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Research article

On $\mathcal{L}$-simulation mappings in partial metric spaces

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Abstract: The class of L-contractive mappings was introduced by Cho [12]. In this paper, we provide some fixed point results for such mappings via a control function introduced by Jleli and Samet [14] in the class of partial metric spaces. Some illustrating examples are given.

Keywords: partial metric space; fixed point; $\mathcal{L}$-simulation; θ-function
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1. Introduction

An abstraction of the classical distance goes back to ancient mathematician Euclid's postulate. The notion of a metric, an Euclidean distance, was axiomatically formulated by Fréchet and Haussdorff at the beginning of nineteen century. Since then this notion has been refined, generalized and improved in several directions. Among all, we focus on a partial metric, defined by Matthews [21] by relaxing the self-distance axiom. Indeed, a partial metric is a very important extension of the metric, which is very useful in the framework of theoretical computer science, in particular, domain theory and semantic.

Matthews [21] proved the analog of the Banach's contraction principle [10] in the setting of partial metric spaces. This pioneer work [21] initiates an attractive trend, see e.g. [1–9, 11, 13, 15–19, 22–25, 27–31].

For the readability of the paper, we recollect fundamental definitions and results.

Definition 1.1. *For a non-empty set X, if a function $p : X \times X \to [0,\infty)$ that satisfies:*

(PM1) $p(\theta,\theta) = p(\theta,\vartheta) = p(\vartheta,\vartheta) \Longleftrightarrow \theta = \vartheta$;
(PM2) $p(\theta,\theta) \leq p(\theta,\vartheta)$;
(PM3) $p(\theta,\vartheta) = p(\vartheta,\theta)$;
(PM4) $p(\theta,\eta) + p(\vartheta,\vartheta) \leq p(\theta,\vartheta) + p(\vartheta,\eta)$.

for all $\theta,\vartheta,\eta \in X$, then it is called a "partial metric" and (X, p) denotes a "partial metric space", in short, PMS.

Let $\{\xi_n\}$ be a sequence in a PMS (X, p). We say that $\{\xi_n\}$ converges to $\xi \in X$ if the following inequality holds:

$$p(\xi,\xi) = \lim_{n\to\infty} p(\xi_n,\xi).$$

A sequence $\{\xi_n\}$ is called "fundamental" (Cauchy), if $\lim_{n,m\to\infty} p(\xi_n,\xi_m)$ exists and is finite. A PMS (X, p) is called complete if each fundamental sequence in X converges to a point $\xi \in X$ such that

$$p(\xi,\xi) = \lim_{n,m\to\infty} p(\xi_n,\xi_m).$$

We need to mention that the strong correlation between metric and partial metric. A function $d_p : X \times X \to [0,\infty)$, defined by

$$d_p(\xi,\eta) = 2p(\xi,\eta) - p(\xi,\xi) - p(\eta,\eta)$$

for all $\xi,\eta \in X$, forms a metric on X.

A sequence in the PMS (X, p) is Cauchy if and only if it is a Cauchy sequence in the metric space (X, d_p). Also, (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(\xi_n,\xi) = 0 \Leftrightarrow p(\xi,\xi) = \lim_{n\to\infty} p(\xi_n,\xi) = \lim_{n,m\to\infty} p(\xi_n,\xi_m).$$

The following lemmas are useful.

Lemma 1.1. *[7, 8, 16] Let (X, p) be a PMS. We have*
(1) *if $p(\xi,\eta) = 0$, then $\xi = \eta$;*
(2) *if $\xi \neq \eta$, then $p(\xi,\eta) > 0$.*

Lemma 1.2. *[7, 8, 16] Let $\xi_n \to \xi$ as $n \to \infty$ in the PMS (X, p) such that $p(\xi,\xi) = 0$. Then $p(\xi_n,\eta) \to p(\xi,\eta)$ as $n \to \infty$ for each $\eta \in X$.*

In [14], for a mapping $\theta : (0,\infty) \to (1,\infty)$, the concept of θ-contractions was defined as
(θ_1) θ is non-decreasing;
(θ_2) For any positive sequence $\{u_n\}$

$$\lim_{n\to\infty} \theta(u_n) = 1 \text{ if and only if } \lim_{n\to\infty} u_n = 0^+;$$

