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Research article

Simpson's type integral inequalities for κ-fractional integrals and their applications

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Abstract: In this paper, some new inequalities of Simpson's type are set up for the classes of functions whose derivatives of absolute are preinvex by means of κ-fractional integrals. Additionally, by extraordinary choices of n and κ, we give some diminished outcomes. Meanwhile, we also provide the inequalities for $\mathcal{F}$-divergence measures and in probabilistic versions.

Keywords: Simpson's type inequality; s-preinvex functions; κ-fractional integrals
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1. Introduction

The following inequality is named the Simpson's integral inequality:

$$\left|\frac{1}{6}\Big[\varphi(\varepsilon)+4\varphi\Big(\frac{\varepsilon+\zeta}{2}\Big)+\varphi(\zeta)\Big]-\frac{1}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta}\varphi(t)dt\right|\leq\|\varphi^{(4)}\|_{\infty}(\zeta-\varepsilon)^{4}, \tag{1.1}$$

where $\varphi:[\varepsilon,\zeta]\to\mathbb{R}$ is a four-order differentiable mapping on (ε,ζ) and $\|\varphi^{(4)}\|_{\infty}=\sup_{t\in(\varepsilon,\zeta)}|\varphi^{(4)}(t)|<\infty$.
Considering the Simpson-type inequality via mappings of different classes, many results involving the ordinary integrals can be found in [1, 3–5, 18, 19] and the references therein.

Definition 1.1. *A set $\Omega \subseteq \mathbb{R}^n$ is named as invex set with respect to the mapping $\eta : \Omega \times \Omega \to \mathbb{R}^n$, if*

$$\varepsilon + \eta(\zeta, \varepsilon) \in \Omega$$

holds, for all $\varepsilon, \zeta \in \Omega$ and $\theta \in [0, 1]$.

Definition 1.2. *A mapping $\varphi : \Omega \to \mathbb{R}$ is called preinvex respecting η, if*

$$\varphi(\varepsilon + \eta(\zeta, \varepsilon)) \leq (1 - \theta)\varphi(\varepsilon) + \theta\varphi(\zeta),$$

holds for all $\varepsilon, \zeta \in \Omega$ and $\theta \in [0, 1]$.

The preinvex function is an important substantive generalization of the convex function. For the properties, applications, integral inequalities and other aspects of preinvex functions, see [5, 7–14] and the references therein.

Definition 1.3. *Let $s \in (0, 1)$ and a mapping $\varphi : \Omega \to \mathbb{R}$ is called s-preinvex respecting η, if*

$$\varphi(\varepsilon + \eta(\zeta, \varepsilon)) \leq (1 - \theta^s)\varphi(\varepsilon) + \theta^s\varphi(\zeta),$$

holds for all $\varepsilon, \zeta \in \Omega$ and $\theta \in [0, 1]$.

Fractional calculus has recently been the focus of mathematics and many related sciences. In addition to defining new integral-derivative operators and their properties, many researchers have achieved important results in applied mathematics, engineering and statistics. It is appropriate for the reader to review articles [2, 15–17] for some recent studies. We end this section by reciting a well known κ-fractional integral operators in the literature.

Definition 1.4. *([6]) Let $\varphi \in L[\varepsilon, \zeta]$, the κ-fractional integrals $J^{\alpha,\kappa}_{\varepsilon^+}$ and $J^{\alpha,\kappa}_{\zeta^-}$ of order $\alpha > 0$ are defined by*

$$J^{\alpha,\kappa}_{\varepsilon^+} = \frac{1}{\kappa\Gamma_\kappa(\alpha)} \int_{\varepsilon}^{x} (x - \theta)^{\frac{\alpha}{\kappa}-1}\varphi(\theta)d\theta, \quad (0 \leq \alpha < x < \zeta) \tag{1.2}$$

and

$$J^{\alpha,\kappa}_{\zeta^+} = \frac{1}{\kappa\Gamma_\kappa(\alpha)} \int_{x}^{\zeta} (\theta - x)^{\frac{\alpha}{\kappa}-1}\varphi(\theta)d\theta, \quad (0 \leq \alpha < x < \zeta), \tag{1.3}$$

respectively, where $\kappa > 0$, and Γ_κ is the κ-gamma function defined as $\Gamma_\kappa(x) := \int_0^\infty \theta^{x-1}e^{-\frac{\theta^\kappa}{\kappa}}d\theta, \quad \Re(x) > 0$, with the properties $\Gamma_\kappa(x + \kappa) = x\Gamma_\kappa(x)$ and $\Gamma_\kappa(\kappa) = 1$.

This paper aims to obtain estimation type results of Simpson-type inequality related to κ- fractional integral operators. Next, we derive the results with the boundedness of the derivative and with a Lipschitzian condition for the derivative of the considered mapping to derive integral inequalities with new bounds. Application of our results to random variables are also provided.

2. Main results

Throughout of this work, let $\Omega \subseteq \mathbb{R}$ be an open subset respecting $\eta : \Omega \times \Omega \to \mathbb{R} \setminus \{0\}$ and $\varepsilon, \zeta \in \Omega$ with $\varepsilon < \zeta$.

To prove our results, we obtain a new integral identity as following:

Lemma 2.1. *Let $\varphi : \Omega = [\varepsilon + \eta(\zeta, \varepsilon)] \to \mathbb{R}$ be a differentiable function on $(\varepsilon + \eta(\zeta, \varepsilon))$ with $\eta(\zeta, \varepsilon) > 0$. If $\varphi' \in L[\varepsilon, \varepsilon + \eta(\zeta, \varepsilon)]$, $n \geq 0$ and $\alpha, \kappa > 0$, then the following identity holds:*

$$
\begin{aligned}
&\Psi(\varepsilon, \zeta; n, \alpha, \kappa) \qquad (2.1)\\
:\ &= \frac{\eta(\zeta, \varepsilon)}{2(n+1)} \Big[\int_0^1 \Big(\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3} \Big) \varphi'\Big(\varepsilon + \frac{1-\theta}{n+1} \eta(\zeta, \varepsilon)\Big) d\theta \\
&+ \int_0^1 \Big(\frac{\theta^{\frac{\alpha}{\kappa}} - 2(1-\theta)^{\frac{\alpha}{\kappa}}}{3} \Big) \Big] \varphi'\Big(\varepsilon + \frac{n+\theta}{n+1} \eta(\zeta, \varepsilon)\Big) d\theta \Big],
\end{aligned}
$$

