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Research article

Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions

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Abstract: In this paper, we prove the existence and uniqueness of a positive solution of nonlinear Hadamard fractional differential equations with integral boundary conditions. In the process we employ the Schauder and Banach fixed point theorems and the method of upper and lower solutions to show the existence and uniqueness of a positive solution. Finally, an example is given to illustrate our results.

Keywords: fractional differential equations; positive solutions; upper and lower solutions; existence; uniqueness; fixed point theorems
Mathematics Subject Classification: 26A33, 34A12, 34G20

1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of the positivity of such solutions for fractional differential equations (FDE) have received the attention of many authors, see [1–20] and the references therein.

Fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have been studied extensively by several researchers. However, the literature on Hadamard type fractional differential equations is not yet as enriched. The fractional derivative due to Hadamard, introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent.

Zhang in [1] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)),\ 0 < t \leq 1, \\ x(0) = 0, \end{cases}$$

where D^{α} is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional differential equation boundary value problem

$$\begin{cases} D^{\alpha}x(t) + f(t, x(t)) = 0,\ 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases}$$

has been investigated in [2], where $1 < \alpha \le 2$, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions have been established.

In [3], Matar discussed the existence and uniqueness of the positive solution of the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)),\ 0 < t \le 1, \\ x(0) = 0,\ x'(0) = \theta > 0, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $1 < \alpha \le 2$, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the author obtained positivity results.

The integral boundary value problems of the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x(t) + f(t, x(t)) = D^{\beta}g(t, x(t)),\ 0 < t < 1, \\ x(0) = 0,\ x(1) = \dfrac{1}{\Gamma(\alpha-\beta)}\displaystyle\int_0^1 (1-s)^{\alpha-\beta-1} g(s, x(s))\, ds, \end{cases}$$

has been investigated in [4], where $1 < \alpha \le 2$, $0 < \beta < \alpha$, $g, f : [0,1] \times [0,\infty) \to [0,\infty)$ are given continuous functions, g is non-decreasing on x. By employing the method of the upper and lower solutions and the Schauder and Banach fixed point theorems, the authors obtained positivity results.

Benchohra and Lazreg in [5] studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^{\alpha}x(t) = f(t, x(t), \mathfrak{D}_1^{\alpha}x(t)),\ t \in [1, T], \\ x(1) = x_0, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ is the Hadamard fractional derivatives of order $0 < \alpha \le 1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. Inspired and motivated by the works mentioned above, we concentrate on the positivity of solutions for the nonlinear Hadamard fractional differential equation with integral boundary conditions

$$\begin{cases} \mathfrak{D}_1^{\alpha}x(t) + f(t, x(t)) = \mathfrak{D}_1^{\beta}g(t, x(t)),\ 1 < t < e, \\ x(1) = 0,\ x(e) = \dfrac{1}{\Gamma(\alpha-\beta)}\displaystyle\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\beta-1} g(s, x(s))\, \frac{ds}{s}, \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, $0 < \beta \le \alpha - 1$, $g, f : [1, e] \times [0,\infty) \to [0,\infty)$ are given continuous functions, g is non-decreasing on x and f is not required any monotone assumption. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use Schauder and Banach fixed point theorems.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. In section 3, we give and prove our main results on positivity, and we provide an example to illustrate our results.

2. Preliminaries

Let $X = C([1, e])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1, e]$, endowed with the maximum norm. Define the subspace $\mathcal{A} = \{x \in X : x(t) \geq 0,\ t \in [1, e]\}$ of X. By a positive solution $x \in X$, we mean a function $x(t) > 0$, $1 \leq t \leq e$.

Let $a, b \in \mathbb{R}^+$ such that $b > a$. For any $x \in [a, b]$, we define the upper-control function

$$U(t, x) = \sup \{f(t, \lambda) : a \leq \lambda \leq x\},$$

and lower-control function

$$L(t, x) = \inf \{f(t, \lambda) : x \leq \lambda \leq b\}.$$

Obviously, $U(t, x)$ and $L(t, x)$ are monotonous non-decreasing on the argument x and $L(t, x) \leq f(t, x) \leq U(t, x)$.

