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Research article

Multi-dimensional Legendre wavelets approach on the Black-Scholes and Heston Cox Ingersoll Ross equations

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Abstract: The one dimension Legendre Wavelet is a numerical method to solve one dimension equation. In this paper Black-Scholes equation (B-S), that has applied via single asset American option and Heston Cox- Ingersoll- Ross equation (HCIR), as partial differential equations have been studied in the form of stochastic model at first. The Black-Scholes and Heston Cox- Ingersoll- Ross Stochastic differential equations (SDE) models are converted to partial differential equations with a basic lemma in stochastic differential equation which called Ito lemma including derivatives and integration calculus in stochastic differential equations. Multi-dimensional Legendre wavelets method is based upon the expanded properties of Legendre wavelets from high order that is utilized to reduce these equations in to a system of algebraic equations. In fact the properties of Legendre wavelets are leads to reduce the PDEs problems to solution the ODEs systems. To ability and efficiency of the proposed techniques, numerical results and comparison with the other numerical method named Adomian decomposition method (ADM) for different values of parameters are tabulated and plotted.

Keywords: Black-Scholes equation; Heston Cox-Ingersoll-Ross equation; finance equations; Legendre wavelet method; stochastic differential equation; option pricing
Mathematics Subject Classification: 03F07, 11CXX, 35EXX

1. Introduction

Modeling derivative products in mathematical finance usually starts with a system of stochastic differential equation that correspond to state variables like stock, interest rate and volatility. For financial analysts, economic auditors and risk managers in financial organizations, securities analysis is very important, thus in last decades they have tried to design new models that help to change markets react and solve some problems in the market. Modeling challenges has created a common link between applied mathematicians and financial researchers as the production of knowledge in this area [4, 15].

As an important model in the original Black–Scholes option valuation theory the volatility and the interest rate are both assumed constant. A popular expansion of the Black–Scholes model has been proposed by Heston. This expansion allows also for correlation between the volatility and the asset price, so that one can capture skewness effects [16]. In addition to the volatility the values of many practically important securities are sensitive to interest rates, necessitating the construction of interest rate models [17].

Consider an agent who at each time has a portfolio valued at $X(t)$. This portfolio invests in a money market account paying a constant rate of interest r, and in a stock modeled by the geometric Brownian motion,

$$dS(t) = \alpha S(t)dt + \delta S(t)dW(t), \tag{1.1}$$

Suppose at each time t, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t), t \geq 0$. The remainder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market account [12].

In addition to, we consider the following model which presents the system of three SDEs which they have composed from different assets called hybrid model. The asset price model in following can be viewed as an extension of the popular Heston stochastic volatility model where the interest rate is not constant but also follows a stochastic process, described here by the Hull–White model.

$$dS_\tau = R_\tau S_\tau d_\tau + S_\tau \sqrt{V_\tau} dW_\tau^1,\ S(0) > 0, \tag{1.2}$$

$$dV_\tau = k(\eta - V_\tau)d\tau + \delta_1 \sqrt{V_\tau} dW_\tau^2, V(0) > 0,$$

$$dR_\tau = a(b(\tau) - R_\tau)d\tau + \delta_2 \sqrt{R_\tau} dW_\tau^3, R(0) > 0,$$

In this model for $0 < \tau \leq T$ with $T > 0$ the given maturity time of an option. The stocks process S_τ, V_τ, and R_τ denote random variables that represent the asset price, its variance, and the interest rate, at time τrespectively. The variance process V_τ was proposed by Heston and interest rate process R_τ, by Cox, Ingersoll and Ross in Eq.(1.2). Parameters $k, \eta, \delta_1, \delta_2$, and a, are given positive real numbers and $dW_\tau^1, dW_\tau^2, dW_\tau^3$ are Brownian motions with given correlation factors $\rho_1, \rho_2, \rho_3 \in [-1, 1]$. Also b, is a given deterministic positive function of time which can be chosen such that to match the current term structure of interest rates [12, 13].

The introduction of Legendre wavelet method for the variational problems by Razzaghi and Yousefi in 2000 and 2001 [3,11,14], several works applying this method were born. The Legendre wavelets method (LWM) is transformed a boundary value problem (BVP) into a system of algebraic equations [14]. The unknown parameter of this system is the vector that its components are the decomposition coefficients of the BVP solution into Legendre wavelets basis. In this paper we apply the multi- Legendre wavelet method to solve B-S and HCIR partial differential equations.

This paper is organized as follows: In section 2, PDE models of Black-Scholes equation and Heston-Cox-Ingersoll-Ross equation have derived from SDE models, Section 3 has described properties and formulation of Legendre wavelet and Multi-dimensional Legendre Wavelet, Section4, metodology of multi-dimensional LWM is applied on PDEs that use in section 2. In section 5, numerical results and figures are presented and section 6, Conclusion is used to review.

2. Deriving PDE models of B-S and HCIR

In his section we discuss about the B-S PDE model and HCIR PDE model that they have derived from SDE models [1,2,19].

Theorem 2.1. *(Ito-Doeblin formula for Brownian motion). Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x), f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continues and let $W(t)$ be a Brownian motion. Then for every $T \geq 0$,*

$$f(t, W(t)) = f(0, W(0)) + \int_0^T f_t(t, W(t))dt \tag{2.1}$$

$$+\int_0^T f_x(t, W(t))dt + \int_0^T f_{xx}(t, W(t))dt,$$

Proof: see [6].

Remark 2.1: In the Ito- Doeblin formula we can write in differential form [6],

$$df(t, W(t)) = f_t(t, W(t)) + f_t(t, W(t))dW(t) + \frac{1}{2}f_t(t, W(t))dW(t)dW(t), \tag{2.2}$$

$$+f_{tx}(t, W(t))dtdW(t) + \frac{1}{2}f_{xx}(t, W(t))dtdt,$$

But

$$dW(t)dW(t) = d(t), d(t)dW(t) = dW(t)d(t) = 0, d(t)d(t) = 0, \tag{2.3}$$

And the Ito- Doeblin formula in differential form simplifies to

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}dt. \tag{2.4}$$

Definition 2.1: let $W(t), t \geq 0$ be a Brownian motion and let $F(t), t \geq 0$ be an associated filtration. An Ito process as the form following

$$X(t) = X(0) + \int_0^t \triangle(u)dW(u) + \int_0^t \theta(u)d(u), \tag{2.5}$$

Where $X(0)$ is nonrandom and $\triangle(u)$ and $\theta(u)$ are adapted stochastic process.

Lemma 2.1: the quadratic variation of the Ito process Eq.(2.5) is

$$[X, X](t) = \int_0^t \triangle^2(u)du. \tag{2.6}$$

Proof: see [6].