(θ_3) there exists $(s,t) \in (0,1) \times (0,\infty)$ such that

$$\lim_{u\to 0^+} \frac{\theta(u) - 1}{u^s} = t.$$

Let F be the class of functions θ verifying (θ_1)-(θ_3). Very recently, Cho [12] introduced the concept of $\mathcal{L}$-simulation mappings. Let $\mathcal{L}$ be the family of all mappings $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ such that
(ξ_1) $\xi(1,1) = 1$;
(ξ_2) $\xi(u,v) < \frac{v}{u}$ for all $u, v > 1$;
(ξ_3) For any sequences $\{u_n\}$, $\{v_n\}$ in $(1,\infty)$ with $u_n < v_n$ for $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n > 1 \text{ implies } \limsup_{n\to\infty} \xi(u_n, v_n) < 1.$$

Any $\xi \in \mathcal{L}$ is said an $\mathcal{L}$-simulation function. Note that $\xi(u,u) < 1$ for each $u > 1$.

Let $\mathcal{Z}$ be the set of simulation functions in the sense of Khojasteh, Shukla and Radenović [20].

Definition 1.2. *[20] A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(u,v) < v - u$ for all $u, v > 0$;
(ζ_3) if $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \ell \in (0,\infty)$, then

$$\limsup_{n\to\infty} \zeta(u_n, v_n) < 0.$$

We mention that any $\mathcal{L}$-simulation function can be deduced from simulation functions of Khojasteh, Shukla and Radenović [20]. Indeed, for $u, v \geq 0$, let $\xi(e^u, e^v) = e^{\zeta(u,v)}$. If $\zeta \in \mathcal{Z}$, then $\xi \in \mathcal{L}$.

In this paper, we obtain some fixed point results for $\mathcal{L}$-simulation mappings using θ-functions in the class of partial metric spaces. Some consequences are also derived. Moreover, we present some examples in support of the given results.

2. Main results

First, let Θ be the set of **continuous** functions $\theta : (0,\infty) \to (1,\infty)$ satisfying (θ_1) and (θ_2). Since the range of θ is in $(1,\infty)$, we will consider $\mathcal{L}$-simulation functions instead of simulation functions [20] and we will get many consequences in this setting. Our first main result is the following theorem.

Theorem 2.1. *Let (X,p) be a complete PMS and $T : X \to X$ be a given mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that for all $x, y \in X$ with $p(Tx,Ty) \neq 0$ and $p(x,y) \neq 0$,*

$$\xi\left(\theta(p(Tx,Ty)), \theta(p(x,y))\right) \geq 1. \tag{2.1}$$

Then T has a unique fixed point.

Proof. Define a sequence $\{\xi_n\}$ by $\xi_n = T^n\xi_0$ for all $n \geq 0$. If $p(\xi_n, \xi_{n+1}) = 0$ for some n, then $\xi_n = \xi_{n+1} = T\xi_n$, that is, ξ_n is a fixed point of T and so the proof is completed. Suppose from now on that $p(\xi_n, \xi_{n+1}) > 0$ for all $n = 0, 1, \ldots$

Step 1: We shall prove that

$$\lim_{n\to\infty} p(\xi_n, \xi_{n+1}) = 0.$$

First, from the condition (2.1), we have

$$\begin{aligned} 1 &\leq \xi\left(\theta(p(T\xi_{n-1}, T\xi_n)), \theta(p(\xi_{n-1}, \xi_n))\right) \\ &< \frac{\theta(p(\xi_{n-1}, \xi_n))}{\theta(p(T\xi_{n-1}, T\xi_n))} = \frac{\theta(p(\xi_{n-1}, \xi_n))}{\theta(p(\xi_n, \xi_{n+1}))}. \end{aligned}$$

Consequently, we obtain that

$$\theta(p(\xi_n, \xi_{n+1})) < \theta(p(\xi_{n-1}, \xi_n)),$$

which implies for all $n = 1, 2, 3, \ldots$,

$$p(\xi_n, \xi_{n+1}) \leq p(\xi_{n-1}, \xi_n).$$

Hence $\{p(\xi_n, \xi_{n+1})\}$ is a decreasing sequence, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(\xi_n, \xi_{n+1}) = r.$$

Assume that $r \neq 0$. It follows from (θ_2) that

$$\lim_{n\to\infty} \theta(p(\xi_n, \xi_{n+1})) \neq 1,$$

and so

$$\lim_{n\to\infty} \theta(p(\xi_n, \xi_{n+1})) > 1, \text{ for all } n = 1, 2, 3, \ldots$$