where

$$
\begin{aligned}
&\Psi(\varepsilon, \zeta; n, \alpha, \kappa) \\
:\ &= \frac{1}{6} \Big[\varphi(\varepsilon) + \varphi(\varepsilon + \eta(\zeta, \varepsilon)) + 2\varphi\Big(\varepsilon + \frac{1}{n+1} \eta(\zeta, \varepsilon)\Big) + 2\varphi\Big(\varepsilon + \frac{n}{n+1} \eta(\zeta, \varepsilon)\Big) \\
&- \frac{\Gamma_\kappa(\alpha+\kappa)(n+1)^{\frac{\alpha}{\kappa}}}{6(\eta(\zeta, \varepsilon))^{\frac{\alpha}{\kappa}}} \Big[J^{\alpha,\kappa}_{(\varepsilon)^+} \varphi\Big(a + \frac{1}{n+1} \eta(\zeta, \varepsilon)\Big) \Big] + J^{\alpha,\kappa}_{(\varepsilon+\eta(\zeta,\varepsilon))^-} \varphi\Big(\varepsilon + \frac{n}{n+1} \eta(\zeta, \varepsilon)\Big) \Big] \\
&- \frac{\Gamma_\kappa(\alpha+\kappa)(n+1)^{\frac{\alpha}{\kappa}}}{3(\eta(\zeta, \varepsilon))^{\frac{\alpha}{\kappa}}} \Big[J^{\alpha,\kappa}_{(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon))^+} \varphi(\varepsilon + \eta(\zeta, \varepsilon)) + J^{\alpha,\kappa}_{(\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon))^+} \varphi(\varepsilon) \Big] \Big].
\end{aligned}
$$

Proof. Integration by parts, one can have

$$
\begin{aligned}
I_1 &= \int_0^1 \Big(\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3} \Big) \varphi'\Big(\varepsilon + \frac{1-\theta}{n+1} \eta(\zeta, \varepsilon)\Big) d\theta \\
&= \frac{n+1}{3\eta(\zeta, \varepsilon)} \varphi(\varepsilon) + \frac{2(n+1)}{3\eta(\zeta, \varepsilon)} \varphi\Big(\varepsilon + \frac{1}{n+1} \eta(\zeta, \varepsilon)\Big) \\
&- \frac{\alpha(n+1)^{\frac{\alpha}{\kappa}+1}}{3\kappa(\eta(\zeta, \varepsilon))^{\frac{\alpha}{\kappa}+1}} \int_{\varepsilon}^{\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)} \Big(\varepsilon + \frac{1}{n+1} \eta(\zeta, \varepsilon) - x\Big)^{\frac{\alpha}{\kappa}-1} \varphi(x) dx \\
&- \frac{2\alpha(n+1)^{\frac{\alpha}{\kappa}+1}}{3\kappa(\eta(\zeta, \varepsilon))^{\frac{\alpha}{\kappa}}} \int_{\varepsilon}^{\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)} (x - \varepsilon)^{\frac{\alpha}{\kappa}-1} \varphi(x) dx
\end{aligned}
$$

and

$$
I_2 = \int_0^1 \Big(\frac{\theta^{\frac{\alpha}{\kappa}} - 2(1-\theta)^{\frac{\alpha}{\kappa}}}{3} \Big) \Big] \varphi'\Big(\varepsilon + \frac{n+\theta}{n+1} \eta(\zeta, \varepsilon)\Big) d\theta
$$

$$
\begin{aligned}
&= \frac{n+1}{3(\eta(\zeta,\varepsilon))}\varphi(\varepsilon+\eta(\zeta,\varepsilon)) + \frac{2(n+1)}{3\eta(\zeta,\varepsilon)}\varphi(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)) \\
&-\frac{\alpha(n+1)^{\frac{\alpha}{\kappa}}+1}{3\kappa(\eta(\zeta,\varepsilon))^{\frac{\alpha}{\kappa}+1}} \int_{\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)}^{\varepsilon+\eta(\zeta,\varepsilon)} \Big(x-(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon))\Big)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx \\
&-\frac{\alpha(n+1)^{\frac{\alpha}{\kappa}}+1}{3\kappa(\eta(\zeta,\varepsilon))^{\frac{\alpha}{\kappa}+1}} \int_{\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)}^{\varepsilon+\eta(\zeta,\varepsilon)} \Big((\varepsilon+\eta(\zeta,\varepsilon))-x\Big)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx.
\end{aligned}
$$

By adding I_1 and I_2 and multiplying the both sides $\frac{\eta(\zeta,\varepsilon)}{2(n+1)}$, we can write

$$
\begin{aligned}
I_1+I_2 \quad &= \frac{1}{6}\Big[\varphi(\varepsilon)+\varphi(\varepsilon+\eta(\zeta,\varepsilon))+2\varphi(\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon))+2\varphi(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon))\Big] \\
&-\frac{\alpha(n+1)^{\frac{\alpha}{\kappa}}}{6\kappa(\eta(\zeta,\varepsilon))^{\frac{\alpha}{\kappa}}}\Big[\int_{\varepsilon}^{\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)} (\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon))^{\frac{\alpha}{\kappa}-1}\varphi(x)dx \\
&+\int_{\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)}^{\varepsilon+\eta(\zeta,\varepsilon)} \Big(x-(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon))\Big)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx\Big] \\
&-\frac{\alpha(n+1)^{\frac{\alpha}{\kappa}}}{3\kappa(\eta(\zeta,\varepsilon))^{\frac{\alpha}{\kappa}}}\Big[\int_{\varepsilon}^{\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)} (\varepsilon+\eta(\zeta,\varepsilon)-x)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx+\int_{\varepsilon}^{\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)} (x-\varepsilon)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx\Big].
\end{aligned}
$$