Definition 2.2: let $X(t), t \geq 0$ be an Ito process as described in Definition 2.1 and let $\Gamma(t), t \geq 0$ be an adapted process. We define the integral with respect to an Ito process,

$$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)\triangle(u)dW(u) + \int_0^t \Gamma(u)\theta(u)du. \tag{2.7}$$

Theorem 2.2. *(Ito-Doeblin formula for an Ito process). Let $X(t), t \geq 0$ be an Ito process as described in Definition 2.1 and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x), f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continues. Then for every $T \geq 0$,*

$$f(T,X(T)) = f(0,X(0)) + \int_0 (T) f_t(t,X(t))dt \tag{2.8}$$

$$+ \int_0^T f_x(t,X(t))dX(t) + \frac{1}{2}\int_0^T f_{xx}(t,X(t))d[X,X](t) =$$

$$f(0,X(0)) + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))\triangle(t)dW(t)dt$$

$$+ \int_0^T f_x(t,X(t))\theta dt + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\triangle(t)^2(t)$$

We may rewrite Eq.(2.8),

$$df(t,x(t)) = f_t(t,X(t))dt + f_x(t,X(t))dX(t) + \frac{1}{2}f_{xx}(t,X(t))dX(t)dX(t). \tag{2.9}$$

proof: See [6].

A SDE has defined as te following

$$dX(u) = \beta(u,X(u))du + \gamma(u,X(u))dW(u). \tag{2.10}$$

The functions $\beta(u,X(u))$ and $\gamma(u,X(u))$ called drift and diffusion respectively. Where $t \geq 0$ final goal in these problems is to find a stochastic proces$X(T)$ which is governed by

$$X(t) = x, \tag{2.11}$$

$$X(t) = X(t) + \int_t^T \beta(u,X(u))du + \int_t^T \gamma(u,X(u))dW(u), \tag{2.12}$$

Under conditions of the functions $\beta(u,x)$ and $\gamma(u,x)$ there exists a unique process $X(T), T \geq t$ Satisfying Eq.(2.11) and Eq.(2.12) and generally this is a SDE form of following

$$dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \delta(u)X(u))dW(u), \tag{2.13}$$

Where $a(u), b(u), \delta(u)$ and $\gamma(u)$ are non-random functions of time and we permit on the right- hand side of Eq.(2.10) is the randomness inherent in the solution $X(u)$ and in the driving Brownian motion $W(u)$.Then the stochastic differential equation for geometric Brownian motion is

$$dS(u) = \alpha S(u)du + \gamma S(u)dW(u), \tag{2.14}$$

The rate of change $\frac{dS}{S}$, can be decomposed in to two parts in Eq.(2.14): One is deterministic, the other one is random.
The deterministic part can be modeled by $\frac{dS}{S} = \mu dt$, where μ is a measure of the growth rate of the asset and random part modeled by a Brownian motion $\delta dW(t)$ that δ is the variance of the return and is called the volatility. The overall asset price model is then given by

$$\frac{dS}{S} = \mu dt + \gamma dW(t), \tag{2.15}$$

Eq.(2.15), is a SDE and it can be expressed as follows

$$dS(t) = \mu(S(t))dt + \delta(S(t))dW_s(t). \tag{2.16}$$

An important lemma for finding their solution is the following Ito lemma.

Lemma 2.2: Suppose $S(t)$ satisfies the stochastic differential Eq.(2.16), and $u(S, t)$ is a smooth function. Then $u(S(t), t)$ satisfies the following stochastic differential equation [8].

$$du = (u_t + \mu u_s + \frac{1}{2}\delta^2 u_{ss})dt + \delta u_s dW_s. \tag{2.17}$$

By expanding Eq.(2.17) with applying different variables in Ito lemma to earn B-S PDE and HCIR PDE equations ,as following Black-Scholes PDE equation [1].

Option transaction contract is divided into different types which the most important types are call option and put option. call option contract means that buyer have to buy the certain asset with given price in certain time, as the same way the put option contract leads to sale of asset with given price and certain time by seller. Maturity validity includes American option and European option [5]. Where $W(t)$, is a standard Brownian motion. We also assume that interest rate are constant so that one unit of currency invested in the cash account at time 0, will be worth

$$B(t) = exp(rt),$$

at time t. We will value of call option or put option at time t. By Ito Lemma we know that,

$$du(S, t) = (\mu S(t)\frac{\partial u}{\partial S} + \frac{\partial u}{\partial t} + \frac{1}{2}\delta^2 S^2(t)\frac{\partial^2 u}{\partial S^2})dt + \delta S(t)\frac{\partial u}{\partial S}dW(t), \tag{2.18}$$

Let us now consider a self-financing trading strategy where at each time t, we hold units x_t, of the cash account and y_t, units of the stock. Then p_t, value of this strategy satisfies,

$$p_t = x_t B_t + y_t S_t, \tag{2.19}$$

We will choose x_t and y_t in such a way that the strategy replicates the value of the option the self-financing assumption implies that

$$dp_t = x_t dB_t + y_t dS_t, \tag{2.20}$$
$$= r x_t B_t dt + y_t(\mu S_t dt + \delta S_t dWt) \tag{2.21}$$
$$= (r x_t B_t dt + y_t \mu S_t)dt + y_t \delta S_t dWt,$$

Note that Eq.(2.20) is consistent with our earlier definition of financing. In particular gains or losses on the portfolio are due entirely to gain or losses in the underlying securities i.e. the cash account and stock and not due to change in holding x_t, and y_t.

Returning to our derivation, we can equate times in Eq.(2.18), to obtain

$$y_t = \frac{\partial u}{\partial S}, \tag{2.22}$$
$$r x_t B_t = \frac{\partial u}{\partial S} + \frac{1}{2}\delta^2 S^2 \frac{\partial^2(u)}{\partial S^2},$$

If we set

$$C_0 = P_0,$$

that C and P are call option and put option in finance, the interval value of strategy, then it must be the case that

$$C_t = P_t,$$

since call option or put option have the same dynamics this is true by construction after we equate terms in Eq.(2.19) , with the corresponding terms in Eq.(2.22).

Substituting Eq.(2.22) into Eq.(2.19), we obtain the Black-Scholes PDE

$$rS_t \frac{\partial u}{\partial S} + \frac{\partial u}{\partial t} + \frac{1}{2}\delta^2 S^2 \frac{\partial^2(u)}{\partial S^2} - ru = 0. \tag{2.23}$$

In order to solve Eq.(2.23), boundary conditions must be provided in the case of call option or put option.

By applying system of Eq.(1.1) in Eq.(2.17), for construction of PDE model we need part of deterministic section of a stochastic differential equation model as,

$$\begin{aligned} du(S,v,r,t) &= u_t(S,v,r,t)dt + u_s(S,v,r,t)d(S(t)) + u_r(S,v,r,t)d(r(t)) + u_v(S,v,r,t)d(v(t)) \\ &+\frac{1}{2}u_{ss}(S,v,r,t)d(S(t))d(S(t)) + \frac{1}{2}u_{rr}(S,v,r,t)d(r(t))d(r(t)) + \frac{1}{2}u_{vv}(S,v,r,t)d(v(t))d(v(t)) \\ &+u_{sv}(S,v,r,t)d(S(t))d(v(t)) + u_{sr}(S,v,r,t)d(S(t))d(r(t)) + u_{vr}(S,v,r,t)d(v(t))d(r(t)). \end{aligned} \tag{2.24}$$

Heston-Cox-Ingersoll-Ross PDE equation [6–9],

$$\begin{aligned} \frac{du}{dt} &= rS\frac{du}{dS} + k(\eta - v)\frac{du}{dv} + a(b(t-T) - r)\frac{du}{dr} + \frac{1}{2}S^2 v\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\delta 1^2 v\frac{\partial^2 u}{\partial v^2} \\ &+\frac{1}{2}\delta_2^2 r\frac{\partial^2 u}{\partial r^2} + \rho_{12}\delta_1 S v\frac{\partial^2 u}{\partial S\partial v} + \rho_{13}S\sqrt{vr}\frac{\partial^2 u}{\partial S\partial r} + \rho_{23}\delta_1\delta_2\sqrt{vr}\frac{\partial^2 u}{\partial v\partial r} - ru. \end{aligned} \tag{2.25}$$

By Eq.(2.25) we are obtained PDE model of the HCIR equation.