From (ξ_3),

$$1 \leq \limsup_{n\to\infty} \xi(\theta(p(\xi_n, \xi_{n+1})), \theta(p(\xi_{n-1}, \xi_n)) < 1,$$

which is contradiction. This implies that

$$\lim_{n\to\infty} p(\xi_n, \xi_{n+1}) = 0. \tag{2.2}$$

Step 2: Now, we shall show that $\{\xi_n\}$ is a bounded sequence in (X, p). We argue by contradiction. If $\{\xi_n\}$ is not bounded, then there exists a subsequence $\{\xi_{n(k)}\}$ of $\{\xi_n\}$ such that $n(1) = 1$ and for all $k = 1, 2, 3, \ldots, n(k+1)$ is the minimum integer greater than $n(k)$ with

$$p(\xi_{n(k+1)}, \xi_{n(k)}) > 1 \quad \text{and} \quad p(\xi_l, \xi_{n(k)}) \leq 1, \tag{2.3}$$

for all $n(k) \leq l \leq n(k+1) - 1$. We have

$$\begin{aligned} 1 &< p(\xi_{n(k+1)}, \xi_{n(k)}) \leq p(\xi_{n(k+1)}, \xi_{n(k+1)-1}) + p(\xi_{n(k+1)-1}, \xi_{n(k)}) \\ &\leq p(\xi_{n(k+1)}, \xi_{n(k+1)-1}) + 1. \end{aligned}$$

Taking the limit as $k \to \infty$ and using (2.2), we get

$$\lim_{k\to\infty} p(\xi_{n(k+1)}, \xi_{n(k)}) = 1. \tag{2.4}$$

Again, using (2.3), (2.4) and $(PM4)$, we have

$$\lim_{k\to\infty} p(\xi_{n(k+1)-1}, \xi_{n(k)-1}) \leq \lim_{k\to\infty} \left[p(\xi_{n(k+1)-1}, \xi_{n(k+1)}) + p(\xi_{n(k+1)}, \xi_{n(k)}) + p(\xi_{n(k)}, \xi_{n(k)-1})\right]$$

$$\leq 1, \tag{2.5}$$

and

$$\begin{aligned} 1 &= \lim_{k\to\infty} p(\xi_{n(k+1)}, \xi_{n(k)}) \leq \lim_{k\to\infty} \left[p(\xi_{n(k+1)}, \xi_{n(k+1)-1}) + p(\xi_{n(k+1)-1}, \xi_{n(k)-1}) + p(\xi_{n(k)-1}, \xi_{n(k)}) \right] \\ &\leq \lim_{k\to\infty} p(\xi_{n(k+1)-1}, \xi_{n(k)-1}). \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we have

$$\lim_{k\to\infty} p(\xi_{n(k+1)-1}, \xi_{n(k)-1}) = 1. \tag{2.7}$$

It follows from (2.7) and (θ_2) that

$$\lim_{k\to\infty} \theta(p(\xi_{n(k+1)-1}, \xi_{n(k)-1})) > 1. \tag{2.8}$$

Having in mind from (2.3) that $p(\xi_{n(k+1)-1}, \xi_{n(k)}) > 0$ and $p(\xi_{n(k+1)}, \xi_{n(k)}) > 0$, so by using condition (2.1), we have

$$\begin{aligned} 1 &\leq \xi(\theta(p(T\xi_{n(k+1)-1}, T\xi_{n(k)-1})), \theta(p(\xi_{n(k+1)-1}, \xi_{n(k)-1}))) \\ &= \xi(\theta(p(\xi_{n(k+1)}, \xi_{n(k)})), \theta(p(\xi_{n(k+1)-1}, \xi_{n(k)-1}))) \\ &< \frac{\theta(p(\xi_{n(k+1)-1}, \xi_{n(k)-1}))}{\theta(p(\xi_{n(k+1)}, \xi_{n(k)}))}. \end{aligned}$$

This implies that

$$\theta(p(\xi_{n(k+1)}, \xi_{n(k)})) < \theta(p(\xi_{n(k+1)-1}, \xi_{n(k)-1})). \tag{2.9}$$

From (2.8), (2.9) and (ξ_3), we have

$$1 \leq \limsup_{k\to\infty} \xi(\theta(p(\xi_{n(k+1)}, \xi_{n(k)})), \theta(p(\xi_{n(k+1)-1}, \xi_{n(k)-1}))) < 1,$$

which is contradiction. Thus, $\{\xi_n\}$ is bounded.