Using the fact that

$$
\begin{aligned}
\frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_{\varepsilon}^{\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)} (x-\varepsilon)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx \quad &= J^{\alpha,\kappa}_{(\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon))^-}\varphi(\varepsilon), \\
\frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_{\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)}^{\varepsilon+\eta(\zeta,\varepsilon)} (\varepsilon+\eta(\zeta,\varepsilon)-x)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx \quad &= J^{\alpha,\kappa}_{(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon))^+}\varphi(\varepsilon+\eta(\zeta,\varepsilon)), \\
\frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_{\varepsilon}^{\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)} (\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)-x)^{\frac{\alpha}{\kappa}-1}\varphi(x)dx \quad &= J^{\alpha,\kappa}_{(\varepsilon)^+}\varphi(\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)), \\
\frac{1}{\kappa\Gamma_\kappa(\alpha)}\int_{\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)}^{\varepsilon+\eta(\zeta,\varepsilon)} (x-(\varepsilon+\frac{1}{n+1}\eta(\zeta,\varepsilon)))^{\frac{\alpha}{\kappa}-1}\varphi(x)dx \quad &= J^{\alpha,\kappa}_{(\varepsilon+\eta(\zeta,\varepsilon))^-}\varphi(\varepsilon+\frac{n}{n+1}\eta(\zeta,\varepsilon)),
\end{aligned}
$$

we get the result. □

Remark 2.1. If we choose $\eta(\zeta,\varepsilon)=\zeta-\varepsilon$ with $\kappa=1$, then under the assumption of Lemma 2.1 one has Lemma 2.1 in [19].

Theorem 2.2. *Let $\varphi:\Omega=[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]\to\mathbb{R}$ be a differentiable function on Ω. If $\varphi'\in L[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]$and $|\varphi'|^\lambda$ for $\lambda>1$ with $\mu^{-1}+\lambda^{-1}=1$ is s-preinvex function, then the following inequality holds for fractional integrals with $\alpha,\kappa>0$, then the following integral inequality holds:*

$$
\begin{aligned}
&|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \qquad (2.2)\\
\le \quad & \frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big(\int_0^1\Big|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big|^{\mu}d\theta\Big)^{\frac{1}{\mu}}
\end{aligned}
$$

$$\times\Big[\big[\big(1-\frac{1}{(n+1)^s(s+1)}|\varphi'(\varepsilon)|^\lambda+\frac{1}{(n+1)^s(s+1)}|\varphi'(\zeta)|^\lambda\big)\big]^{\frac{1}{\lambda}}$$
$$+\big[\frac{1}{(n+1)^s(s+1)}|\varphi'(\varepsilon)|^\lambda+\big(2^{1-s}-\frac{1}{(n+1)^s(s+1)}\big)|\varphi'(\zeta)|^\lambda\big]^{\frac{1}{\lambda}}\Big].$$

Proof. From the integral identity given in Lemma 2.1, the H*ö*lder integral inequality and the s-preinvexity of $|\varphi'(x)|^\lambda$, we have

$$|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \leq \frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big[\int_0^1\Big|\Big(\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big)\Big|\Big|\varphi'\Big(\varepsilon+\frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big)\Big|d\theta \tag{2.3}$$
$$+\Big|\int_0^1\Big(\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big)\Big|\Big|\varphi'\Big(\varepsilon+\frac{n+\theta}{n+1}\eta(\zeta,\varepsilon)\Big)\Big|d\theta\Big]$$
$$\leq \frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big[\Big(\int_0^1\Big|\Big(\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big)\Big|^\mu d\theta\Big)^{\frac{1}{\mu}}$$
$$\times\Big[\int_0^1\Big(\Big(1-\Big(\frac{1-\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|^\lambda+\Big(\frac{1-\theta}{n+1}\Big)^s|\varphi'(\zeta)|^\lambda\Big)d\theta\Big]^{\frac{1}{\lambda}}$$
$$+\Big(\int_0^1\Big|\Big(\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big)\Big|^\mu d\theta\Big)^{\frac{1}{\mu}}\Big[\int_0^1\Big(\Big(1-\Big(\frac{n+\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|^\lambda+\Big(\frac{n+\theta}{n+1}\Big)^s|\varphi'(\zeta)|^\lambda\Big)d\theta\Big]^{\frac{1}{\lambda}}\Big].$$

Using the fact that $1-\chi^s \leq (1-\chi)^s \leq 2^{1-s}-\chi^s$ for $\chi \in [0,1]$ with $s \in (0,1]$, we have

$$\int_0^1\Big(1-\Big(\frac{n+\theta}{n+1}\Big)^s|\varphi'(\varepsilon)|^\lambda+\Big(\frac{n+\theta}{n+1}\Big)|\varphi'(\zeta)|^\lambda\Big)d\theta \tag{2.4}$$
$$\leq \int_0^1\Big[\Big(\frac{1-\theta}{n+1}\Big)|\varphi'(\varepsilon)|^\lambda+\Big(2^{1-s}-\Big(\frac{1-\theta}{n+1}\Big)^s\Big)|\varphi'(\zeta)|^\lambda\Big]d\theta$$
$$= \frac{1}{(n+1)^s(s+1)}|\varphi'(\varepsilon)|^\lambda+\Big(2^{1-s}-\frac{1}{(n+1)^s(s+1)}\Big)|\varphi'(\zeta)|^\lambda$$

and

$$\int_0^1\Big[\Big(1-\Big(\frac{1-\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|^\lambda+\Big(\frac{1-\theta}{n+1}\Big)^s|\varphi'(\zeta)|^\lambda\Big]d\theta \tag{2.5}$$
$$= \Big(1-\frac{1}{(n+1)^s(s+1)}|\varphi'(\varepsilon)|^\lambda+\frac{1}{(n+1)^s(s+1)}|\varphi'(\zeta)|^\lambda\Big).$$

□

Remark 2.2. If we choose $\eta(\zeta,\varepsilon)=\zeta-\varepsilon$ with $\kappa=1$, then under the assumption of Theorem 2.2, one has Theorem 2.3 in [19].