2.1. Fractional differential equations

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [16–18].

Definition 2.1. [16] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function $x : [0, +\infty) \to \mathbb{R}$ is defined as

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) \frac{ds}{s}.$$

Definition 2.2. [16] The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{I}_1^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} x(s) \frac{ds}{s}.$$

Definition 2.3. [16] *T*he Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $x : [0, +\infty) \to \mathbb{R}$ is defined as

$$D^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt}\right)^n \int_0^t (t - s)^{n - \alpha - 1} x(s) \frac{ds}{s},\ n - 1 < \alpha < n.$$

Definition 2.4. [16] *T*he Hadamard fractional derivative of order $\alpha > 0$ for a function $x : [1, +\infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_1^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n - \alpha - 1} x(s) \frac{ds}{s},\ n - 1 < \alpha < n.$$

Lemma 2.5. *[16] Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$. The equality $\left(\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\alpha}x\right)(t)=0$ is valid if and only if*

$$x(t)=\sum_{k=1}^{n}c_k(\log t)^{\alpha-k}\ \textit{for each}\ t\in[1,e],$$

where $c_k\in\mathbb{R}$, $k=1,...,n$ are arbitrary constants.

Lemma 2.6. *[16] For all $\mu>0$ and $\nu>-1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^t\left(\log\frac{t}{s}\right)^{\mu-1}(\log s)^{\nu}\frac{ds}{s}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

The following lemma is fundamental to our results.

Lemma 2.7. *Let $x\in C^1([1,e])$, $x^{(2)}$ and $\frac{\partial g}{\partial t}$ exist, then x is a solution of (1.1) if and only if*

$$x(t)=\int_1^e H(t,s)\,f(s,x(s))\frac{ds}{s}+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}, \tag{2.1}$$

where

$$H(t,s)=\begin{cases}\dfrac{\left(\log t\log\frac{e}{s}\right)^{\alpha-1}-\left(\log\frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}, & 1\leq s\leq t\leq e,\\[2mm] \dfrac{\left(\log t\log\frac{e}{s}\right)^{\alpha-1}}{\Gamma(\alpha)}, & 1\leq t\leq s\leq e.\end{cases} \tag{2.2}$$

Proof. Let x be a solution of (1.1). First we write this equation as

$$\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\alpha}x(t)=\mathfrak{I}_1^{\alpha}\left(-f(t,x(t))+\mathfrak{D}_1^{\beta}g(t,x(t))\right),\ 1<t<e.$$

From Lemma 2.5, we have

$$\begin{aligned}
&x(t)-c_2(\log t)^{\alpha-2}-c_1(\log t)^{\alpha-1}\\
&=-\mathfrak{I}_1^{\alpha}f(t,x(t))+\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\beta}g(t,x(t))\\
&=-\mathfrak{I}_1^{\alpha}f(t,x(t))+\mathfrak{I}_1^{\alpha-\beta}\mathfrak{I}_1^{\beta}\mathfrak{D}_1^{\beta}g(t,x(t))\\
&=-\mathfrak{I}_1^{\alpha}f(t,x(t))+\mathfrak{I}_1^{\alpha-\beta}\left(g(t,x(t))-c_3(\log t)^{\beta-1}\right)\\
&=-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}-c_3\frac{\Gamma(\beta)(\log t)^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}$$

Then

$$\begin{aligned}
x(t)&=c_2+\left(c_1-c_3\frac{\Gamma(\beta)}{\Gamma(\alpha)}\right)(\log t)^{\alpha-1}\\
&-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}.$$

By boundary conditions $x(1)=0$ and $x(e)=\frac{1}{\Gamma(\alpha-\beta)}\int_1^e\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}$, we obtain $c_2=0$ and

$$c_1-\frac{c_3}{\alpha-1}=\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}.$$

Therefore

$$\begin{aligned}x(t)&=\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log t\log\frac{e}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}\\&-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,x(s))\frac{ds}{s}\\&+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}\\&=\int_1^e H(t,s)f(s,x(s))\frac{ds}{s}+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}.\end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

Corollary 2.8. *The function H defined by (2.2) satisfies*
(1) $H(t,s)>0$ for $t,s\in(1,e)$,
(2) $\max_{1\le t\le e}H(t,s)=H(s,s)$, $s\in(1,e)$.