3. Construction of multi-dimensional Legendre wavelet

In this study Legendre wavelet well-addressed in [5,10,18], are employed.The wavelet basis is constructed from a single function, which is called the mother wavelet. These basis functions are called wavelets and they are an orthonormal set. One of the most important wavelets is Legendre wavelets. The Legendre wavelets are obtained from Legendre polynomials. In the past decade special attention has been given to applications of wavelets. The main characteristic of Legendre wavelet is that it reduces to a system of algebraic equation.
The function $\psi(x) \in L^2(R)$ is a mother wavelet and $\psi(x)_{uv} = |u|^{\frac{-1}{2}}\psi(\frac{x-v}{u})$ which $u, v \in R$ and $u \neq 0$ is a family of continuous wavelets. If we choose the dilation parameter $u = a^{-n}$ and the translation parameter$v = ma^{-n}b$ where $a > 1, b > 0, n$ and m are positive integer, we have the discrete orthogonal wavelets set.

$$\{\psi(x)_{n,m} = |a|^{\frac{n}{2}}\psi(a^n x - mb) : m, n \in Z\}$$

The Legendre wavelet is constructed from Legendre function[4,15]. One dimension Legendre wavelets over the interval [0, 1] are defined as,

$$\psi_{m,n}(t) = \sqrt{(m+\frac{1}{2})}2^{\frac{k}{2}}P_m(2^k x - 2n + 1), \& \frac{\widehat{n} - 1}{2^{k-1}} 0, \& o.w \tag{3.1}$$

With $n = 1, 2..., 2k - 1, m = 0, 1, 2, ..., M - 1$. In Eq.(3.1) P_m are ordinary Legendre functions of order defined over the interval [-1,1]. Legendre wavelet is an orthonormal set as,

$$\int_0^1 \psi(x)_{n,m}\psi(x)_{n',m'}dx = \delta_{n,n'}\delta_{m,m'}, \tag{3.2}$$

The function $f(x) \in L^2([0, 1] \times [0, 1])$, may be expanded as the following,

$$f(x) \cong \sum_{n=1}^{2^k-1}\sum_{m=0}^{M-1} c_{n,m}\psi(x)_{n,m}, \tag{3.3}$$

Now we defined four dimensions Legendre wavelets in $L^4([0, 1] \times [0, 1] \times [0, 1] \times [0, 1])$, as the form

$$\psi_{n_{1,2,3,4},m_{1,2,3,4}}(x) = \begin{cases} A_{m_{1,2,3,4}}(\prod_{i=1}^4 P_{m_i}(2^{k_i}x_i - 2n_i + 1)), & \frac{n_i-1}{2^{k_i-1}} \leq x_i \leq \frac{n_i}{2^{n_i-1}}, i = 1, .., 4 \\ 0, & o.w \end{cases} \tag{3.4}$$

that

$$A_{m_{1,2,3,4}} = \sqrt{\prod_{i=1}^{4}(m_i + \frac{1}{2})2^{\frac{\Sigma_{i=1}^4 k_i}{2}}}, m_i = 0, 1, ..., M_i - 1, n_i = 1, 2, ..., 2^{k_1-1}, i = 1, .., 4$$

Four dimensions Legendre wavelets are orthonormal set over $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$.

$$\int_0^1\int_0^1\int_0^1\int_0^1 \psi_{n_{1,2,3,4},m_{1,2,3,4}}(x, y, z, q)\psi_{n'_{1,2,3,4},m'_{1,2,3,4}}(x, y, z, q)dxdydzdq = \tag{3.5}$$

$$\delta_{n_1,n'_1}, \delta_{n_2,n'_2}, \delta_{n_3,n'_3}, \delta_{n_4,n'_4}, \delta_{m_1,m'_1}, \delta_{m_2,m'_2}, \delta_{m_3,m'_3}, \delta_{m_4,m'_4}.$$

The function $u(x,y,z,q) \in L^4([0,1]\times[0,1]\times[0,1]\times[0,1])$, can be expanded as fallows

$$u(x,y,z,q) = X(x)Y(y)Z(z)Q(q) \cong \prod_{i=1}^{4}(\sum_{n_i=1}^{2^{k_i}-1}\sum_{m_i=1}^{2^{M_i}-1} c_{n_i,m_i}\psi_{n_i,m_i}(x,y,z,q)). \tag{3.6}$$

Where

$$c_{n_{1,2,3,4},m_{1,2,3,4}} = \int_0^1\int_0^1\int_0^1\int_0^1 X(x)Y(y)Z(z)Q(q)\psi_{n_{1,2,3,4},m_{1,2,3,4}}dxdydzdq.$$

The truncated version of Eq.(3.6), can be expressed as the following,

$$u(x,y,z,q) = C^T\Psi(x,y,z,q), \tag{3.7}$$

Where C and $\Psi(x,y,z,q)$, are coefficients matrix and wavelets vector matrix, respectively. The dimensions of those are $\{\prod_{i=1}^{4}(2^{k_i-1}M_i)\}\times 1$, and are given as the form

$$C = [c_{10101010},...,c_{1010101M_1-1},c_{10101020},...,c_{1010102M_2-1},c_{10101030},...,c_{1010103M_3-1},$$
$$c_{1010102^{k_1-1}0},...,c_{1010102^{k_1-1}M_1-1},...,c_{1110101M_1-1},...,c_{1120},...,c_{112M_1-1},$$
$$...,c_{2^{k_1-1}M_{1-1}2^{k_2-1}02^{k_3-1}02^{k_4-1}0},...,c_{2^{k_1-1}M_{1-1}2^{k_2-1}M_{2-1}2^{k_3-1}M_{3-1}2^{k_4-1}M_{4-1},...}]^T,$$

and also

$$\Psi = [\psi_{10101010},...,\psi_{1010101M_1-1},\psi_{10101020},...,\psi_{1010102M_2-1},\psi_{10101030},...,\psi_{1010103M_3-1},$$
$$\psi_{1010102^{k_1-1}0},...,\psi_{1010102^{k_1-1}M_1-1},...,\psi_{1110101M_1-1},...,\psi_{1120},...,\psi_{112M_1-1},$$
$$...,\psi_{2^{k_1-1}M_{1-1}2^{k_2-1}02^{k_3-1}02^{k_4-1}0},...,\psi_{2^{k_1-1}M_{1-1}2^{k_2-1}M_{2-1}2^{k_3-1}M_{3-1}2^{k_4-1}M_{4-1},...]^T}.$$

The integration of the product of four Legendre wavelet function vectors is obtained as

$$\int_0^1\int_0^1\int_0^1\int_0^1 \Psi(x,y,z,q)\Psi^T(x,y,z,q) = I. \tag{3.8}$$

Where I, is diagonal unit matrix.(All of procedures in section 3, are established)

Lemma 3.1: if $h \in C([0,T], L^2[0,1])$ then the functions $C_{n,m}(t)$ are continuous on $[0,1]$ [13].