Corollary 2.1. *If we choose $\alpha = \kappa = 1$, and $n = s = 1$, then under the assumption of Theoremm 2.2 we have*

$$\Big|\frac{1}{6}\Big[\varphi(\varepsilon) + 4\varphi(\frac{2\varepsilon + \eta(\zeta,\varepsilon)}{2}) + \varphi(\varepsilon + \eta(\zeta,\varepsilon))\Big] - \frac{1}{\eta(\zeta,\varepsilon)}\int_{\varepsilon}^{\varepsilon+\eta(\zeta,\varepsilon)} \varphi(x)dx\Big|$$
$$\leq \frac{\eta(\zeta,\varepsilon)}{4}\Big[\frac{1+2^{\mu+1}}{3^{\mu+1}(\mu+1)}\Big]^{\frac{1}{\mu}}\Big[\Big(\frac{3}{4}\Big)^{\frac{1}{\lambda}} + \Big(\frac{1}{4}\Big)^{\frac{1}{\lambda}}\Big]\Big[|\varphi'(\varepsilon)| + |\varphi'(\zeta)|\Big].$$

Corollary 2.2. *If we choose $\eta(\zeta,\varepsilon) = \zeta - \varepsilon$ with $\alpha = \kappa = 1$, and $n = s = 1$, then under the assumption of Theoremm 2.2 we have*

$$\Big|\frac{1}{6}\Big[\varphi(\varepsilon) + 4\varphi(\frac{\varepsilon+\zeta}{2}) + \varphi(\zeta)\Big] - \frac{1}{\zeta-\varepsilon}\int_{\varepsilon}^{\zeta} \varphi(x)dx\Big|$$
$$\leq \frac{\zeta-\varepsilon}{4}\Big[\frac{1+2^{\mu+1}}{3^{\mu+1}(\mu+1)}\Big]^{\frac{1}{\mu}}\Big[\Big(\frac{3}{4}\Big)^{\frac{1}{\lambda}} + \Big(\frac{1}{4}\Big)^{\frac{1}{\lambda}}\Big]\Big[|\varphi'(\varepsilon)| + |\varphi'(\zeta)|\Big].$$

Proof. The proof of the last inequality is obtained by using the fact that

$$\sum_{i=1}^{n}(\omega_i + \nu_i)^j \leq \sum_{i=1}^{n}(\omega_i)^j + \sum_{i=1}^{n}(\nu_i)^j$$

for $0 \leq j \leq 1$, $\omega_1, \omega_2, \omega_3, ..., \omega_n \geq 0$; $\nu_1, \nu_2, \nu_3, ..., \nu_n \geq 0$. □

Theorem 2.3. *Let $\varphi : \Omega = [\varepsilon, \varepsilon + \eta(\zeta,\varepsilon)] \to \mathbb{R}$ be a differentiable function on Ω. If $\varphi' \in L[\varepsilon, \varepsilon + \eta(\zeta,\varepsilon)]$and $|\varphi'|$ is s-preinvex function, then the following inequality holds for fractional integrals with $\alpha, \kappa > 0$, then the following integral inequality holds:*

$$|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \leq \frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big\{(\Delta_1 + \Delta_3)|\varphi'(\varepsilon)| + (\Delta_1 + \Delta_2)|\varphi'(\zeta)|\Big\}, \tag{2.6}$$

where

$$\Delta_1 = \frac{2 - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+s+1}}{3(n+1)^s(\frac{\alpha}{\kappa}+s+1)} + \frac{1}{3(n+1)^s}\Big[\beta(\frac{\alpha}{\kappa}+1, s+1) - 2\beta(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}; \frac{\alpha}{\kappa}+1, s+1)\Big],$$

$$\Delta_2 = \frac{2^{1-s}\{3 - 2\Big(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1} - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}\}}{3(\frac{\alpha}{\kappa}+1)} - \frac{2 - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+s+1}}{3(n+1)^s(\frac{\alpha}{\kappa}+s+1)}$$
$$+\frac{1}{3(n+1)^s}\Big[2\beta(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}; \frac{\alpha}{\kappa}+1, s+1) - \beta(\frac{\alpha}{\kappa}+1, s+1)\Big]$$

and

$$\Delta_3 = \frac{\{3 - 2\Big(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1} - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}\}}{3(\frac{\alpha}{\kappa}+1)} - \frac{2 - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+s+1}}{3(n+1)^s(\frac{\alpha}{\kappa}+s+1)}$$
$$+\frac{1}{3(n+1)^s}\Big[2\beta(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}; \frac{\alpha}{\kappa}+1, s+1) - \beta(\frac{\alpha}{\kappa}+1, s+1)\Big].$$

Proof. From Lemma 2.1 and using the s-preinvexity of $|\varphi'(x)|$, we have

$$\begin{aligned} &|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \qquad (2.7)\\ \leq\ & \frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big[\int_0^1\Big|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big|\Big[\Big(1-\Big(\frac{1-\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|+\Big(\frac{1-\theta}{n+1}\Big)^s|\varphi'(\zeta)|\Big]d\theta \\ &+\int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big[\Big(1-\Big(\frac{n+\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|+\Big(\frac{n+\theta}{n+1}\Big)^s|\varphi'(\zeta)|\Big]d\theta\Big]\end{aligned}$$

From (2.7), we have

$$\begin{aligned} &\int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big[\Big(1-\Big(\frac{n+\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|+\Big(\frac{n+\theta}{n+1}\Big)^s|\varphi'(\zeta)|\Big]d\theta \qquad (2.8)\\ \leq\ & |\varphi'(\varepsilon)|\int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big(\frac{1-\theta}{n+1}\Big)^s d\theta+|\varphi'(\zeta)|\int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big(2^{1-s}-\frac{1-\theta}{n+1}\Big)^s d\theta,\end{aligned}$$

By taking into account

$$\begin{aligned}\Delta_1 &= \int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big(\frac{1-\theta}{n+1}\Big)^s d\theta \qquad (2.9)\\ &= \frac{2-4\Big(1-\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+s+1}}{3(n+1)^s(\frac{\alpha}{\kappa}+s+1)}+\frac{1}{3(n+1)^s}\Big[\beta(\frac{\alpha}{\kappa}+1,s+1)-2\beta(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1};\frac{\alpha}{\kappa}+1,s+1)\Big],\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big(2^{1-s}-\frac{1-\theta}{n+1}\Big)^s d\theta \qquad (2.10)\\ &= \frac{2^{1-s}\{3-2\Big(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}-4\Big(1-\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+1}\}}{3(\frac{\alpha}{\kappa}+1)}-\frac{2-4\Big(1-\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+s+1}}{3(n+1)^s(\frac{\alpha}{\kappa}+s+1)}\\ &\quad+\frac{1}{3(n+1)^s}\Big[2\beta(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1};\frac{\alpha}{\kappa}+1,s+1)-\beta(\frac{\alpha}{\kappa}+1,s+1)\Big]\end{aligned}$$

and

$$\begin{aligned}\Delta_3 &= \int_0^1\Big|\frac{\theta^{\frac{\alpha}{\kappa}}-2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big|\Big(1-\frac{1-\theta}{n+1}\Big)^s d\theta \qquad (2.11)\\ &= \frac{\{3-2\Big(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}-4\Big(1-\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}\}}{3(\frac{\alpha}{\kappa}+1)}-\frac{2-4\Big(1-\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\kappa}{\alpha}}+1}\Big)^{\frac{\alpha}{\kappa}+s+1}}{3(n+1)^s(\frac{\alpha}{\kappa}+s+1)}\\ &\quad+\frac{1}{3(n+1)^s}\Big[2\beta(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1};\frac{\alpha}{\kappa}+1,s+1)-\beta(\frac{\alpha}{\kappa}+1,s+1)\Big],\end{aligned}$$