Step 3: Here, we shall show that $\{\xi_n\}$ is a Cauchy sequence in (X, p). It suffices to prove that $\{\xi_n\}$ is Cauchy in the metric space (X, d_p). Consider,

$$H_n = \sup\{d_p(\xi_i, \xi_j) : i \geq j \geq n\}.$$

It is clear that

$$0 \leq H_{n+1} \leq H_n \leq \ldots < H_1.$$

Hence, there exists $R \geq 0$ such that

$$\lim_{n\to\infty} H_n = R. \tag{2.10}$$

Assume that $R \neq 0$. For each positive integer k, there exist $n(k) \geq m(k) \geq k$ such that

$$H_k - \frac{1}{k} < d_p(\xi_{n(k)}, \xi_{m(k)}) \leq H_k.$$

Taking $k \to \infty$ and using (2.10), we get

$$\lim_{k\to\infty} d_p(\xi_{n(k)}, \xi_{m(k)}) = \lim_{k\to\infty} H_k = R > 0. \tag{2.11}$$

In view of (2.2), we have

$$\lim_{n\to\infty} p(\xi_n, \xi_n) = 0. \tag{2.12}$$

By definition of d_p and using (2.12), we get

$$\lim_{k\to\infty} p(\xi_{n(k)}, \xi_{m(k)}) = 2 \lim_{k\to\infty} d_p(\xi_{n(k)}, \xi_{m(k)}) = 2R. \tag{2.13}$$

From condition (2.1),

$$\begin{aligned} 1 &\leq \xi(\theta(p(T\xi_{n(k)-1}, T\xi_{m(k)-1})), \theta(p(\xi_{n(k)-1}, \xi_{m(k)-1}))) \\ &= \xi(\theta(p(\xi_{n(k)}, \xi_{m(k)})), \theta(p(\xi_{n(k)-1}, \xi_{m(k)-1}))) \\ &< \frac{\theta(p(\xi_{n(k)-1}, \xi_{m(k)-1}))}{\theta(p(\xi_{n(k)}, \xi_{m(k)}))}. \end{aligned}$$

This implies that

$$\theta(p(\xi_{n(k)}, \xi_{m(k)})) < \theta(p(\xi_{n(k)-1}, \xi_{m(k)-1})).$$

Since θ is non-decreasing, we get that

$$p(\xi_{n(k)}, \xi_{m(k)}) < p(\xi_{n(k)-1}, \xi_{m(k)-1}). \tag{2.14}$$

By $(PM4)$, we have

$$\begin{aligned} p(\xi_{n(k)}, \xi_{m(k)}) &< p(\xi_{n(k)-1}, \xi_{m(k)-1}) \\ &\leq p(\xi_{n(k)-1}, \xi_{n(k)}) + p(\xi_{n(k)}, \xi_{m(k)}) + p(\xi_{m(k)}, \xi_{m(k)-1}). \end{aligned}$$

From (2.2) and (2.13), we get

$$\lim_{k\to\infty} p(\xi_{n(k)}, \xi_{m(k)}) = \lim_{k\to\infty} p(\xi_{n(k)-1}, \xi_{m(k)-1}) = 2R > 0. \tag{2.15}$$

Applying (2.15) in (ξ_3), we have

$$1 \leq \xi(\theta(p(\xi_{n(k)}, \xi_{m(k)})), \theta(p(\xi_{n(k)-1}, \xi_{m(k)-1}))) < 1,$$

which is a contradiction. This proves that $R = 0$. We deduce that

$$\lim_{n,m\to\infty} p(\xi_n, \xi_m) = 0. \tag{2.16}$$

Then $\{\xi_n\}$ is Cauchy in (X, p).

Step 4: Existence and uniqueness of a fixed point of T.

The sequence $\{\xi_n\}$ is Cauchy in the complete PMS (X, p), so there exists $u \in X$ such that

$$\lim_{n\to\infty} p(\xi_n, x) = p(u, u) = \lim_{n,m\to\infty} p(\xi_n, \xi_m).$$

By (2.16),

$$\lim_{n\to\infty} p(\xi_n, x) = p(u, u) = 0. \tag{2.17}$$

We shall prove that u is the unique fixed point of T. Without loss of generality, we may assume that for infinitely many n,

$$p(\xi_{n-1}, u) \neq 0 \quad \text{and} \quad p(\xi_n, Tu) \neq 0. \tag{2.18}$$

Let $x, y \in X$ such that $p(Tx, Ty) \neq 0$ and $p(x, y) \neq 0$. Applying (2.1), we have

$$1 \leq \xi\,(\theta(p(Tx, Ty)), \theta(p(x, y))) < \frac{\theta(p(x, y)}{\theta(p(Tx, Yy)}.$$