Lemma 3.2: if $h \in C^1((0,T), L^2[0,1])$ then the function $C_{n,m}(t)$ coefficient belong to $C^1(0,T)$,

furthermore if $\frac{\partial h}{\partial t} \in L^2((0,T), L^2[0,1])$ then,

$$\frac{dC_{n,m}(t)}{dt} = \int_0^1 \frac{\partial h(t,x)}{\partial t}\psi_{n,m}(x)dx. \tag{3.9}$$

Proof: see [13].

3.1. Operational matrix of integration

The integration matrix for one variable x, define as follows

$$\int_0^x \Psi(x', y, z, q)dx' = P_x \Psi(x, y, z, q). \tag{3.10}$$

In which

$$P_x = \frac{1}{M_1 2^{k_1}} \begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix},$$

P_x, is a $2^{k_1-1}2^{k_2-1}2^{k_3-1}2^{k_4-1}M_1M_2M_3M_4 \times 2^{k_1-1}2^{k_2-1}2^{k_3-1}2^{k_4-1}M_1M_2M_3M_4$ matrix and L, F and O, are $2^{k_1-1} \times M_1 \times 2^{k_2-1} \times M_2 \times 2^{k_3-1} \times M_3 \times 2^{k_4-1} \times M_4$ matrices [14]. (In case of two variables Eq.(3.10), is established clearly).

Theorem 3.1. *A function $f(x, y, z, l) \in L^4([0, 1)\times[0, 1)\times[0, 1)\times[0, 1))$ is with bounded second derivative say $|f''(x, y, z, l) \leq M$, can be expanded as an infinite sum of Legendre wavelets and the series converges uniformly to the function $f(x, y, z, l)$ that is [15],*

$$f(x, y, z, l) = \sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_3=1}^{\infty}\sum_{m_3=0}^{\infty}\sum_{n_4=1}^{\infty}\sum_{m_4=0}^{\infty} c_{n_{1,..,4},m_{1,..,4}} \psi_{n_{1,..,4},m_{1,..,4}}(x, y, z, l), \tag{3.11}$$

Proof:
we have proved this theorem for multi-dimension states as following,

$$c_{n_{1,..,4},m_{1,..,4}} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x, y, z, l)dxdydzdl =$$

$$\int_{\frac{\widehat{n_1}-1}{2^k}}^{\frac{\widehat{n_1}+1}{2^k}} \int_{\frac{\widehat{n_2}-1}{2^k}}^{\frac{\widehat{n_2}+1}{2^k}} \int_{\frac{\widehat{n_3}-1}{2^k}}^{\frac{\widehat{n_3}+1}{2^k}} \int_{\frac{\widehat{n_4}-1}{2^k}}^{\frac{\widehat{n_4}+1}{2^k}} f(x, y, z, l)(\frac{2m_1+1}{2})^{\frac{1}{2}}(\frac{2m_2+1}{2})^{\frac{1}{2}}(\frac{2m_3+1}{2})^{\frac{1}{2}}(\frac{2m_4+1}{2})^{\frac{1}{2}}$$

$$2^{\frac{k_1+k_2+k_3+k_4}{2}} P(2^{k_1}x - \widehat{n})P(2^{k_2}x - \widehat{n})P(2^{k_3}x - \widehat{n})P(2^{k_4}x - \widehat{n})dxdydzdl.$$

Now let,

$$2^{k_1}x - \widehat{n} = t_1, 2^{k_2}x - \widehat{n} = t_2, 2^{k_3}x - \widehat{n} = t_3, 2^{k_4}x - \widehat{n} = t_4,$$

then,

$$dx = \frac{1}{2^{k_1}}dt_1, dy = \frac{1}{2^{k_2}}dt_2, dz = \frac{1}{2^{k_3}}dt_3, dl = \frac{1}{2^{k_4}}dt_4.$$

$$c_{n_{1,..,4},m_{1,..,4}} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\frac{\widehat{n_1}+t_1}{2^{k_1}})f(\frac{\widehat{n_2}+t_2}{2^{k_2}})f(\frac{\widehat{n_3}+t_3}{2^{k_3}})f(\frac{\widehat{n_4}+t_4}{2^{k_4}})(\frac{2m_1+1}{2})^{\frac{1}{2}}$$
$$(\frac{2m_2+1}{2})^{\frac{1}{2}}(\frac{2m_3+1}{2})^{\frac{1}{2}}(\frac{2m_4+1}{2})^{\frac{1}{2}}2^{\frac{k_1+k_2+k_3+k_4}{2}}P(t_1)P(t_2)P(t_3)P(t_4)\frac{dxdydzdl}{2^{k_1+k_2+k_3+k_4}} =$$

with get out the constants as followes

$$(\frac{2m_1+1}{2^{k_1+1}})^{\frac{1}{2}}(\frac{2m_2+1}{2^{k_2+1}})^{\frac{1}{2}}(\frac{2m_3+1}{2^{k_3+1}})^{\frac{1}{2}}(\frac{2m_4+1}{2^{k_4+1}})^{\frac{1}{2}}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\frac{\widehat{n_1}+t_1}{2^{k_1}})$$
$$f(\frac{\widehat{n_2}+t_2}{2^{k_2}})f(\frac{\widehat{n_3}+t_3}{2^{k_3}})f(\frac{\widehat{n_4}+t_4}{2^{k_4}})P_{m_1}(t_1)P_{m_2}(t_2)P_{m_3}(t_3)P_{m_4}(t_4)dt_1dt_2dt_3dt_4 =$$

and polynomials are substituted

$$(\frac{1}{2^{k_1+1}(2m_1+1)})^{\frac{1}{2}}(\frac{1}{2^{k_2+1}(2m_2+1)})^{\frac{1}{2}}(\frac{1}{2^{k_3+1}(2m_3+1)})^{\frac{1}{2}}(\frac{1}{2^{k_4+1}(2m_4+1)})^{\frac{1}{2}}$$
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\frac{\widehat{n_1}+t_1}{2^{k_1}})f(\frac{\widehat{n_2}+t_2}{2^{k_2}})f(\frac{\widehat{n_3}+t_3}{2^{k_3}})f(\frac{\widehat{n_4}+t_4}{2^{k_4}})d(P_{m_1+1}(t_1)-P_{m_1-1}(t_1))$$
$$d(P_{m_2+1}(t_2)-P_{m_2-1}(t_2))d(P_{m_3+1}(t_3)-P_{m_3-1}(t_3))d(P_{m_4+1}(t_4)-P_{m_4-1}(t_4)) =$$
$$(\frac{2m_1+1}{2^{k_1+1}})^{\frac{1}{2}}(\frac{2m_2+1}{2^{k_2+1}})^{\frac{1}{2}}(\frac{2m_3+1}{2^{k_3+1}})^{\frac{1}{2}}(\frac{2m_4+1}{2^{k_4+1}})^{\frac{1}{2}}f(\frac{\widehat{n_1}+t_1}{2^{k_1}})f(\frac{\widehat{n_2}+t_2}{2^{k_2}})f(\frac{\widehat{n_3}+t_3}{2^{k_3}})f(\frac{\widehat{n_4}+t_4}{2^{k_4}})$$
$$(P_{m_1+1}(t_1)-P_{m_1-1}(t_1))(P_{m_2+1}(t_2)-P_{m_2-1}(t_2))(P_{m_3+1}(t_3)-P_{m_3-1}(t_3))$$
$$(P_{m_4+1}(t_4)-P_{m_4-1}(t_4))|_{-1}^{1}|_{-1}^{1}|_{-1}^{1}|_{-1}^{1} = -(\frac{1}{2^{3k_1+1}(2m_1+1)})^{\frac{1}{2}}$$