□

Theorem 2.4. *Let $\varphi : \Omega = [\varepsilon, \varepsilon + \eta(\zeta, \varepsilon)] \to \mathbb{R}$ be a differentiable function on Ω. If $\varphi' \in L[\varepsilon, \varepsilon + \eta(\zeta, \varepsilon)]$and $|\varphi'|$ is s-preinvex function, then the following inequality holds for fractional integrals with $\alpha, \kappa > 0$, then the following integral inequality holds:*

$$\begin{aligned} & |\Psi(\varepsilon, \zeta; n, \alpha, \kappa)| \\ \leq\ & \frac{\eta(\zeta, \varepsilon)}{2(n+1)}(\Delta_0)^{1-\frac{1}{\lambda}}\Big\{\Big(\Delta_1|\varphi'(\varepsilon)|^\lambda + \Delta_3|\varphi'(\zeta)|^\lambda\Big)^{\frac{1}{\lambda}} + \Big(\Delta_1|\varphi'(\varepsilon)|^\lambda + \Delta_2|\varphi'(\zeta)|^\lambda\Big)^{\frac{1}{\lambda}}\Big\}, \end{aligned} \tag{2.12}$$

where

$$\Delta_0 = \frac{\{3 - 2\Big(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1} - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}\}}{3(\frac{\alpha}{\kappa}+1)}.$$

Proof. By using Lemma 2.1 and the Hölder's integral inequality for $\lambda > 1$ we have

$$\begin{aligned} & |\Psi(\varepsilon, \zeta; n, \alpha, \kappa)| \\ \leq\ & \frac{\eta(\zeta, \varepsilon)}{2(n+1)}\Bigg[\Bigg(\int_0^1 \Bigg|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3}\Bigg| d\theta\Bigg)^{1-\frac{1}{\lambda}} \\ & \times\Bigg[\int_0^1 \Bigg|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3}\Bigg|\Bigg|\varphi'\Big(\varepsilon + \frac{1-\theta}{n+1}\eta(\zeta, \varepsilon)\Big)\Bigg|^\lambda d\theta\Bigg)^{\frac{1}{\lambda}} \\ & + \int_0^1 \Bigg|\frac{\theta^{\frac{\alpha}{\kappa}} - 2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Bigg|\Bigg|\varphi'\Big(\varepsilon + \frac{n+\theta}{n+1}\eta(\zeta, \varepsilon)\Big)\Bigg|^\lambda d\theta\Bigg)^{\frac{1}{\lambda}}\Bigg] \end{aligned}$$

Using s-preinvexity of $|\varphi'|$, we get

$$\begin{aligned} & |\Psi(\varepsilon, \zeta; n, \alpha, \kappa)| \\ \leq\ & \frac{\eta(\zeta, \varepsilon)}{2(n+1)}\Bigg[\Bigg(\int_0^1 \Bigg|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3}\Bigg| d\theta\Bigg)^{1-\frac{1}{\lambda}} \\ & \times\Bigg[\int_0^1 \Bigg|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3}\Bigg|\Bigg|\Bigg[\Bigg(\Big(1 - \Big(\frac{1-\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|^\lambda + \Big(\frac{1-\theta}{n+1}\Big)^\lambda|\varphi'(\zeta)|^\lambda\Bigg)d\theta\Bigg]^{\frac{1}{\lambda}} \\ & + \int_0^1 \Bigg|\frac{\theta^{\frac{\alpha}{\kappa}} - 2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Bigg|\Bigg|\Bigg[\Bigg(\Big(1 - \Big(\frac{n+\theta}{n+1}\Big)^s\Big)|\varphi'(\varepsilon)|^\lambda + \Big(\frac{n+\theta}{n+1}\Big)^\lambda|\varphi'(\zeta)|^\lambda\Bigg)d\theta\Bigg]^{\frac{1}{\lambda}}, \Bigg] \end{aligned} \tag{2.13}$$

using the fact that

$$\begin{aligned} \Delta_0 &= \int_0^1 \Bigg|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3}\Bigg| d\theta \\ &= \frac{\{3 - 2\Big(\frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1} - 4\Big(1 - \frac{2^{\frac{\kappa}{\alpha}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+1}\}}{3(\frac{\alpha}{\kappa}+1)}. \end{aligned} \tag{2.14}$$

A combination of (2.9)–(2.11) with (2.14) into (2.13) gives the desired inequality (2.12). The proof is completed. □

Corollary 2.3. *If we take $n = s = 1$ and $\alpha = \kappa = 1$, then under the assumption of Theorem 2.4, we have*

$$\Big|\frac{1}{6}\Big(\varphi(\varepsilon) + 4\varphi(\frac{2\varepsilon + \eta(\zeta,\varepsilon)}{2}) + \varphi(\varepsilon + \eta(\zeta,\varepsilon))\Big) - \frac{1}{\eta(\zeta,\varepsilon)}\int_{\varepsilon}^{\varepsilon+\eta(\zeta,\varepsilon)}\varphi(x)dx\Big|$$
$$\leq \frac{\eta(\zeta,\varepsilon)}{4}(\frac{5}{18})^{1-\frac{1}{\lambda}}\Big[(\frac{61}{324})^{\frac{1}{\lambda}} + (\frac{29}{324})^{\frac{1}{\lambda}}\Big][|\varphi'(\varepsilon)| + |\varphi'(\zeta)|].$$