In this case, we have

$$p(Tx, Ty) < p(x, y). \tag{2.19}$$

By (2.18) and (2.19), $p(\xi_n, Tu) < p(\xi_{n-1}, u)$ for infinitely many n. Using (2.17) and Lemma 1.2, we have

$$0 \leq p(u, Tu) \leq \lim_{n\to\infty} p(\xi_{n-1}, u) = 0.$$

Thus, $p(Tu, u) = 0$, that is, $u = Tu$. Finally, we show the uniqueness of the fixed point. Let $\xi_1, \xi_2 \in X$ be two distinct fixed points of T. Then $\xi_1 \neq \xi_2$, so $p(T\xi_1, T\xi_2) = p(\xi_1, \xi_2) > 0$. From condition (2.1),

$$1 \leq \xi(\theta(p(T\xi_1, T\xi_2)), \theta(p(\xi_1, \xi_2))) < \frac{\theta(p(\xi_1, \xi_2))}{\theta(p(\xi_1, \xi_2))} = 1.$$

It is a contradiction, so $\xi_1 = \xi_2$. □

Since each metric space is a PMS, the following is the analog of Theorem 2.1 in the setting of metric spaces.

Corollary 2.1. *Let (X, d) be a complete metric metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\xi \in \mathcal{L}$ and $\theta \in \Theta$ such that for all $x, y \in X$ with $d(Tx, Ty) \neq 0$,*

$$\xi\,(\theta(d(Tx, Ty)), \theta(d(x, y))) \geq 1. \tag{2.20}$$

Then T has a unique fixed point.

Remark 2.1. *Corollary 2.1 is a proper generalization of Banach contraction mapping principle,by taking $\zeta(t, s) = \frac{t}{s^k}$ where $k \in (0, 1)$ and $\theta(t) = e^t$.*

The following corollary is a key result which guides us to derive several existing results.

Corollary 2.2. *Let (X, p) be a complete PMS and $T : X \to X$ be a given mapping such that for all $x, y \in X$ with $p(Tx, Ty) \neq 0$,*

$$p(Tx, Ty) \leq p(x, y) - \varphi(p(x, y)) \tag{2.21}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing and lower semi-continuous such that $\varphi^{-1}(\{0\}) = \{0\}$. Then T has a unique fixed point.

Proof. From condition (2.21), we have

$$e^{p(Tx,Ty)} \leq e^{p(x,y)-\varphi(p(x,y))}.$$

Putting $\theta(t) = e^t$, we get

$$\theta(p(Tx, Ty)) \leq \frac{\theta(p(x, y))}{e^{\varphi(p(x,y))}}.$$

Also, define $\varphi(t) = \ln(\psi(\theta(t)))$, where $\psi : [1, \infty) \to [1, \infty)$ is nondecreasing and lower semi-continuous such that $\psi^{-1}(\{1\}) = \{1\}$, we get

$$\theta(p(Tx, Ty)) \leq \frac{\theta(p(x, y))}{\psi(\theta(p(x, y)))}.$$

By putting $\xi(t, s) = \frac{s}{t\psi(s)}$, we get

$$1 \leq \frac{\theta(p(x, y))}{\theta(p(Tx, Ty))\psi(\theta(p(x, y)))} = \xi(\theta(p(Tx, Ty)), \theta(p(x, y))).$$

By Theorem 2.1, T has a unique fixed point.

□

Remark 2.2. *In Corollary 2.2, the condition $p(x, y) \neq 0$ is not required. Because, if $p(Tx, Ty) \neq 0$ in (2.21), necessarily we have $p(x, y) \neq 0$.*

To illustrate Theorem 2.1, we present the following examples.

Example 2.1. *Let $X = [0, 1] \cup [3, 4]$ and $p : X \times X \to [0, \infty)$ be defined by $p(x, y) = \max\{x, y\}$. It is clear that (X, p) is a complete PMS. Consider the mapping $T : X \to X$ defined by*

$$Tx = \begin{cases} \frac{x}{3}, & x \in [0, 1] \\ \frac{3}{2}, & x \in [3, 4]. \end{cases}$$

Choose $\theta(t) = e^t$ for all $t > 0$, and $\xi(t, s) = \frac{\sqrt{s}}{t}$ for all $t, s \geq 1$.