2.2. Fixed point theorems

In this subsection, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

Definition 2.9. Let $(X,\|.\|)$ be a Banach space and $\Phi: X\to X$. The operator Φ is a contraction operator if there is an $\lambda\in(0,1)$ such that $x,y\in X$ imply

$$\|\Phi x-\Phi y\|\le\lambda\|x-y\|.$$

Theorem 2.10. *[21] Let C be a nonempty closed convex subset of a Banach space X and $\Phi: C\to C$ be a contraction operator. Then there is a unique $x\in C$ with $\Phi x=x$.*

Theorem 2.11. *[21] Let C be a nonempty closed convex subset of a Banach space X and $\Phi: C\to C$ be a continuous compact operator. Then Φ has a fixed point in C.*

3. Main results

In this section, we consider the results of existence problem for many cases of the FDE (1.1). Moreover, we introduce the sufficient conditions of the uniqueness problem of (1.1).

To transform equation (2.1) to be applicable to Schauder fixed point, we define an operator $\Phi : \mathcal{A} \longrightarrow X$ by

$$(\Phi x)(t) = \int_1^e H(t,s) f(s, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} g(s, x(s)) \frac{ds}{s}, \ t \in [1,e], \tag{3.1}$$

where the figured fixed point must satisfy the identity operator equation $\Phi x = x$.

The following assumptions are needed for the next results.

$(H1)$ Let $x^*, x_* \in \mathcal{A}$, such that $a \le x_*(t) \le x^*(t) \le b$ and

$$\begin{cases} \mathfrak{D}_1^{\alpha} x^*(t) + U(t, x^*(t)) \ge \mathfrak{D}_1^{\beta} g(t, x^*(t)), \\ \mathfrak{D}_1^{\alpha} x_*(t) + L(t, x_*(t)) \le \mathfrak{D}_1^{\beta} g(t, x_*(t)), \end{cases}$$

for any $t \in [1,e]$.

$(H2)$ For $t \in [1,e]$ and $x, y \in X$, there exist positive real numbers $\beta_1, \beta_2 < 1$ such that

$$|g(t,y) - g(t,x)| \le \beta_1 \|y - x\|,$$
$$|f(t,y) - f(t,x)| \le \beta_2 \|y - x\|.$$

The functions x^* and x_* are respectively called the pair of upper and lower solutions for Equation (1.1).

Theorem 3.1. *Assume that* $(H1)$ *is satisfied, then the FDE (1.1) has at least one solution* $x \in X$ *satisfying* $x_*(t) \le x(t) \le x^*(t)$, $t \in [1,e]$.

Proof. Let $C = \{x \in \mathcal{A} : x_*(t) \le x(t) \le x^*(t), \ t \in [1,e]\}$, endowed with the norm $\|x\| = \max_{t\in[1,e]} |x(t)|$, then we have $\|x\| \le b$. Hence, C is a convex, bounded, and closed subset of the Banach space X. Moreover, the continuity of g and f implies the continuity of the operator Φ on C defined by (3.1). Now, if $x \in C$, there exist positive constants c_f and c_g such that

$$\max\{f(t, x(t)) : t \in [1,e], \ x(t) \le b\} < c_f,$$

and

$$\max\{g(t, x(t)) : t \in [1,e], \ x(t) \le b\} < c_g.$$

Then

$$\begin{aligned} |(\Phi x)(t)| &\le \int_1^e H(s,s) |f(s, x(s))| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} |g(s, x(s))| \frac{ds}{s} \\ &\le c_f \int_1^e \frac{\left(\log s \log \frac{e}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} \frac{ds}{s} + c_g \frac{(\log t)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\ &\le \frac{\Gamma(\alpha) c_f}{\Gamma(2\alpha)} + \frac{c_g}{\Gamma(\alpha-\beta+1)}. \end{aligned}$$