Constants in terms of polynomials are calculated with the others,

$$(\frac{1}{2^{3k_2+1}(2m_2+1)})^{\frac{1}{2}}(\frac{1}{2^{3k_3+1}(2m_3+1)})^{\frac{1}{2}}(\frac{1}{2^{3k_4+1}(2m_4+1)})^{\frac{1}{2}}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\frac{\widehat{n_1}+t_1}{2^{k_1}})$$
$$f(\frac{\widehat{n_2}+t_2}{2^{k_2}})f(\frac{\widehat{n_3}+t_1}{2^{k_3}})f(\frac{\widehat{n_4}+t_4}{2^{k_4}})d(\frac{P_{m_1+2}(t_1)-P_{m_1}(t_1)}{2m_1+3}-\frac{P_{m_1}(t_1)-P_{m_1-2}(t_1)}{2m_1-1})$$
$$d(\frac{P_{m_2+2}(t_2)-P_{m_2}(t_2)}{2m_2+3}-\frac{P_{m_2}(t_2)-P_{m_2-2}(t_2)}{2m_2-1})d(\frac{P_{m_3+2}(t_3)-P_{m_3}(t_3)}{2m_3+3}-\frac{P_{m_3}(t_3)-P_{m_3-2}(t_3)}{2m_3-1})$$
$$d(\frac{P_{m_4+2}(t_4)-P_{m_4}(t_4)}{2m_4+3}-\frac{P_{m_4}(t_4)-P_{m_4-2}(t_4)}{2m_4-1})$$

that,

$$= |\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f''(\frac{\widehat{n_1}+t_1}{2^{k_1}})f''(\frac{\widehat{n_2}+t_2}{2^{k_2}})f''(\frac{\widehat{n_3}+t_1}{2^{k_3}})f''(\frac{\widehat{n_4}+t_4}{2^{k_4}})$$

$$(\frac{P_{m_1+2}(t_1)-P_{m_1}(t_1)}{2m_1+3}-\frac{P_{m_1}(t_1)-P_{m_1-2}(t_1)}{2m_1-1})(\frac{P_{m_2+2}(t_2)-P_{m_2}(t_2)}{2m_2+3}-\frac{P_{m_2}(t_2)-P_{m_2-2}(t_2)}{2m_2-1})$$
$$(\frac{P_{m_3+2}(t_3)-P_{m_3}(t_3)}{2m_3+3}-\frac{P_{m_3}(t_3)-P_{m_3-2}(t_3)}{2m_3-1})(\frac{P_{m_4+2}(t_4)-P_{m_4}(t_4)}{2m_4+3}-\frac{P_{m_4}(t_4)-P_{m_4-2}(t_4)}{2m_4-1})|^2$$
$$|\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f''(\frac{\widehat{n_1}+t_1}{2^{k_1}})f''(\frac{\widehat{n_2}+t_2}{2^{k_2}})f''(\frac{\widehat{n_3}+t_1}{2^{k_3}})f''(\frac{\widehat{n_4}+t_4}{2^{k_4}})$$
$$(\frac{(2m_1-1)P_{m_1+2}(t_1)-(4m_1+2)P_{m_1}(t_1)+(2m_1+3)}{2m_1+3})...$$

Using integration properties as following,

$$(\frac{(2m_4-1)P_{m_4+2}(t_4)-(4m_4+2)P_{m_4}(t_4)+(2m_1+3)}{2m_4+3})dt_1dt_2dt_3dt_4|^2 \leq$$
$$|\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f''(\frac{\widehat{n_1}+t_1}{2^{k_1}})f''(\frac{\widehat{n_2}+t_2}{2^{k_2}})f''(\frac{\widehat{n_3}+t_1}{2^{k_3}})f''(\frac{\widehat{n_4}+t_4}{2^{k_4}})dt_1dt_2dt_3dt_4|^2\times$$
$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|(\frac{(2m_1-1)P_{m_1+2}(t_1)-(4m_1+2)P_{m_1}(t_1)+(2m_1+3)}{2m_1+3})...dt_1dt_2dt_3dt_4|^2$$
$$2M^2 2M^2 2M^2 2M^2\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(\frac{(2m_1-1)P_{m_1+2}(t_1)-(4m_1+2)P_{m_1}(t_1)+(2m_1+3)}{2m_1+3})$$
$$...dt_1dt_2dt_3dt_4 =$$

Upper bounds are noted as,

$$\frac{16M^{16}}{(2m_1+3)^2(2m_1-1)^2...(2m_4+3)^2(2m_4-1)^2}((2m_1-1)^2\frac{2}{2m_1+5}+$$
$$(4m_1+2)^2\frac{2}{2m_1+1}+(2m_1+3)^2\frac{2}{2m_1-3})...((2m_4-1)^2\frac{2}{2m_4+5}+$$
$$(4m_4+2)^2\frac{2}{2m_4+1}+(2m_4+3)^2\frac{2}{2m_4-3}) <$$

Now $(2m_1-1)^2 < (2m_1+3)^2$, and $(2m_1+1)^2 < (2m_1+3)^2$, then

$$= \frac{331776M^{16}}{(2m_1-1)((2m_1-3)...(2m_4-1)(2m_4-3))^2}.$$

Therefore we have,

$$|c_{n_{1,...,4},m_{1,...,4}}| < (576)^{\frac{1}{2}}\frac{1}{(2n)^{\frac{1}{2}}}\frac{1}{(2m_1-3)^2...(2m_4-3)^2}.$$

Thus the series $\sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_3=1}^{\infty}\sum_{m_3=0}^{\infty}\sum_{n_4=1}^{\infty}\sum_{m_4=0}^{\infty} c_{n_{1,...,4},m_{1,...,4}}$, is absolute convergent, it follows that

$$\sum_{n_1=1}^{\infty}\sum_{m_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_3=1}^{\infty}\sum_{m_3=0}^{\infty}\sum_{n_4=1}^{\infty}\sum_{m_4=0}^{\infty} c_{n_{1,..,4},m_{1,..,4}}\psi_{n_{1,..,4},m_{1,..,4}}(x,y,z,l),$$

convergence to the function $f(x, y, z, l)$, uniformly.
Notice: This result for two dimensions in theorem (3.1), is calculated as upper bound,

$$|c_{n_1n_2m_1m_2}| < (288)^{\frac{1}{2}}\frac{1}{2n^{\frac{5}{2}}}\frac{1}{(2m_1-3)^2(2m_2-3)^2}.$$