Corollary 2.4. *If we take $\eta(\zeta,\varepsilon) = \zeta - \varepsilon$ with $n = s = 1$ and $\alpha = \kappa = 1$, then under the assumption of Theorem 2.4, we have*

$$\Big|\frac{1}{6}\Big(\varphi(\varepsilon) + 4\varphi(\frac{\varepsilon + \zeta}{2}) + \varphi(\zeta)\Big) - \frac{1}{\zeta - \varepsilon}\int_{\varepsilon}^{\zeta}\varphi(x)dx\Big|$$
$$\leq \frac{\zeta - \varepsilon}{4}(\frac{5}{18})^{1-\frac{1}{\lambda}}\Big[(\frac{61}{324})^{\frac{1}{\lambda}} + (\frac{29}{324})^{\frac{1}{\lambda}}\Big][|\varphi'(\varepsilon)| + |\varphi'(\zeta)|].$$

Remark 2.3. If we take $\eta(\zeta,\varepsilon) = \zeta - \varepsilon$ with $\lambda = 1$ and $\kappa = s = 1$, then Theorem 2.4 reduces to Theorem 2.2 in [19].

Remark 2.4. If we take $\eta(\zeta,\varepsilon) = \zeta - \varepsilon$ with $\lambda = 1$, $\kappa = s = 1$, and $\alpha = n = 1$ then Theorem 2.4 reduces to Corollary 1 in [18].

For deriving further results, we deal with the boundedness and the Lipschitzian condition of φ'.

Theorem 2.5. *Let $\varphi : \Omega = [\varepsilon, \varepsilon + \eta(\zeta,\varepsilon)] \to \mathbb{R}$ be a differentiable function on Ω. If $|\varphi'|$ is s-preinvex function, there exists constants $m < \mathcal{M}$ satisfying that $\infty < m \leq \varphi'(x) \leq \mathcal{M} < \infty$ for all $x \in [\varepsilon, \varepsilon + \eta(\zeta,\varepsilon)]$ and $\alpha, \kappa > 0$, and $n \geq 0$, then the following integral inequality holds:*

$$|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \leq \frac{(\mathcal{M} - m)\eta(\zeta,\varepsilon)}{2(n+1)}\Delta_0, \tag{2.15}$$

where Δ_0 given in (2.14).

Proof. From Lemma 2.1, we have

$$\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| = \frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big[\int_0^1\Big(\frac{2(1-\theta)^{\frac{\alpha}{\kappa}} - \theta^{\frac{\alpha}{\kappa}}}{3}\Big)\Big(\varphi'\Big(\varepsilon + \frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big) - \frac{m+\mathcal{M}}{2}\Big)d\theta$$
$$+ \int_0^1\Big(\frac{\theta^{\frac{\alpha}{\kappa}} - 2(1-\theta)^{\frac{\alpha}{\kappa}}}{3}\Big)\Big(\varphi'\Big(\varepsilon + \frac{n+\theta}{n+1}\eta(\zeta,\varepsilon)\Big) - \frac{m+\mathcal{M}}{2}\Big)d\theta\Big] \tag{2.16}$$

Using the fact that $m - \frac{m+\mathcal{M}}{2} \leq \varphi'\Big(\varepsilon + \frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big) - \frac{m+\mathcal{M}}{2} \leq \mathcal{M} - \frac{m+\mathcal{M}}{2}$, one has

$$\Big|\varphi'\Big(\varepsilon + \frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big) - \frac{m+\mathcal{M}}{2}\Big| \leq \frac{\mathcal{M} - m}{2},$$

similarly,

$$\Big|\varphi'\Big(\varepsilon + \frac{n+\theta}{n+1}\eta(\zeta,\varepsilon)\Big) - \frac{m+\mathcal{M}}{2}\Big| \leq \frac{\mathcal{M} - m}{2},$$

Inequality (2.16) implies that

$$\begin{aligned} |\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| &\le \frac{(\mathcal{M}-m)\eta(\zeta,\varepsilon)}{2(n+1)}\int_0^1\left|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\right|d\theta \\ &= \frac{(\mathcal{M}-m)\eta(\zeta,\varepsilon)}{2(n+1)}\Delta_0. \end{aligned}$$

The proof is completed. □

Corollary 2.5. *If we take $\eta(\zeta,\varepsilon)=\zeta-\varepsilon$ with $\alpha=\kappa=1$, then under the assumption of Theorem 2.5, we have*

$$|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \le \frac{7(\mathcal{M}-m)(\zeta-\varepsilon)}{36(n+1)}.$$

Theorem 2.6. *Let $\varphi:\Omega=[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]\to\mathbb{R}$ be a differentiable function on Ω. If φ' satisfies Lipchitz conditions on $[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]$ for certain $\mathfrak{L}>0$, with $\alpha>0,\kappa>0,\ n\ge 0$, then the following inequality holds:*

$$\begin{aligned} &|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \\ \le\ & \frac{\mathfrak{L}(\eta(\zeta,\varepsilon))^2}{2(n+1)^2}\Big[(n-1)\Delta_0-\frac{4\Big(\frac{2^{\frac{\alpha}{\kappa}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+2}-2}{3(\frac{\alpha}{\kappa})+2}+\frac{8}{3}\beta\Big(\frac{2^{\frac{\alpha}{\kappa}}}{2^{\frac{\alpha}{\kappa}}+1};\frac{\alpha}{\kappa}+1,2\Big)-\frac{4}{3}\beta\Big(\frac{\alpha}{\kappa}+1,2\Big)\Big], \end{aligned} \tag{2.17}$$

where Δ_0 given in (2.14).

