AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(3): 663–683.
DOI:10.3934/math.2019.3.663
Received: 25 March 2019
Accepted: 20 May 2019
Published: 17 June 2019

Research article

Solvability and optimal controls of non-instantaneous impulsive stochastic neutral integro-differential equation driven by fractional Brownian motion

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Abstract: In this manuscript, a new class of non-instantaneous impulsive stochastic neutral integro-differential equation driven by fractional Brownian motion (fBm, in short) with state-dependent delay and their stochastic optimal control problem is studied. We utilize the theory of the resolvent operator and a fixed point technique to present the solvability of the stochastic system. Then, the existence of optimal controls is discussed for the proposed stochastic system. Finally, an example is offered to demonstrate the obtained theoretical results.

Keywords: stochastic neutral integro-differential equation; optimal controls; state-dependent delay; fractional Brownian motion; non-instantaneous impulses
Mathematics Subject Classification: 34K45, 60G22, 65C30, 93E20

1. Introduction

Stochastic differential equations have been used with great success in many application areas including biology, epidemiology, mechanics, economics and finance. For the fundamental study of the theory of stochastic differential equations, we refer to [1–4]. Yang and Zhu [5] studied the existence, uniqueness, and stability of mild solutions for the stochastic differential equations with Poisson jumps by using fixed point techniques. The fBm with Hurst parameter $\mathcal{H} \in (0, 1)$ is a self-similar centered Gaussian random process with stationary increments. It admits the long-range dependence properties when $\mathcal{H} > 1/2$. Many exciting applications of fBm have been established in diverse fields such as finance, economics, telecommunications, and hydrology. For more details on fBm, see [6–9] and the references cited therein. Boudaoui et al. [10] studied the existence and continuous dependence of the mild solutions for the impulsive stochastic differential equation driven by fBm.

In recent years, the differential equation with fixed moments of impulses (instantaneous impulses) has become the natural framework for modeling of many evolving processes and phenomena studied in economics, population dynamics, and physics. For more details on differential equations with instantaneous impulses, one can see the papers [11–14] and the references cited therein. Deng et al. [15] discussed the existence and exponential stability for impulsive neutral stochastic functional differential equations driven by fBm with non-compact semigroup. Zhu [16] obtained some sufficient conditions to ensure the pth moment exponential stability of impulsive stochastic functional differential equations with Markovian switching. The action of instantaneous impulses does not describe certain dynamics of evolution processes in pharmacotherapy. For example, consider the following simplified situation concerning the hemodynamic equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs into the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action that starts abruptly and remains active over a finite time interval. For these reasons, Hernández and O'Regan [17] introduced a new class of abstract differential equations with non-instantaneous impulses and they investigated the existence of mild and classical solutions. For comprehensive details on differential equation with non-instantaneous impulses, see [18–20]. The qualitative properties of mild solutions for differential equations with non-instantaneous impulses have been investigated in several papers [21–23] and the references cited therein.

On the other hand, delay differential equation has been gaining much interest and attracting the attention of several researchers, because of its wide applications in various fields of science and engineering such as control theory, heat flow, mechanics, distributed networks, and neural networks, etc. The delay depends on the state variable, is called state-dependent delay. For more details on state-dependent delay, we refer to [24–28]. In the neutral differential equation, the highest order derivative of the state variable appears without delay and with delay. Ezzinbi et al. [29] discussed the existence and regularity of solutions for the neutral functional integro-differential equation with delay. Vijayakumar [30] investigated the approximate controllability for integro-differential inclusions by using the resolvent operators. The optimal control problem plays an important role in many scientific fields, such as engineering, mathematics, and biomedical. When the stochastic differential equation describes the performance index and system dynamics, an optimal control problem reduces to a stochastic optimal control problem. Wei et al. [31] obtained the existence of optimal controls for the impulsive integro-differential equation of mixed type. Jiang et al. [32] discussed the existence of optimal controls for fractional evolution inclusion with Clarke subdifferential and nonlocal conditions. In particular, in [33, 34], the authors analyzed the existence of optimal controls for the fractional differential equations, whereas in [35, 36] the authors investigated the same type of problem for the impulsive fractional stochastic integro-differential equations with delay.