Thus,

$$\|\Phi x\| \le \frac{\Gamma(\alpha) c_f}{\Gamma(2\alpha)} + \frac{c_g}{\Gamma(\alpha-\beta+1)}.$$

Hence, $\Phi(C)$ is uniformly bounded. Next, we prove the equicontinuity of $\Phi(C)$. Let $x \in C$, then for any $t_1, t_2 \in [1, e]$, $t_2 > t_1$, we have

$$\begin{aligned}
&\left|(\Phi x)(t_2) - (\Phi x)(t_1)\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_1^e \left(\log t_2 \log \frac{e}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} - \int_1^e \left(\log t_1 \log \frac{e}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right| \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} - \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \right| \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \left| \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-\beta-1} g(s, x(s)) \frac{ds}{s} - \int_1^{t_1} \left(\log \frac{t_2}{s}\right)^{\alpha-\beta-1} g(s, x(s)) \frac{ds}{s} \right| \\
&\leq \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right) |f(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_1^{t_1} \left(\left(\log \frac{t_2}{s}\right)^{\alpha-\beta-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-\beta-1} \right) |g(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-\beta-1} |g(s, x(s))| \frac{ds}{s} \\
&\leq \frac{c_f}{\Gamma(\alpha+1)} \left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1} + (\log t_2)^{\alpha} - (\log t_1)^{\alpha} \right) \\
&+ \frac{c_g}{\Gamma(\alpha-\beta+1)} \left((\log t_2)^{\alpha-\beta} - (\log t_1)^{\alpha-\beta} \right).
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the previous inequality is independent of x and tends to zero. Therefore, $\Phi(C)$ is equicontinuous. The Arzelè-Ascoli theorem implies that $\Phi : C \longrightarrow X$ is compact. The only thing to apply Schauder fixed point is to prove that $\Phi(C) \subseteq C$. Let $x \in C$, then by hypotheses, we have

$$\begin{aligned}
(\Phi x)(t) &= \int_1^e H(t, s) f(s, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} g(s, x(s)) \frac{ds}{s} \\
&\leq \int_1^e H(t, s) U(s, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} g(s, x^*(s)) \frac{ds}{s} \\
&\leq \int_1^e H(t, s) U(s, x^*(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} g(s, x^*(s)) \frac{ds}{s} \\
&\leq x^*(t),
\end{aligned}$$

and

$$\begin{aligned}
(\Phi x)(t) &= \int_1^e H(t, s) f(s, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} g(s, x(s)) \frac{ds}{s} \\
&\geq \int_1^e H(t, s) L(t, x(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-\beta-1} g(s, x_*(s)) \frac{ds}{s}
\end{aligned}$$

$$\geq \int_1^e H(t,s)\,L(t,x_*(s))\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} g(s,x_*(s))\,\frac{ds}{s}$$
$$\geq x_*(t).$$

Hence, $x_*(t) \leq (\Phi x)(t) \leq x^*(t)$, $t \in [1,e]$, that is, $\Phi(C) \subseteq C$. According to Schauder fixed point theorem, the operator Φ has at least one fixed point $x \in C$. Therefore, the FDE (1.1) has at least one positive solution $x \in X$ and $x_*(t) \leq x(t) \leq x^*(t)$, $t \in [1,e]$. □

Next, we consider many particular cases of the previous theorem.

