bar{y}),$$

subject to the initial conditions

$$\bar{t}(\epsilon = 0) = t, \qquad \bar{x}(\epsilon = 0) = x, \qquad \bar{y}(\epsilon = 0) = y.$$

The one-parameter groups G_i generated by X_i are given as follows

$$G_1 : (t,x,y) \to \Big(t, x+\epsilon, y\Big), \tag{4.60}$$
$$G_2 : (t,x,y) \to \Big((t^\alpha + \epsilon\alpha)^{1/\alpha}, x, y\Big), \tag{4.61}$$
$$G_3 : (t,x,y) \to \Big(t, x, e^\epsilon y\Big), \tag{4.62}$$

$$G_4 :(t,x,y) \to \Big(e^{2\epsilon/\alpha}t, e^{\epsilon}x, y\Big), \tag{4.63}$$

$$G_5 :(t,x,y) \to \Big(t, x + 2\epsilon(t^{\alpha}/\alpha), ye^{-\left(\epsilon x + \epsilon^2 (t^{\alpha}/\alpha)\right)}\Big), \tag{4.64}$$

$$G_6 :(t,x,y) \to \left(\left(\frac{t^{\alpha}}{1-4\epsilon(t^{\alpha}/\alpha)}\right)^{1/\alpha}, \frac{x}{1-4\epsilon(t^{\alpha}/\alpha)}, y\sqrt{1-4\epsilon(t^{\alpha}/\alpha)}\exp\left(\frac{-\epsilon x^2}{1-4\epsilon(t^{\alpha}/\alpha)}\right)\right), \tag{4.65}$$

$$G_{\infty} :(t,x,y) \to \Big(t, x, y + \epsilon\gamma(t,x)\Big). \tag{4.66}$$

Consequently, since G_i is a symmetry, if $y = f(t,x)$ is a solution of the conformable heat equation, so are

$$y_1 = f(t, x-\epsilon), \tag{4.67}$$

$$y_2 = f((t^{\alpha} - \epsilon\alpha)^{1/\alpha}, x), \tag{4.68}$$

$$y_3 = e^{\epsilon} f(t,x), \tag{4.69}$$

$$y_4 = f(e^{-2\epsilon/\alpha}t, e^{-\epsilon}x), \tag{4.70}$$

$$y_5 = f\Big(t, x - 2\epsilon(t^{\alpha}/\alpha)\Big)e^{-\left(\epsilon x - \epsilon^2 (t^{\alpha}/\alpha)\right)}, \tag{4.71}$$

$$y_6 = \frac{f\left(\left(\frac{t^{\alpha}}{1+4\epsilon(t^{\alpha}/\alpha)}\right)^{1/\alpha}, \frac{x}{1+4\epsilon(t^{\alpha}/\alpha)}\right)}{\sqrt{1+4\epsilon(t^{\alpha}/\alpha)}}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon(t^{\alpha}/\alpha)}\right), \tag{4.72}$$

$$y_{\infty} = f(t,x) + \epsilon\gamma(t,x). \tag{4.73}$$

If we let $y = 1$ be a constant solution of the conformable heat equation, we conclude that the function

$$G_6(\epsilon).1 = \frac{1}{\sqrt{1+4\epsilon(t^{\alpha}/\alpha)}}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon(t^{\alpha}/\alpha)}\right), \tag{4.74}$$

is also solution. Furthermore, we can derive other exact solutions of conformable heat equation if we continue this iteration process.

5. Conclusion and comments

In this work, we have shown that the Lie symmetry analysis could be extended to conformable differential equations as it has been done for fractional differential equations based on the Riemann-Liouville or the Caputo approaches. This extension might be more investigated to employ various developed properties of invariance to derive exact solutions of other linear and nonlinear conformable differential equations. In addition, this work will bring new opportunities in studying some non autonomous system of partial differential equations with p-dependent variables and q-independent variables. Finally, it will be important to develop the Lie symmetry analysis for differential equations involving different fractional derivatives such as Caputo-Fabrizio derivative and Atangana-Baleanu derivative.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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