4. Methodology for B-S and HCIR PDEs

In this methodology section, the Black-Scholes model for American put problem have used in the form of moving-boundary problem.
Lemma 4.1: For American options, we have [5, 6]
i) The optimal exercise time for American call option is T and we have

$$C = c,$$

ii) The optimal exercise time for American put option as earlier as possible, i.e. t and we have

$$P \geq p,$$

iii) The put-call parity for American option:

$$S - K < C - P < S - ke^{-t(T-t)},$$

As a consequence,

$$P \leq K.$$

Letting S be the price of an asset A at time t the American early exercise constraint leads to the following model for the value $P(S, t)$ of an American put option to sell the asset A.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\delta^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, 0 \leq t < T \tag{4.1}$$

$$P(S,T) = max(E-S,0), S \geqslant 0,$$

$$\frac{\partial P}{\partial S}(\overline{S},t) = -1,$$

$$P(S,t) = E - S,\ 0 \leq S < S(\bar{t}).$$

Where $S(\bar{t})$ represent the free boundary. The parameters δ, r and E represent the volatility of the underlying asset, the interest rate, and the exercise price of the option. And the partial differential equation (PDE) associated with Eq.(2.25) for the fair values of European- style options forms a time-dependent connection- diffusion- reaction equation with mixed spatial- derivative terms.

$$\frac{\partial u}{\partial t} = \frac{1}{2}S^2 v\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\delta_1^2 v\frac{\partial^2 u}{\partial v^2} + \frac{1}{2}\delta_2^2 r\frac{\partial^2 u}{\partial r^2} + \rho_{12}\delta_1 S v\frac{\partial^2 u}{\partial S\partial v} + \rho_{13}\delta_2 S\sqrt{vr}\frac{\partial^2 u}{\partial S\partial r} \tag{4.2}$$

$$\rho_{23}\delta_1\delta_2\sqrt{vr}\frac{\partial^2 u}{\partial v\partial r} + rS\frac{\partial u}{\partial S} + k(\eta - v)\frac{\partial u}{\partial v} + a(b(T-t) - r)\frac{\partial u}{\partial r} - ru.$$

On the unbounded three- dimensional spatial domain $S > 0, v > 0, r > 0$ with $0 < t \leq T$. $u(S, v, r, t)$ denotes the fair value of a European- style option if at the time $\tau = T - t$ the asset price, its variance, and the interest rate be equal to S, v and r respectively. Eq.(4.2) is known as the Heston- Cox-Ingersoll- Ross (HCIR) PDE. This equation usually appears with initial and boundary conditions which are determined by the specific option under consideration. Initial conditions in European call options equation is usually as follows,

$$u(S, v, r, T) = max(0, S - k). \tag{4.3}$$

With a given strike price $k > 0$.

In this methodology section the Eq.(4.1) and Eq.(4.2), have been solved using two dimensional and four dimensional Legendre wavelets respectively,

First we assume for B-S PDE,

$$\frac{\partial P}{\partial t} = -\alpha\frac{\partial^2 P}{\partial S^2} - \alpha\frac{\partial P}{\partial S} + \alpha P, \tag{4.4}$$

With boundary condition $P(S, 0) = \beta_1(S, t)$, and $\frac{\partial P(S,0)}{\partial t} = \beta_2(S, t)$.

Those coefficients of α_i the same parameters and variables which be in Eq.(4.1), in practice they will be calculate with this technique in follow.

$$\frac{\partial P}{\partial t} = C_1^T(S, t)\Psi(S, t), \tag{4.5}$$

Integrating (4.4) with respect to second variable over $[0, t]$, we get

$$P(S, t) = C_1^T(S, t)P\Psi(S, t) + \beta_2(S, t), \tag{4.6}$$

And also

$$\frac{\partial P}{\partial S} = \frac{dC^T(S, t)}{dS}P\Psi(S, t) + \frac{d\beta_2(S, t)}{dS}, \tag{4.7}$$

$$\frac{\partial^2 P}{\partial S^2} = \frac{d^2C^T(S, t)}{dS^2}P\Psi(S, t) + \frac{d^2\beta_2(S, t)}{dS^2},$$

We have also

$$1 = d^T\Psi(x), x = e^T\Psi(x). \tag{4.8}$$

Substituting Eqs.(4.4) to (4.8) in Eq.(4.1), we obtain

$$C_1^T(S,t)\Psi = -\alpha_1(\frac{d^2C^T(S,t)}{dS^2}P\Psi + \frac{d^2\beta_2(S,t)}{dS^2}d^T\Psi) - \alpha_2(\frac{dC^T(S,t)}{dS}P\Psi + \frac{d\beta_2(S,t)}{dS}d^T\Psi) \tag{4.9}$$
$$+\alpha_3(C_1^T(S,t)P\Psi + \beta_2 d^T\Psi).$$

The solution of B-S PDE, $P(S,t)$ can be calculate when Eq.(4.9) can be solved For unknown coefficients of the vector $C(S,t)$.

$C = [0.12177112177 - 0.2214022140e-1t, -0.1661746162 + 0.3451318952t, 0.7030464532e-1 - 0.1278266279e-1t, -0.9594095941e-1 + 0.1992619926t]^T$.

Let for HCIR PDE in Eq.(4.2), we assume as following,

$$\frac{\partial u}{\partial t} = \alpha_1\frac{\partial^2 u}{\partial S^2} + \alpha_2\frac{\partial^2 u}{\partial v^2} + \alpha_3\frac{\partial^2 u}{\partial r^2} + \alpha_4\frac{\partial^2 u}{\partial S\,\partial v} + \alpha_5\frac{\partial^2 u}{\partial S\,\partial r} + \alpha_6\frac{\partial^2 u}{\partial v\partial r} + \alpha_7\frac{\partial u}{\partial S} \tag{4.10}$$
$$+\alpha_8\frac{\partial u}{\partial v} + \alpha_9\frac{\partial u}{\partial r} + \alpha_{10}u.$$

Also with boundary condition

$$u(S,v,r,t) = \beta_1(S,v,r,t),$$

and

$$\frac{\partial u(S,v,r,0)}{\partial t} = \beta_2(S,v,r,t).$$

Parameters of Eq.(4.2), by using method as following,

$$\frac{\partial u}{\partial t} = C_1^T(S,v,r,t)\Psi(S,v,r,t), \tag{4.11}$$

Integrating right hand side in Eq.(4.10), with respect to differential equations over [0, t] we get As well as

$$\frac{\partial u}{\partial S} = \frac{dC^T(S,v,r,t)}{dS}P\Psi(S,v,r,t) + \frac{d\beta^2(S,v,r,t)}{dS}, \tag{4.12}$$
$$\frac{\partial u}{\partial v} = \frac{dC^T(S,v,r,t)}{dv}P\Psi(S,v,r,t) + \frac{d\beta^2(S,v,r,t)}{dv},$$
$$\frac{\partial u}{\partial r} = \frac{dC^T(S,v,r,t)}{dr}P\Psi(S,v,r,t) + \frac{d\beta^2(S,v,r,t)}{dr}.$$