Corollary 3.2. *Assume that there exist continuous functions k_1, k_2, k_3 and k_4 such that*

$$0 < k_1(t) \leq g(t,x(t)) \leq k_2(t) < \infty,\ (t,x(t)) \in [1,e]\times[0,+\infty), \tag{3.2}$$

and

$$0 < k_3(t) \leq f(t,x(t)) \leq k_4(t) < \infty,\ (t,x(t)) \in [1,e]\times[0,+\infty). \tag{3.3}$$

Then, the FDE (1.1) has at least one positive solution $x \in X$. Moreover,

$$\int_1^e H(t,s)\,k_3(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_1(s)\frac{ds}{s}$$
$$\leq x(t)$$
$$\leq \int_1^e H(t,s)\,k_4(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_2(s)\frac{ds}{s}. \tag{3.4}$$

Proof. By the given assumption (3.3) and the definition of control function, we have $k_3(t) \leq L(t,x) \leq U(t,x) \leq k_4(t)$, $(t,x(t)) \in [1,e]\times[a,b]$. Now, we consider the equations

$$\begin{cases} \mathfrak{D}_1^{\alpha}x(t) + k_3(t) = \mathfrak{D}_1^{\beta}k_1(t), \\ x(1)=0,\ x(e) = \frac{1}{\Gamma(\alpha-\beta)}\int_1^e \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_1(s)\dfrac{ds}{s}, \\ \mathfrak{D}_1^{\alpha}x(t) + k_4(t) = \mathfrak{D}_1^{\beta}k_2(t), \\ x(1)=0,\ x(e) = \frac{1}{\Gamma(\alpha-\beta)}\int_1^e \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_2(s)\dfrac{ds}{s}. \end{cases} \tag{3.5}$$

Obviously, Equations (3.5) are equivalent to

$$x(t) = \int_1^e H(t,s)\,k_3(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_1(s)\frac{ds}{s},$$
$$x(t) = \int_1^e H(t,s)\,k_4(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_2(s)\frac{ds}{s}.$$

Hence, the first implies

$$x(t) - \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} k_1(s)\frac{ds}{s}$$
$$= \int_1^e H(t,s)\,k_3(s)\frac{ds}{s} \leq \int_1^e H(t,s)\,L(s,x(s))\frac{ds}{s},$$

and the second implies

$$\begin{aligned}
& x(t)-\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}k_2(s)\frac{ds}{s}\\
&=\int_1^e H(t,s)\,k_4(s)\frac{ds}{s}\geq\int_1^e H(t,s)\,U(s,x(s))\frac{ds}{s},
\end{aligned}$$

which are the upper and lower solutions of Equations (3.5), respectively. An application of Theorem 3.1 yields that the FDE (1.1) has at least one solution $x\in X$ and satisfies Equation (3.4). □

Corollary 3.3. *Assume that (3.2) holds and* $0<\sigma<k(t)=\lim_{x\to\infty}f(t,x)<\infty$ *for* $t\in[1,e]$*. Then the FDE (1.1) has at least a positive solution* $x\in X$*.*

Proof. By assumption, if $x>\rho>0$, then $0\leq|f(t,x)-k(t)|<\sigma$ for any $t\in[1,e]$. Hence, $0<k(t)-\sigma\leq f(t,x)\leq k(t)+\sigma$ for $t\in[1,e]$ and $\rho<x<+\infty$. Now if $\max\{f(t,x):t\in[1,e],\ x\leq\rho\}\leq\nu$, then $k(t)-\sigma\leq f(t,x)\leq k(t)+\sigma+\nu$ for $t\in[1,e]$, and $0<x<+\infty$. By Corollary 3.3, the FDE (1.1) has at least one positive solution $x\in X$ satisfying

$$\begin{aligned}
&\int_1^e H(t,s)\,k(s)\frac{ds}{s}-\frac{\sigma\left((\log t)^{\alpha-1}-(\log t)^{\alpha}\right)}{\Gamma(\alpha+1)}\\
&+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}k_1(s)\frac{ds}{s}\\
&\leq x(t)\\
&\leq\int_1^e H(t,s)\,k(s)\frac{ds}{s}+\frac{(\sigma+\nu)\left((\log t)^{\alpha-1}-(\log t)^{\alpha}\right)}{\Gamma(\alpha+1)}\\
&+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}k_2(s)\frac{ds}{s}.
\end{aligned}$$