To the best of our knowledge, there is no paper discussing the solvability and optimal controls of a non-instantaneous impulsive stochastic system driven by fBm with state-dependent delay. In order to fill this gap, we consider the following non-instantaneous impulsive stochastic neutral integro-differential equation driven by fBm with state-dependent delay:

$$\begin{cases} d\mathfrak{D}(t,z_t) = \mathcal{A}\Big[\mathfrak{D}(t,z_t) + \int_0^t \mathcal{G}(t-s)\mathfrak{D}(s,z_s)ds\Big]dt + C(t)v(t)dt + \mathcal{F}_2(t, z_{\rho(t,z_t)})dB^{\mathcal{H}}(t) \\ \qquad\qquad\qquad t \in (p_j, t_{j+1}],\ j = 0,1,\dots,\mathcal{M}, \\ z(t) = \mathcal{E}_j(t,z_t),\quad t \in (t_j, p_j],\ j = 1,2,\dots,\mathcal{M}, \\ z_0 = \Omega \in \mathfrak{B}, \end{cases} \tag{1.1}$$

where $z(\cdot)$ takes values in a real separable Hilbert space Z, $\mathcal{A}$ is the generator of a C_0-semigroup of operators $\{\mathfrak{R}(t) : t \geq 0\}$ on Z. $B^{\mathcal{H}} = \{B^{\mathcal{H}}(t) : t \geq 0\}$ is a fBm with Hurst index $\mathcal{H} \in (1/2, 1)$, takes values in a Hilbert space Y. The initial data $\Omega = \{\Omega(t),\ t \in (-\infty, 0]\}$ is a $\mathfrak{B}$-valued, $\mathcal{F}_0$-adapted random variable, which not dependent on $B^{\mathcal{H}}$, where $\mathfrak{B}$ abstract phase space. The history valued function $z_t : (-\infty, 0] \to Z$ is defined as $z_t(\theta) = z(t+\theta)$ for all $\theta \in (-\infty, 0]$ belongs to $\mathfrak{B}$. The control function v takes value from a separable reflexive Hilbert space $\mathcal{T}$, and C is linear operator from $\mathcal{T}$ into Z. $0 = t_0 = p_0 < t_1 < p_1 < \cdots < t_{\mathcal{M}} < p_{\mathcal{M}} < t_{\mathcal{M}+1} = b < \infty$ are prefixed numbers, $\mathcal{J}_1 = [0, b]$. Suppose that $\mathcal{G}(t)$, $t \in \mathcal{J}_1$ is a linear and bounded operator. The function $\mathfrak{D} : \mathcal{J}_1 \times \mathfrak{B} \to Z$ is defined by $\mathfrak{D}(t, \psi) = \psi(0) - \mathcal{F}_1(t, \psi)$, $\psi \in \mathfrak{B}$ and $\mathcal{F}_1 : \mathcal{J}_1 \times \mathfrak{B} \to Z, \mathcal{F}_2 : \mathcal{J}_1 \times \mathfrak{B} \to L_2^0(Y, Z)$, where $L_2^0(Y, Z)$ is space of all Q-Hilbert-Schmidt operators from Y into Z, $\mathcal{E}_j : (t_j, p_j] \times \mathfrak{B} \to Z$, $j = 1, 2, \ldots, \mathcal{M}$ and $\rho : \mathcal{J}_1 \times \mathfrak{B} \to (-\infty, b]$ are suitable functions and they will be specified later.

The manuscript is structured as follows. Section 2 introduces preliminary facts and some notations. In Section 3, we discussed the solvability of the stochastic system and Section 4 is devoted to the investigation of the existence of optimal control pairs of the Lagrange problem corresponding to the proposed stochastic system. In Section 5, an example is provided to illustrate the applications of the obtained results. The last section is devoted to our conclusions.