Proof. From Lemma 2.1, we have

$$\begin{aligned} &\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \\ &=\frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big[\int_0^1\Big(\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big)\Big[\varphi'\Big(\varepsilon+\frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big)-\varphi'\Big(\varepsilon+\frac{n+\theta}{n+1}\eta(\zeta,\varepsilon)\Big)\Big]d\theta. \end{aligned}$$

Since φ' is Lipschitz condition on $[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]$, for certain $\mathfrak{L}>0$, we have

$$\Big|\varphi'\Big(\varepsilon+\frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big)-\varphi'\Big(\varepsilon+\frac{n+\theta}{n+1}\eta(\zeta,\varepsilon)\Big)\Big|\le\mathfrak{L}\eta(\zeta,\varepsilon)\Big(\frac{n-1+2t}{n+1}\Big)$$

Thus

$$\begin{aligned} &|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)| \\ &\le\frac{\eta(\zeta,\varepsilon)}{2(n+1)}\Big[\int_0^1\Big|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big|\Big|\varphi'\Big(\varepsilon+\frac{1-\theta}{n+1}\eta(\zeta,\varepsilon)\Big)-\varphi'\Big(\varepsilon+\frac{n+\theta}{n+1}\eta(\zeta,\varepsilon)\Big|\Big]d\theta \\ &\le\frac{\mathfrak{L}(\eta(\zeta,\varepsilon))^2}{2(n+1)}\int_0^1\Big|\frac{2(1-\theta)^{\frac{\alpha}{\kappa}}-\theta^{\frac{\alpha}{\kappa}}}{3}\Big|\Big(\frac{n-1+2t}{n+1}\Big)d\theta \\ &=\frac{\mathfrak{L}(\eta(\zeta,\varepsilon))^2}{2(n+1)^2}\Big[(n-1)\Delta_0-\frac{4\Big(\frac{2^{\frac{\alpha}{\kappa}}}{2^{\frac{\alpha}{\kappa}}+1}\Big)^{\frac{\alpha}{\kappa}+2}-2}{3(\frac{\alpha}{\kappa})+2}+\frac{8}{3}\beta\Big(\frac{2^{\frac{\alpha}{\kappa}}}{2^{\frac{\alpha}{\kappa}}+1};\frac{\alpha}{\kappa}+1,2\Big)-\frac{4}{3}\beta\Big(\frac{\alpha}{\kappa}+1,2\Big)\Big]. \end{aligned}$$

The proof is completed. □

Corollary 2.6. *If we take $\eta(\zeta,\varepsilon)=\zeta-\varepsilon$ with $\alpha=\kappa=1$, then under the assumption of Theorem 2.6, we have*

$$|\Psi(\varepsilon,\zeta;n,\alpha,\kappa)|\leq\frac{\mathfrak{L}(\zeta-\varepsilon)^2}{2(n+1)^2}\Big[\frac{7n-11}{18}+\frac{22}{135}+\frac{8}{3}\beta\Big(\frac{2}{3};2,2\Big)\Big].$$

3. Applications

3.1. $\mathcal{F}$-divergence measure

Let the set ψ and the δ-finite measure ϱ be given, and let the set of all probability densities on ϱ to be defined on $\Lambda:=\{z|z:\psi\to\mathbb{R},z(x)>0,\int_\psi z(x)d\varrho(x)=1\}$.
Let $\mathcal{F}:(0,\infty)\to\mathbb{R}$ be given mapping and consider $\mathcal{D}_{\mathcal{F}}(z,w)$ be defined by

$$\mathcal{D}_{\mathcal{F}}(z,w):=\int\limits_{\psi}z(x)\mathcal{F}\Big(\frac{z(x)}{w(x)}\Big)d\varrho(x),\quad z,w\in\Lambda. \tag{3.1}$$

If $\mathcal{F}$ is convex, then (3.1) is called as the Csiszar $\mathcal{F}$-divergence.
Consider the following Hermite-Hadamard divergence

$$\mathcal{D}_{HH}^{\mathcal{F}}(z,w):=\int\limits_{\psi}\frac{\int_{1}^{\frac{w(x)}{z(x)}}\mathcal{F}(t)dt}{\frac{w(x)}{z(x)}-1}d\varrho(x),\quad w,z\in\Lambda, \tag{3.2}$$

where $\mathcal{F}$ is convex on $(0,\infty)$ with $\mathcal{F}(1)=0$. Note that $\mathcal{D}_{HH}^{\mathcal{F}}(z,w)\geq0$ with the equality holds if and only if $w=z$.

Proposition 3.1. *Suppose all assumptions of Corollary 2.2 hold for $(0,\infty)$ and $\mathcal{F}(1)=0$, if $w,z\in\Lambda$, then the following inequality holds:*

$$\Big|\frac{1}{6}\Big[\mathcal{D}_{\mathcal{F}}(w,z)+4\int\limits_{\psi}z(x)\mathcal{F}\Big(\frac{z(x)+w(x)}{2w(x)}d\varrho(x)\Big)\Big]-\mathcal{D}_{HH}^{\mathcal{F}}(w,z)\Big| \tag{3.3}$$

$$\leq\frac{(1+2^{\lambda+1})^{\frac{1}{\lambda}}}{4(3^{\lambda+1}(1+\lambda))^{\frac{1}{\lambda}}}\Big[\Big(\frac{3}{4}\Big)^{\frac{1}{\lambda}}+\Big(\frac{1}{4}\Big)^{\frac{1}{\lambda}}\Big]\times$$

$$\Big[|\mathcal{F}'(1)|\int\limits_{\psi}|z(x)-w(x)|d\varrho(x)+\int\limits_{\psi}|z(x)-w(x)|\Big|\mathcal{F}'\frac{z(x)}{w(x)}\Big|d\varrho(x)\Big].$$

Proof. Let $\Theta_1=\{x\in\psi:z(x)=w(x)\}$. $\Theta_2=\{x\in\psi:z(x)<w(x)\}$ and $\Theta_3=\{x\in\psi:z(x)>w(x)\}$ and Obviously, if $x\in\Theta_1$, then equality holds in (3.3).

Now if $x\in\Theta_2$, then using Corollary 2.2 for $\varepsilon=\frac{z(x)}{w(x)}$ and $\zeta=1$, multiplying both sides of the obtained results by $w(x)$ and then integrating over Θ_2, we obtain

$$\Big|\frac{1}{6}\Big[4\int\limits_{\psi_2}w(x)\mathcal{F}\Big(\frac{w(x)+z(x)}{2w(x)}\Big)d\varrho(x)+\int\limits_{\psi_2}w(x)\mathcal{F}\Big(\frac{z(x)}{w(x)}\Big)d\varrho(x)\Big]-\int\limits_{\psi_2}w(x)\frac{\int_{1}^{\frac{z(x)}{w(x)}}\mathcal{F}(t)dt}{\frac{z(x)}{w(x)}-1}d\varrho(x)\Big| \tag{3.4}$$

$$\leq \frac{(1+2^{\lambda+1})^{\frac{1}{\lambda}}}{4(3^{\lambda+1}(1+\lambda))^{\frac{1}{\lambda}}}\Big[\Big(\frac{3}{4}\Big)^{\frac{1}{\lambda}}+\Big(\frac{1}{4}\Big)^{\frac{1}{\lambda}}\Big]\times$$
$$\Big[|\mathcal{F}'(1)|\int_{\psi_2}|w(x)-z(x)|d\varrho(x)+\int_{\psi_2}|w(x)-z(x)|\Big|\mathcal{F}'\frac{z(x)}{w(x)}\Big|d\varrho(x)\Big].$$