□

Corollary 3.4. *Assume that* $0<\sigma<f(t,x(t))\leq\gamma x(t)+\eta<\infty$ *for* $t\in[1,e]$*, and* σ*,* η *and* γ *are positive constants with* $\frac{\Gamma(\alpha)\gamma}{\Gamma(2\alpha)}<1$*. Then, the FDE (1.1) has at least one positive solution* $x\in C([1,e])$*.*

Proof. Consider the equation

$$\begin{cases}\mathfrak{D}_1^{\alpha}x(t)+(\gamma x(t)+\eta)=\mathfrak{D}_1^{\beta}g(t,x(t)),\ 1<t<e,\\ x(1)=0,\ x(e)=\frac{1}{\Gamma(\alpha-\beta)}\int_1^e\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}.\end{cases}\tag{3.6}$$

Equation (3.6) is equivalent to integral equation

$$x(t)=\int_1^e H(t,s)\,(\gamma x(s)+\eta)\frac{ds}{s}+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\beta-1}g(s,x(s))\frac{ds}{s}.$$

Let ω be a positive real number and $\phi=\frac{\Gamma(\alpha)\gamma}{\Gamma(2\alpha)}<1$ such that $\omega>(1-\phi)^{-1}\left(\frac{\eta\Gamma(\alpha)}{\Gamma(2\alpha)}+c_g\right)$. Then, the set $B_\omega=\{x\in X:|x(t)|\leq\omega,\ 1\leq t\leq e\}$ is convex, closed, and bounded subset of $C([1,e])$. The operator

$F : B_\omega \longrightarrow B_\omega$ given by

$$(Fx)(t) = \int_1^e H(t,s)(\gamma x(s) + \eta)\,\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} g(s, x(s))\,\frac{ds}{s},$$

is compact as in the proof of Theorem 3.1. Moreover,

$$|(Fx)(t)| \le \frac{\Gamma(\alpha)\gamma}{\Gamma(2\alpha)}\|x\| + \frac{\eta\Gamma(\alpha)}{\Gamma(2\alpha)} + c_g.$$

If $x \in B_\omega$, then

$$|(Fx)(t)| \le \phi\omega + (1-\phi)\,\omega = \omega,$$

that is $\|Fx\| \le \omega$. Hence, the Schauder fixed theorem ensures that the operator F has at least one fixed point in B_ω, and then Equation (3.6) has at least one positive solution $x^*(t)$, where $1 < t < e$. Therefore, if $t \in [1, e]$ one can asserts that

$$\begin{aligned} x^*(t) &= \int_1^e H(t,s)(\gamma x^*(s) + \eta)\,\frac{ds}{s} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} g(s, x^*(s))\,\frac{ds}{s} \\ &= \gamma\int_1^e H(t,s)\,x^*(s)\frac{ds}{s} + \frac{\eta\left((\log t)^{\alpha-1} - (\log t)^{\alpha}\right)}{\Gamma(\alpha+1)} \\ &\quad + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} g(s, x^*(s))\,\frac{ds}{s}. \end{aligned}$$

The definition of control function implies

$$\begin{aligned} &\int_1^e H(t,s)\,U(s, x^*(s))\frac{ds}{s} \\ &\le \int_1^e H(t,s)(\gamma x^*(s) + \eta)\,\frac{ds}{s} \\ &= x^*(t) - \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} g(s, x^*(s))\,\frac{ds}{s}, \end{aligned}$$

then x^* is an upper positive solution of the FDE (1.1). Moreover, one can consider

$$x_*(t) = \frac{\sigma\left((\log t)^{\alpha-1} - (\log t)^{\alpha}\right)}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} g(s, x_*(s))\,\frac{ds}{s},$$

as a lower positive solution of Equation (1.1). By Theorem 3.1, the FDE (1.1) has at least one positive solution $x \in C([1, e])$, where $x_*(t) \le x(t) \le x^*(t)$. □

The last result is the uniqueness of the positive solution of (1.1) using Banach contraction principle.