2. Preliminaries

In this section, we briefly review some basic definitions and notations that will be used in the subsequent sections. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space, where $\mathcal{F}_t$ the σ-algebra is generated by $\{B^{\mathcal{H}}(s),\ s \in [0, t]\}$. By $L(Y, Z)$, we denote the space of bounded linear operator from Y into Z. For convenience, the same notation $\|.\|$ is used to denote the norms in Z, Y, $L(Y, Z)$. The collection of all square integrable, strongly measurable, Z-valued random variables, denoted by $L^2(\Omega, Z)$, which is a Banach space. $L^2_{\mathcal{F}_0}(\Omega, Z) = \{f \in L^2(\Omega, Z) : f \text{ is } \mathcal{F}_0 - \text{measurable}\}$ is subspace of $L^2(\Omega, Z)$. We denote by $\mathcal{PC}([r_1, r_2], Z)$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable process from $[r_1, r_2]$ into Z.

Definition 2.1. *Given $\mathcal{H} \in (0, 1)$, a centered Gaussian and continuous random process $\beta^{\mathcal{H}} = \{\beta^{\mathcal{H}}(t),\ t \geq 0\}$ with covariance function*

$$\mathbb{E}[\beta^{\mathcal{H}}(\varrho_1), \beta^{\mathcal{H}}(\varrho_2)] = \frac{1}{2}(\varrho_1^{2\mathcal{H}} + \varrho_2^{2\mathcal{H}} - |\varrho_1 - \varrho_2|^{2\mathcal{H}}),$$

is called one dimensional fBm and $\mathcal{H}$ is the Hurst parameter.

The fBm $\beta^{\mathcal{H}}(t)$ with $1/2 < \mathcal{H} < 1$ has the following integral representation

$$\beta^{\mathcal{H}}(t) = \int_0^t \mathfrak{K}_{\mathcal{H}}(t, \varrho) dw(\varrho),$$

where $w(\varrho)$ is a Wiener process or Brownian motion and the kernel $\mathfrak{K}_{\mathcal{H}}(t, \varrho)$ is defined as

$$\mathfrak{K}_{\mathcal{H}}(t, \varrho) = \mathfrak{P}_{\mathcal{H}} \varrho^{1/2-\mathcal{H}} \int_\varrho^t (\tau - \varrho)^{\mathcal{H}-3/2} \tau^{\mathcal{H}-1/2} d\tau, \text{ for } t > \varrho.$$

We put $\mathfrak{K}_{\mathcal{H}}(t, \varrho) = 0$ if $t \leq \varrho$. Notice that $\dfrac{\partial \mathfrak{K}_{\mathcal{H}}}{\partial t}(t, \varrho) = \mathfrak{P}_{\mathcal{H}}(t/\varrho)^{\mathcal{H}-1/2}(t - \varrho)^{\mathcal{H}-3/2}$. Here, $\mathfrak{P}_{\mathcal{H}} = [\mathcal{H}(2\mathcal{H} - 1)/\xi(2 - 2\mathcal{H}, \mathcal{H} - 1/2)]^{1/2}$ and $\xi(\cdot, \cdot)$ is Beta function. For $\Psi \in L^2([0, b])$, it is well known from [37] that the Wiener-type integral of the function Ψ w.r.t fBm $\beta^{\mathcal{H}}$ is defined by

$$\int_0^b \Psi(\varrho)d\beta^{\mathcal{H}}(\varrho) = \int_0^b \mathfrak{R}^*_{\mathcal{H}}\Psi(\varrho)dw(\varrho),$$

where $\mathfrak{R}^*_{\mathcal{H}}\Psi(\varrho) = \int_\varrho^b \Psi(t)\frac{\partial \mathfrak{R}_{\mathcal{H}}}{\partial t}(t,\varrho)dt$.