We have

$$\frac{\partial^2 u}{\partial S^2} = \frac{d^2C^T(S,v,r,t)}{dS^2}P\Psi(S,v,r,t) + \frac{d^2\beta^2(S,v,r,t)}{dS^2}, \tag{4.13}$$

$$\frac{\partial^2 u}{\partial v^2} = \frac{d^2 C^T(S,v,r,t)}{dv^2} P\Psi(S,v,r,t) + \frac{d^2\beta^2(S,v,r,t)}{dv^2},$$
$$\frac{\partial^2 u}{\partial r^2} = \frac{d^2 C^T(S,v,r,t)}{dr^2} P\Psi(S,v,r,t) + \frac{d^2\beta^2(S,v,r,t)}{dr^2}.$$

And also

$$\frac{\partial^2 u}{\partial S\,\partial v} = \frac{d}{dS}\frac{dC^T(S,v,r,t)}{dv} P\Psi(S,v,r,t) + \frac{d}{dS}\frac{d\beta^2(S,v,r,t)}{dv}, \tag{4.14}$$
$$\frac{\partial^2 u}{\partial S\,\partial r} = \frac{d}{dS}\frac{dC^T(S,v,r,t)}{dr} P\Psi(S,v,r,t) + \frac{d}{dS}\frac{d\beta^2(S,v,r,t)}{dr},$$
$$\frac{\partial^2 u}{\partial v \partial r} = \frac{d}{dv}\frac{dC^T(S,v,r,t)}{dr} P\Psi(S,v,r,t) + \frac{d}{dv}\frac{d\beta^2(S,v,r,t)}{dr}.$$

Substituting (4.11) to (4.14) in (48), we obtain

$$C^T(S,v,r,t)\Psi = \alpha_1\left(\frac{\partial^2 C^T(S,v,r,t)}{\partial S^2} P\Psi + \frac{d^2\beta_2(S,v,r,t)}{dS^2} d^T\Psi\right) \tag{4.15}$$
$$+\alpha_2\left(\frac{\partial^2 C^T(S,v,r,t)}{\partial v^2} P\Psi + \frac{d^2\beta_2(S,v,r,t)}{dv^2} d^T\Psi\right) + \alpha_3\left(\frac{\partial^2 C^T(S,v,r,t)}{\partial r^2} P\Psi + \frac{d^2\beta_2(S,v,r,t)}{dr^2} d^T\Psi\right)$$
$$+\alpha_4\left(\frac{d}{dS}\frac{dC^T(S,v,r,t)}{dv} P\Psi + \frac{d}{dS}\frac{d\beta_2(S,v,r,t)}{dv} d^T\Psi\right) + \alpha_5\left(\frac{d}{dS}\frac{dC^T(S,v,r,t)}{dr} P\Psi + \frac{d}{dS}\frac{d\beta_2(S,v,r,t)}{dr} d^T\Psi\right)$$
$$+\alpha_6\left(\frac{d}{dv}\frac{dC^T(S,v,r,t)}{dr} P\Psi + \frac{d}{dv}\frac{d\beta_2(S,v,r,t)}{dr} d^T\Psi\right) + \alpha_7\left(\frac{\partial C^T(S,v,r,t)}{\partial S} P\Psi + \frac{d\beta_2(S,v,r,t)}{dS} d^T\Psi\right)$$
$$+\alpha_8\left(\frac{\partial C^T(S,v,r,t)}{\partial v} P\Psi + \frac{d\beta_2(S,v,r,t)}{dv} d^T\Psi\right) + \alpha_9\left(\frac{\partial C^T(S,v,r,t)}{\partial r} P\Psi + \frac{d\beta_2(S,v,r,t)}{dr} d^T\Psi\right)$$
$$-\alpha_{10}(C^T(S,v,r,t)P\Psi + \beta_2 d^T\Psi).$$

This system can be solved for unknown coefficients, which are calculated,

$C[1,1] = 0.1816780437e - 2 \times sqrt(v) \times s \times t + .2310089049 - 0.3191757882e - 2 \times s \times t^2 - 0.2243167683e - 2 \times sqrt(v) \times s - 0.3663030704 \times s \times t + 0.5319596471e - 3 \times t^2 + 0.4571378943 \times s + 0.8386506682e - 2 \times t$,

$C[1,2] = -0.4720134035e - 3 \times sqrt(v) \times s \times t + 0.8292430227e - 3 \times s \times t^2 + 0.5827920594e - 3 \times sqrt(v) \times s + 0.9516832935e - 1 \times s \times t - 0.1382071705e - 3 \times t^2 - 0.1187679088 \times s - 0.2178878351e - 2 \times t + 0.1997897471$,

$C[1,3] = 0, C[1,4] = 0, \quad C[1,5] = 0.2097293811,$

$C[1,6] = 0.1705107084, \quad C[1,7] = 0, C[1,8] = 0,$

$C[1,9] = -0.04660652914,$ $C[1,10] = -0.3789126853e-1,$

$C[1,11] = 0, C[1,12] = 0,$ $C[1,13] = -0.4036243822e-1,$

$C[1,14] = -0.3281480113e-1,$ $C[1,15] = 0, C[1,16] = 0.$

5. Numerical results

In this section, values of parameters are used in B-S PDE model (Figure 1): $k_1 = k_2 = 1, r = 0.2, E = 0.2, \delta = 0.001$, and HCIR PDE model (Figures 2 and 3): $k = 0.1, \eta = 0.12, \rho_{12} = 0.6, \rho_{13} = 0.2, \rho_{23} = 0.4, \delta_1 = 0.04, \delta_2 = 0.03$, respectively. Appling values of parameters in models, numerical results are calculated. Due to the accuracy and convergence of spectral method which has been reviewed in [0, 1], the numerical results obtained from this method can be assured. Therefore it can be used for future scams to compare with the same methods.

Table 1. Corresponding solution of Black-Scholes PDE by LWM, ADM.

t	$P_{(LWM,M=2)}$	$P_{(LWM,M=4)}$	$P_{(ADM)}$	$Error$
0.1	0.3060753725	0.3060954392	0.3068021827	0.000726810
0.2	0.02615249781	0.02664386902	0.02613346987	0.000019028
0.4	0.04431200702	0.04942003420	0.04430950409	0.000005503
0.6	0.05847852762	0.05882487996	0.05862488320	0.000146355
0.8	0.08565205963	0.08534345864	0.08534345864	0.000308601
1	0.05083260299	0.0509281709	0.05075733333	0.000075268

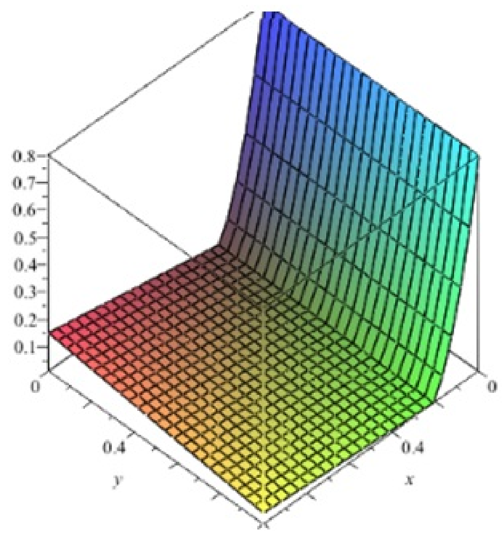

Figure 1. Figure of solution B-S PDE model by multi-LWM.