Theorem 3.5. *Assume that* $(H1)$ *and* $(H2)$ *are satisfied and*

$$\left(\frac{\Gamma(\alpha)\beta_2}{\Gamma(2\alpha)} + \frac{\beta_1}{\Gamma(\alpha-\beta+1)}\right) < 1. \tag{3.7}$$

Then the FDE (1.1) has a unique positive solution $x \in C$.

Proof. From Theorem 3.1, it follows that the FDE (1.1) has at least one positive solution in C. Hence, we need only to prove that the operator defined in (3.1) is a contraction on C. In fact, for any $x, y \in C$, we have

$$
\begin{aligned}
&|(\Phi x)(t) - (\Phi y)(t)| \\
&\leq \int_1^e H(t,s)\,|f(s,x(s)) - f(s,y(s))|\,\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha-\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-\beta-1} |g(s,x(s)) - g(s,y(s))|\,\frac{ds}{s} \\
&\leq \left(\frac{\Gamma(\alpha)\beta_2}{\Gamma(2\alpha)} + \frac{\beta_1}{\Gamma(\alpha-\beta+1)}\right)\|x-y\|.
\end{aligned}
$$

Hence, the operator Φ is a contraction mapping by (3.7). Therefore, the FDE (1.1) has a unique positive solution $x \in C$. □

Finally, we give an example to illustrate our results.

Example 3.6. We consider the nonlinear Hadamard fractional differential equation

$$
\begin{cases}
\mathfrak{D}_1^{\frac{7}{5}} x(t) - \mathfrak{D}_1^{\frac{1}{5}} \dfrac{1+x(t)}{2+x(t)} = \dfrac{1}{3+t}\left(3 + \dfrac{tx(t)}{2+x(t)}\right), \ 1 < t < e, \\
x(1) = 0,\ x(e) = \dfrac{1}{\Gamma(6/5)}\int_1^e \left(\log\dfrac{e}{s}\right)^{\frac{1}{5}} \dfrac{1+x(s)}{2+x(s)}\dfrac{ds}{s},
\end{cases}
\tag{3.8}
$$

where $g(t,x) = \frac{1+x}{2+x}$ and $f(t,x) = \frac{1}{3+t}\left(3 + \frac{tx}{2+x}\right)$. Since g is non-decreasing on x,

$$
\lim_{x\longrightarrow\infty} \frac{1+x}{2+x} = \lim_{x\longrightarrow\infty} \frac{1}{3+t}\left(3 + \frac{tx}{2+x}\right) = 1,
$$

and

$$
\frac{1}{2} \leq g(t,x) \leq 1,\ \frac{3}{3+e} \leq f(t,x) \leq 1,
$$

for $(t,x) \in [1,e] \times [0,+\infty)$, hence by Corollary 3.2, (3.8) has a positive solution which verifies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ where

$$
\overline{x}(t) = \frac{(\log t)^{\frac{2}{5}} - (\log t)^{\frac{7}{5}}}{\Gamma\left(\frac{12}{5}\right)} + \frac{(\log t)^{\frac{6}{5}}}{\Gamma\left(\frac{11}{5}\right)},
$$

and

$$
\underline{x}(t) = \frac{3\left((\log t)^{\frac{2}{5}} - (\log t)^{\frac{7}{5}}\right)}{(3+e)\,\Gamma\left(\frac{12}{5}\right)} + \frac{(\log t)^{\frac{6}{5}}}{2\Gamma\left(\frac{11}{5}\right)},
$$

are respectively the upper and lower solutions of (3.8). Also, we have

$$
\frac{\Gamma(\alpha)\beta_2}{\Gamma(2\alpha)} + \frac{\beta_1}{\Gamma(\alpha-\beta+1)} \simeq 0.353 < 1,
$$

then by Theorem 3.5, (3.8) has a unique positive solution which is bounded by $\underline{x}$ and $\overline{x}$.

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Conflict of interest

The author declares no conflict of interest.

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