Similarly if $x\in\Theta_3$, then using Corollary 2.2 for $\varepsilon=1$ and $\zeta=\frac{z(x)}{w(x)}$, multiplying both sides of the obtained results by $w(x)$ and then integrating over Θ_3, we obtain

$$\Big|\frac{1}{6}\Big[4\int_{\psi_3}w(x)\mathcal{F}\Big(\frac{w(x)+z(x)}{2w(x)}\Big)d\varrho(x)+\int_{\psi_3}w(x)\mathcal{F}\Big(\frac{z(x)}{w(x)}\Big)d\varrho(x)\Big]-\int_{\psi_3}w(x)\frac{\int_1^{\frac{z(x)}{w(x)}}\mathcal{F}(t)dt}{\frac{z(x)}{w(x)}-1}d\varrho(x)\Big| \tag{3.5}$$
$$\leq \frac{(1+2^{\lambda+1})^{\frac{1}{\lambda}}}{4(3^{\lambda+1}(1+\lambda))^{\frac{1}{\lambda}}}\Big[\Big(\frac{3}{4}\Big)^{\frac{1}{\lambda}}+\Big(\frac{1}{4}\Big)^{\frac{1}{\lambda}}\Big]\times$$
$$\Big[|\mathcal{F}'(1)|\int_{\psi_3}|z(x)-w(x)|d\varrho(x)+\int_{\psi_3}|z(x)-w(x)|\Big|\mathcal{F}'\frac{z(x)}{w(x)}\Big|d\varrho(x)\Big].$$

Adding inequalities (3.4) and (3.5) and then utilizing triangular inequality, we get the result. □

Proposition 3.2. *Suppose all assumptions of Corollary 2.4 holds for* $(0,\infty)$ *and* $f(1)=0$*, if* $w,z\in\Lambda$*, then the following inequality holds:*

$$\Big|\frac{1}{6}\Big[\mathcal{D}_{\mathcal{F}}(w,z)+4\int_{\psi}z(x)\mathcal{F}\Big(\frac{z(x)+w(x)}{2w(x)}d\varrho(x)\Big)\Big]-\mathcal{D}^{\mathcal{F}}_{HH}(w,z)\Big| \tag{3.6}$$
$$\leq \Big(\frac{5}{18}\Big)^{\frac{1}{\lambda}}\Big[\Big(\frac{61}{324}\Big)^{\frac{1}{\lambda}}+\Big(\frac{29}{324}\Big)^{\frac{1}{\lambda}}\Big]\times$$
$$\Big[|\mathcal{F}'(1)|\int_{\psi}\frac{|z(x)-w(x)|}{4}d\varrho(x)+\int_{\psi}\frac{|z(x)-w(x)|}{4}\Big|\mathcal{F}'\frac{z(x)}{w(x)}\Big|d\varrho(x)\Big].$$

Proof. The proof is similar as one has done for Corollary 2.2. □

3.2. Probability density functions

Let $\mathfrak{G}:[\varepsilon,\varepsilon+\eta(\zeta,\varepsilon)]\to[0,1]$ be the probability density function of a continuous random variable Y with the cumulative distribution function of $\mathfrak{G}$

$$F(y)=P(Y\leq y)=\int_{\varepsilon}^{x}\mathfrak{G}(t)dt.$$

As we know $E(y)=\int_{\varepsilon}^{\varepsilon+\eta(\zeta,\varepsilon)}tdF(t)=(\varepsilon+\eta(\zeta,\varepsilon))-\int_{\varepsilon}^{\varepsilon+\eta(\zeta,\varepsilon)}F(t)$

Proposition 3.3. *By Corollary 2.1, we get the inequality*

$$\Big|\frac{1}{6}\Big[4P\Big(Y\leq\frac{2\varepsilon+\eta(\zeta,\varepsilon)}{2}\Big)+1\Big]-\frac{1}{\eta(\zeta,\varepsilon)}(\varepsilon+\eta(\zeta,\varepsilon)-E(y))\Big|$$

$$\leq \frac{\eta(\zeta,\varepsilon)}{4}\Big[\frac{1+2^{\mu+1}}{3^{\mu+1}(\mu+1)}\Big]^{\frac{1}{\mu}}\Big[\Big(\frac{3}{4}\Big)^{\frac{1}{\lambda}}+\Big(\frac{1}{4}\Big)^{\frac{1}{\lambda}}\Big]\Big[|\mathfrak{G}(\varepsilon)|+|\mathfrak{G}(\zeta)|\Big].$$

Proposition 3.4. *By Corollary 2.3, we get the inequality*

$$\Big|\frac{1}{6}\Big[4P\Big(Y\leq\frac{2\varepsilon+\eta(\zeta,\varepsilon)}{2}\Big)+1\Big]-\frac{1}{\eta(\zeta,\varepsilon)}(\varepsilon+\eta(\zeta,\varepsilon)-E(y))\Big|$$
$$\leq \frac{\eta(\zeta,\varepsilon)}{4}\Big(\frac{5}{18}\Big)^{1-\frac{1}{\lambda}}\Big[\Big(\frac{61}{324}\Big)^{\frac{1}{\lambda}}+\Big(\frac{29}{324}\Big)^{\frac{1}{\lambda}}\Big][|\mathfrak{G}(\varepsilon)|+|\mathfrak{G}(\zeta)|].$$

Same applications can be found for Corollary 2.2 and Corollary 2.4, respectively. We leave it to the interested readers.

4. Conclusion

In this paper, we have established several Simpson's type inequalities via κ-fractional integrals in terms of preinvex functions. We also have obtained the inequalities applied to $\mathcal{F}$-divergence measures and application for probability density functions. These results can be viewed as refinement and significant improvements of the previously known for [18, 19] and preinvex functions. Applications can be provided in terms of the obtained results to special means. The ideas and techniques of this paper may be attracted to interested readers.

Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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