To prove that T is a $\mathcal{L}$-simulation with respect to ξ, let $x, y \in X$ be such that $p(x, y) \neq 0$ and $p(Tx, Ty) \neq 0$. Then the case $x = y = 0$ is excluded. Here, we have the following possible cases:
Case 1. $x, y \in [0, 1]$ and $(x, y) \neq (0, 0)$. We have $\theta(p(Tx, Ty)) = e^{\frac{1}{3}\max\{x,y\}}$ and $\theta(p(x, y)) = e^{\max\{x,y\}}$. Hence

$$\xi(\theta(p(Tx, Ty)), \theta(p(x, y))) = \frac{e^{\frac{1}{2}\max\{x,y\}}}{e^{\frac{1}{3}\max\{x,y\}}} > 1.$$

Case 2. $x, y \in [3, 4]$. Then $\theta(p(Tx, Ty)) = e^{\frac{3}{2}}$, $\theta(p(x, y)) = e^{\max\{x,y\}}$ and

$$\xi(\theta(p(Tx, Ty)), \theta(p(x, y))) = \frac{e^{\frac{1}{2}\max\{x,y\}}}{e^{\frac{3}{2}}} \geq 1.$$

Case 3. $y \in [3,4]$ *and* $x \in [0,1]$. *Here,* $\theta(p(Tx,Ty)) = e^{\frac{3}{2}}$ *and* $\theta(p(x,y)) = e^y$. *Then*

$$\xi(\theta(p(Tx,Ty)),\theta(p(x,y))) = \frac{e^{\frac{1}{2}y}}{e^{\frac{3}{2}}} \geq 1$$

for each $y \in [3,4]$.
Case 4. $x \in [3,4]$ *and* $y \in [0,1]$. *Then* $\theta(p(Tx,Ty)) = e^{\frac{3}{2}}$ *and* $\theta(p(x,y)) = e^x$. *Then*

$$\xi(\theta(p(Tx,Ty)),\theta(p(x,y))) = \frac{e^{\frac{1}{2}x}}{e^{\frac{3}{2}}} \geq 1,$$

for all $x \in [3,4]$. *Then* T *is a* $\mathcal{L}$*-simulation with respect to* ξ. *Hence all conditions of Theorem 2.1 are satisfied, and then* T *has a unique fixed point, which is 0.*

On the other hand, if we replace the partial metric p by the usual metric d and choose $x = 1$ *and* $y = 3$, *then* $\theta(d(Tx,Ty)) = e^{\frac{7}{6}}$ *and* $\theta(d(x,y)) = e^2$. *Hence*

$$\xi(\theta(d(Tx,Ty)),\theta(d(x,y))) = \frac{e^1}{e^{\frac{7}{6}}} < 1.$$

Thus, T *is not a* $\mathcal{L}$*-simulation with respect to* ξ *in the usual metric space.*

Example 2.2. *Let* $X = [0,\infty)$ *and* $p : X \times X \to [0,\infty)$ *be defined by* $p(x,y) = \max\{x,y\}$. *It is clear that* (X,p) *is a complete PMS. Consider the mapping* $T : X \to X$ *defined by*

$$Tx = \begin{cases} \frac{x}{4}, & x \leq 3 \\ \frac{1}{3}, & x > 3. \end{cases}$$

Choose $\theta(t) = e^t$ *for all* $t > 0$, *and* $\xi(t,s) = \frac{s^{\frac{1}{3}}}{t}$ *for all* $t,s \geq 1$.

To prove that T *is a* $\mathcal{L}$*-simulation with respect to* ξ, *let* $x,y \in X$ *be such that* $p(x,y) \neq 0$ *and* $p(Tx,Ty) \neq 0$. *The case* $x = y = 0$ *is excluded. Here, we have the following possible cases:*
Case 1. $x,y \in [0,3]$ *and* $(x,y) \neq (0,0)$. *We have* $\theta(p(Tx,Ty)) = e^{\frac{1}{4}\max\{x,y\}}$ *and* $\theta(p(x,y)) = e^{\max\{x,y\}}$. *Then*

$$\xi(\theta(p(Tx,Ty)),\theta(p(x,y))) = \frac{e^{\frac{1}{3}\max\{x,y\}}}{e^{\frac{1}{4}\max\{x,y\}}} > 1.$$

Case 2. $x,y \in (3,\infty)$. *Then* $\theta(p(Tx,Ty)) = e^{\frac{1}{3}}$ *and* $\theta(p(x,y)) = e^{\max\{x,y\}}$. *Then*

$$\xi(\theta(p(Tx,Ty)),\theta(p(x,y))) = \frac{e^{\frac{1}{3}\max\{x,y\}}}{e^{\frac{1}{3}}} \geq 1.$$