Let the operator $Q \in L(Y, Y)$ is defined by $Qe_i = \lambda_i e_i$, where $\{\lambda_i \geq 0 : i = 1, 2, \dots, \}$ are real numbers with trace $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i < \infty$ and $\{e_i,\ i = 1, 2, \dots, \}$ is a complete orthonormal basis in Y. Next, we define the infinite dimensional fBm $B^{\mathcal{H}}$ on Y with covariance Q as

$$B^{\mathcal{H}}(t) = B^{\mathcal{H}}_Q(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \beta_i^{\mathcal{H}}(t),$$

where $\beta_i^{\mathcal{H}}(t)$ are real, independent fBm. Now, we define the separable Hilbert space $L_2^0(Y, Z)$ of all Q-Hilbert-Schmidt operators from Y into Z with norm $\|\psi\|^2_{L_2^0} = \sum_{i=1}^{\infty} \| \sqrt{\lambda_i}\psi e_i\|^2 < \infty$ and the inner product $\langle \psi_1, \psi_2\rangle_{L_2^0} = \sum_{i=1}^{\infty}\langle \psi_1 e_i, \psi_2 e_i\rangle$. The Wiener integral of function $\Upsilon : \mathcal{J}_1 \to L_2^0(Y, Z)$ w.r.t fBm $B^{\mathcal{H}}$ is defined by

$$\int_0^t \Upsilon(s)dB^{\mathcal{H}}(s) = \sum_{i=1}^{\infty}\int_0^t \sqrt{\lambda_i}\Upsilon(s)e_i d\beta_i^{\mathcal{H}}(s) = \sum_{i=1}^{\infty}\int_0^t \sqrt{\lambda_i}\mathfrak{R}^*_{\mathcal{H}}(\Upsilon e_i)(s)dw_i(s). \tag{2.1}$$

Lemma 2.1. *[6] If $\Upsilon : \mathcal{J}_1 \to L_2^0(Y, Z)$ satisfies $\int_0^b \|\Upsilon(s)\|^2_{L_2^0}ds < \infty$, then equation* (2.1) *is well-defined and Z-valued random variable and we get*

$$\mathbb{E}\Big\|\int_0^t \Upsilon(s)dB^{\mathcal{H}}(s)\Big\|^2 \leq 2\mathcal{H}t^{2\mathcal{H}-1}\int_0^t \|\Upsilon(s)\|^2_{L_2^0}ds. \tag{2.2}$$

Now, we introduce the space $\mathcal{PC}(Z)$ formed by all $\mathcal{F}_t$-adapted measurable, Z-valued stochastic processes $\{z(t) : t \in \mathcal{J}_1\}$ such that z is continuous at $t \neq t_j$, $z(t_j^-) = z(t_j)$ and $z(t_j^+)$ exists for all $j = 1, 2, \dots, \mathcal{M}$, endowed with the norm $\| z \|_{\mathcal{PC}} = \left(\sup_{t\in\mathcal{J}_1}\mathbb{E}\|z(t)\|^2\right)^{1/2}$. Then $(\mathcal{PC}(Z), \|\cdot\|_{\mathcal{PC}})$ is Banach space.

In the following, let $\mathcal{T}$ is a separable reflexive Hilbert space from which the controls v take the values. Operator $C \in L^\infty(\mathcal{J}_1, L(\mathcal{T}, Z))$, where $L^\infty(\mathcal{J}_1, L(\mathcal{T}, Z))$ denote the space of operator-valued functions which are measurable in the strong operator topology and uniformly bounded on the interval $\mathcal{J}_1$, endowed with the norm $\|\cdot\|_\infty$. Let $L^2_{\mathcal{F}}(\mathcal{J}_1, \mathcal{T})$ denote the space of all measurable and $\mathcal{F}_t$-adapted, $\mathcal{T}$-valued stochastic processes satisfying the condition $\mathbb{E}\int_0^b \|v(t)\|^2_{\mathcal{T}}dt < \infty$, and endowed with the norm $\| v \|_{L^2_{\mathcal{F}}} = (\mathbb{E}\int_0^b \|v(t)\|^2_{\mathcal{T}}dt)^{1/2}$. Let $\mathcal{U}$ be a non-empty closed bounded convex subset of $\mathcal{T}$. We define the admissible control set

$$\mathcal{U}_{ad} = \{v \in L^2_{\mathcal{F}}(\mathcal{J}_1, \mathcal{T}) \mid v(t) \in \mathcal{U}\ a.e.\ t \in \mathcal{J}_1\}.$$

Then, $Cv \in L^2(\mathcal{J}_1, Z)$ for all $v \in \mathcal{U}_{ad}$.