Table 2. Corresponding solution of Heston-Cox-Ingersoll-Ross PDE by LWM, ADM.

(t, s)	$U_{(LWM,M=1)}$	$U_{(LWM,M=2)}$	$U_{(ADM)}$	$Error$
$(5, 50)$	$0.1950128206e+3$	$0.1955141009e+3$	$0.1951180661e+3$	0.0003960348
$(10, 100)$	$0.2721204331e+4$	$0.2725170985e+4$	$0.2789681554e+4$	0.0064510569
$(12, 150)$	$0.6421235221e+4$	$0.6430612189e+4$	$0.6448380051e+4$	0.0017767862
$(15, 200)$	$0.1461103897e+5$	$0.1470153697e+5$	$0.1486797294e+5$	0.0016643597
$(17, 250)$	$0.2471265331e+5$	$0.2474067533e+5$	$0.2471107229e+5$	0.0002960304
$(19, 300)$	$0.3852234601e+5$	$0.3858603408e+5$	$0.3807240307e+5$	0.0051363101
$(22, 380)$	$0.6881012512e+5$	$0.6890201411e+5$	$0.6860384740e+5$	0.0029816671

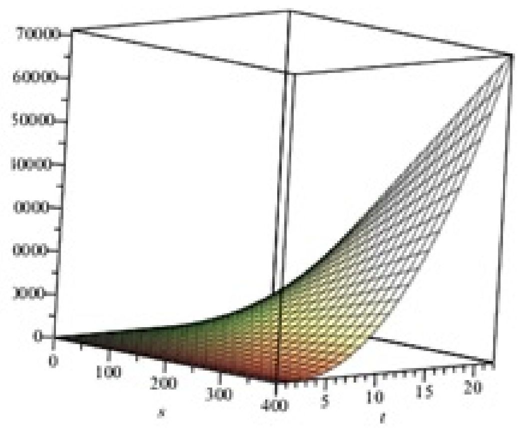

Figure 2. Figure of solution HCIR PDE model by multi-LWM.

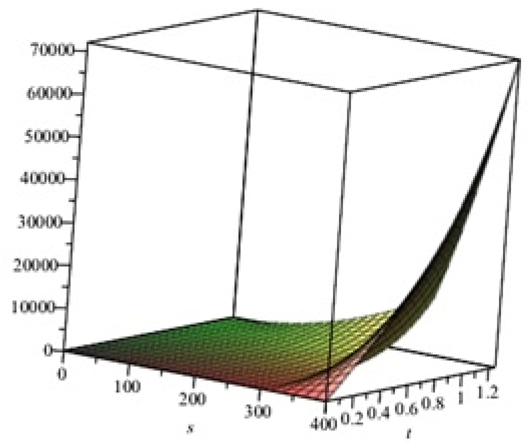

Figure 3. Figure of solution HCIR PDE model by ADM.

By choosing $M = 2, 4$ and $k = 1$ in Eq.(2.23) and $M = 1, 2$ in Eq.(2.25) the models are solved from the coefficients vector of C that was found. Then the results were compared by Adomian decomposition method (ADM), as a semi-analytical method [20–21]. Tables 1 and 2 give values of pricing options solutions in B-S and HCIR models.

6. Conclusion

The multi-dimensional Legendre wavelet method has been applied to solve the Black-Scholes and Heston-Cox-Ingersoll-Ross models which these PDE equations have been derived from stochastic differential equations by using one of the important stochastic calculous Lemma named Ito. Generally the present method (LWM) reduces the partial differential equation into a set of algebraic equation and it can be useful in high degree derivatives problems, therefore considering the properties and using structure of LWM leads to save the time. Multi- dimensional Legendre wavelet is orthonormal set and its integration of the product is a diagonal matrix as in the case of the one dimension Legendre wavelet. In addition the Theorem 3.1 has been proved in paper which it could be determine an upper bound for coefficients of vector C. Also in this work results of LWM have been compared with the Adomian Decomposition Method (ADM) as a semi-analytical method and they have been shown for different parameters in Tables 1–2 and Figures 1–3, finally.

Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

References

1. J. C. Hull, *Fundamentals of Futures and Options Markets and Derivagem Package*, 6th Edition, Prentice Hall, 2007.
2. S. G. Kou, *A Jump-Diffusion Model for Option Pricing*, Manage. Sci., **48** (2002), 1086–1101.
3. S. L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies, **6** (1993), 327–343.
4. L. B. G. Andersen, V. Piterbarg, *Interest Rate Modeling*, 1st ed., Atlantic Financial Press, 2010.
5. E. S. Shreve, *Stochastic calculus for finance*, Springer, 2000.
6. D. Duffie, J. Pan, K. Singleton, *Transform Analysis and Asset Pricing for Affine Jump-Diffusions*, Econometrica, **68** (2000), 1343–1376.
7. S. Heston, *A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, Review of Financial Studies, **6** (1993), 327–343.
8. M. Razzaghi, S. Yousefi, *Legendre wavelets direct method for variational problems*, Math. Comput. Simulat., **53** (2000), 185–192.
9. M. Razzaghi, S. Yousefi, *Legendre wavelets method for constrained optimal control problems*, Math. Method. Appl. Sci., **25** (2002), 529–539.

10. M. Razzaghi, S. Yousefi, *The Legendre wavelets operational matrix of integration*, Int. J. Syst. Sci., **32** (2001), 495–502.

11. M. Haugh, IEOR E4706 Foundations of financial engineering, 2016.

12. L. A. Grzelak, C. W. Oosterlee, *On the Heston model with stochastic interest rates*, SIAM J. Financ. Math., **2** (2011), 255–286.

13. T. Haentjens, *ADI FD schemes for the numerical solution of the three-dimensional Heston-Cox-Ingersoll-Ross PDE*, AIP Conference Proceedings, **1479** (2012), 2195–2199.

14. T. Haentjens & K. J. In't Hout, *Alternating direction implicit finite difference schemes for the Heston-Hull-White PDE*, The Journal of Computational Finance, **16** (2012), 83.

15. R. J. Adler, *The Geometry of Random Fields*, Wiley & Sons, 1981.

16. R. Y. Chang, M. L. Wang, *Shifted Legendre directs method for Variational problems*, J. Optimiz. Theory App., **39** (1983), 299–307.

17. C. F. Chen, C. H. Hsiao, *A Walsh series direct method for solving variational problems*, J. Franklin I., **300** (1975), 265–280.

18. M. Razzaghi, *Fourier series direct method for variational problems*, Int. J. Control, **48** (1988), 887–895.

19. N. Ablaoui-Lahmar, O. Belhamitib and S. M. Bahri, *A New Legendre Wavelets Decomposition Method for Solving PDEs*, Malaya Journal of Matematik (MJM), **2** (2014), 72–81.

20. G. Adomian, *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl., **135** (1988), 501–544.

21. J. Biazar, F. Goldoust, F. Mehrdoust, *On the Numerical Solutions of Heston Partial Differential Equation*, Mathematical Sciences Letters, **4** (2015), 63.