Case 3. $x \in [0,3]$ *and* $y \in (3,\infty)$. *Here,* $\theta(p(Tx,Ty)) = e^{\max\{\frac{x}{4},\frac{1}{3}\}}$ *and* $\theta(p(x,y)) = e^y$. *Now,*

$$\xi(\theta(p(Tx,Ty)),\theta(p(x,y))) = \frac{e^{\frac{y}{3}}}{e^{\max\{\frac{x}{4},\frac{1}{3}\}}}.$$

We have

$$\xi(\theta(p(Tx,Ty)),\theta(p(x,y))) = \begin{cases} e^{\frac{y-1}{3}} > 1, & \text{if } \max\{\frac{x}{4},\frac{1}{3}\} = \frac{1}{3} \\ e^{\frac{y}{3}-\frac{x}{4}} > 1, & \text{if } \max\{\frac{x}{4},\frac{1}{3}\} = \frac{x}{4}. \end{cases}$$

Case 4. $y \in [0,3]$ *and* $x \in (3,\infty)$. *This case follows from case 3 by replacing x and y as* $p(x,y) = p(y,x)$. *Then* T *is a* $\mathcal{L}$*-simulation with respect to* ξ. *Hence all conditions of Theorem 2.1 are satisfied, and then* T *has a unique fixed point, which is 0.*

To illustrate Corollary 2.2, we present the following example.

Example 2.3. *Consider the PMS* (X,p) *where* $X = [0,\infty)$ *and* $p(x,y) = \max\{x,y\}$. *Consider the mapping* $T : X \to X$ *defined by*

$$Tx = \begin{cases} \frac{x}{3}, & x \le 2 \\ 1, & x > 2. \end{cases}$$

Define $\varphi : [0,\infty) \to [0,\infty)$ *by*

$$\varphi(t) = \begin{cases} \frac{t}{5}, & t \le 2 \\ \frac{1}{2}, & t > 2. \end{cases}$$

Now for $x,y \in X$ *with* $p(Tx,Ty) \neq 0$, *we get* $(x,y) \neq (0,0)$. *We have the following 3 cases:*
Case 1. If $x,y \le 2$ *and* $(x,y) \neq (0,0)$, *then* $p(Tx,Ty) = \frac{1}{3}\max\{x,y\}$ *and* $\varphi(p(x,y)) = \frac{1}{5}\max\{x,y\}$. *Thus we have*

$$p(Tx,Ty) \le p(x,y) - \varphi(p(x,y)).$$

Case 2. If $x,y > 2$, *then* $p(Tx,Ty) = 1$ *and* $p(x,y) = \max\{x,y\} > 2$. *Also,* $\varphi(p(x,y)) = \frac{1}{2}$. *Hence*

$$p(Tx,Ty) \le p(x,y) - \varphi(p(x,y)).$$

Case 3. If $x \le 2$ *and* $y > 2$ *or* $y \le 2$ *and* $x > 2$. *Without loss of generality, let us assume* $x \le 2$ *and* $y > 2$. *Then* $p(Tx,Ty) = 1$ *and* $p(x,y) = y > 2$. *Also,* $\varphi(p(x,y)) = \frac{1}{2}$. *We obtain that*

$$p(Tx,Ty) \le p(x,y) - \varphi(p(x,y)).$$

Thus all conditions of Corollary 2.2 are satisfied, and then T *has a unique fixed point, which is 0.*

Open Problem

Romaguera [26] defined the concepts of 0-Cauchyness and 0-completeness of a partial metric space as follows:

Definition 2.1. *[26] Let* (X,p) *be a partial metric space and* $\{x_n\}$ *be any sequence in* X *and* $x \in X$. *Then:*
(i) *The sequence* $\{x_n\}$ *is called* 0*-Cauchy if*

$$\lim_{n,m\to\infty} p(x_n,x_m) = 0.$$

(ii) (X,p) *is called* 0*-complete if for every* 0*-Cauchy sequence* $\{x_n\}$ *in* X, *there exists* $x \in X$ *such that*

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x) = 0.$$

It is straightforward that if the partial metric type (X, p) is complete, then it is 0-complete. Then since 0-completeness is more general than completeness, it would be better to prove or disapprove Theorem 2.1 in the class of 0-complete partial metric spaces.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. T. Abdeljawad, H. Aydi, E. Karapinar, *Coupled fixed points for Meir-Keeler contractions in ordered partial metric spaces*, Math. Probl. Eng., **2012** (2012).
2. M. Abbas, B. Ali, C. Vetro, *A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces*, Topol. Appl., **160** (2013), 553–563.
3. P. Agarwal, M. A. Alghamdi, N. Shahzad, *Fixed point theory for cyclic generalized contractions in partial metric spaces*, Fixed Point Theory A., **2012** (2012), 40.
4. I. Altun, A. Erduran, *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory A., **2011** (2011), 508730.
5. E. Ameer, H. Aydi, M. Arshad, et al. *Hybrid multivalued type contraction mappings in α_K-complete partial b-metric spaces and applications*, Symmetry, **11** (2019), 86.
6. H. Aydi, M. A. Barakat, Z. D. Mitrović, et al. *A Suzuki type multi-valued contraction on weak partial metric spaces and application*, J. Inequal. Appl., **2018** (2018), 270.
7. H. Aydi, E. Karapinar, W. Shatanawi, *Coupled fixed point results for (ψ, φ)-weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl., **62** (2011), 4449–4460.
8. H. Aydi, E. Karapinar, *New Meir-Keeler type tripled fixed point theorems on ordered partial metric spaces,* Math. Probl. Eng., **2012** (2012), 1–17.
9. H. Aydi, M. Abbas, C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topol. Appl., **159** (2012), 3234–3242.
10. B. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1992), 133–181.
11. L. j. Ćirić, B. Samet, H. Aydi, et al. *Common fixed points of generalized contractions on partial metric spaces and an application*, Appl. Math. Comput., **218** (2011), 2398–2406.
12. S. H. Cho, *Fixed point theorems for $\mathcal{L}$-contractions in generalized metric spaces*, Abstr. Appl. Anal., **2018** (2018), 1–6.
13. S. Gulyaz, E. Karapinar, *A coupled fixed point result in partially ordered partial metric spaces through implicit function*, Hacet. J. Math. Stat., **42** (2013), 347–357.

14. M. Jleli, B. Samet, *new generalization of the Banach contraction principle*, J. Inequal. Appl., **2014** (2014), 38.
15. E. Karapınar, R. P. Agarwal, H. Aydi, *Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces*, Mathematics, **6** (2018), 256.
16. E. Karapinar, W. Shatanawi, K. Tas, *Fixed point theorem on partial metric spaces involving rational expressions*, Miskolc Math. Notes, **14** (2013), 135–142.
17. E. Karapinar, S. Romaguera, *Nonunique fixed point theorems in partial metric spaces*, Filomat, **27** (2013), 1305–1314.
18. E. Karapinar, I. Erhan, A. Ozturk, *Fixed point theorems on quasi-partial metric spaces*, Math. Comput. Model., **57** (2013), 2442–2448.
19. W. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Springer International Publishing, Switzerland, 2014.
20. F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat, **29** (2015), 1189–1194.
21. S. G. Matthews, *Partial metric topology*, Ann. NY Acad. Sci., **728** (1994), 183–197.
22. S. J. O'Neill, *Partial metrics, valuations and domain theory,* Ann. NY Acad. Sci., **806** (1996), 304–315.
23. L. Pasicki, *Dislocated quasi-metric and generalized contractions*, Fixed Point Theory, **19** (2018), 359–368.
24. S. Radenović, *Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension*, The Allahabad Mathematical Society, **31** (2016), 39–55.
25. S. Radenović, *Coincidence point results for generalized weakly* (ψ,φ)*-contractive mappings in ordered partial metric spaces*, J. Indian Math. Soc., **3** (2014), 319–333.
26. S. Romaguera, *Fixed point theorems for generalized contraction on partial metric spaces*, Topol. Appl., **159** (2012), 194–199.
27. W. Shatanawi, M. Postolache, *Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces*, Fixed Point Theory A., **2013** (2013), 54.
28. W. Shatanawi, S. Manro, *Fixed point results for cyclic* (ψ,ϕ,A,B)*-contraction in partial metric spaces*, Fixed Point Theory A., **2012** (2012), 165.
29. W. Shatanawi, H. K. Nashine, N. Tahat, *Generalization of some coupled fixed point results on partial metric spaces*, International Journal of Mathematics and Mathematical Sciences, **2012** (2012), 1–10.
30. W. Shatanawi, B. Samet, M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Model., **55** (2012), 680–687.
31. S. Shukla, S. Radenović, *Some common fixed point theorems for* F*-contraction type mappings in 0-complete partial metric spaces*, Journal of Mathematics, **2013** (2013).

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