In this paper, we assume that the phase space $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into Z and subsequent conditions are satisfied.

[A1]: If $z : (-\infty, e + b] \to Z,\ b > 0$ is such that $z|_{[e,e+b]} \in \mathcal{PC}([e, e + b], Z)$ and $z_e \in \mathfrak{B}$, then for each $t \in [e, e + b]$ the subsequent conditions are satisfied:

1. $z_t \in \mathfrak{B}$.
2. $\|z(t)\| \leq \mathcal{K}_1 \|z_t\|_{\mathfrak{B}}$.
3. $\|z_t\|_{\mathfrak{B}} \leq \mathcal{K}_2(t-e) \sup\{\|z(s)\| : e \leq s \leq t\} + \mathcal{K}_3(t-e)\|z_e\|_{\mathfrak{B}}$, where $\mathcal{K}_1$ is a positive constant, $\mathcal{K}_2, \mathcal{K}_3 : [0,+\infty) \to [1,+\infty)$, $\mathcal{K}_2$ is a continuous function, $\mathcal{K}_3$ is a locally bounded function and $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are independent of $z(\cdot)$.

[A2]: For the function $z(\cdot)$ in [A1], the function $t \to z_t$ is continuous from $[e, e+b]$ into $\mathfrak{B}$.
[A3]: The phase space $\mathfrak{B}$ is complete.
For more details on phase space, we refer to [38, 39].

Lemma 2.2. *[21] Let $z : (-\infty, b] \to Z$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $z_0 = \Omega(t) \in L^2_{\mathcal{F}_0}(\Omega, \mathfrak{B})$ and $z|_{\mathcal{J}_1} \in \mathcal{PC}(Z)$, then*

$$\|z_s\|_{\mathfrak{B}} \leq \mathcal{K}_3^* \, \mathbb{E}\|\Omega\|_{\mathfrak{B}} + \mathcal{K}_2^* \sup_{s \in \mathcal{J}_1} \mathbb{E}\|z(s)\|,$$

where $\mathcal{K}_2^* = \sup_{t \in \mathcal{J}_1} \mathcal{K}_2(t)$, $\mathcal{K}_3^* = \sup_{t \in \mathcal{J}_1} \mathcal{K}_3(t)$.

Definition 2.2. *A one parameter family $\{\mathfrak{R}(t) : t \geq 0\}$ of bounded linear operators, is called resolvent operator for*

$$\frac{dz}{dt} = \mathcal{A}\Big[z(t) + \int_0^t \mathcal{G}(t-\kappa) z(\kappa) d\kappa\Big], \tag{2.3}$$

if

1. *$\mathfrak{R}(0) = I$ and $\|\mathfrak{R}(t)\| \leq N e^{\beta t}$ for some constants β and $N \geq 1$.*
2. *For all $z \in Z$, $\mathfrak{R}(t)z$ is strongly continuously for $t \in \mathcal{J}_1$.*
3. *For all $t \in \mathcal{J}_1$, $\mathfrak{R}(t) \in L(X)$. For all $x \in X$, $\mathfrak{R}(\cdot)x \in C^1(\mathcal{J}_1, Z) \cap C(\mathcal{J}_1, X)$ and*

$$\begin{aligned}
\frac{d}{dt}\mathfrak{R}(t)x &= \mathcal{A}\Big[\mathfrak{R}(t)x + \int_0^t \mathcal{G}(t-\kappa)\mathfrak{R}(\kappa) x d\kappa\Big] \\
&= \mathfrak{R}(t)\mathcal{A}x + \int_0^t \mathfrak{R}(t-\kappa)\mathcal{A}\mathcal{G}(\kappa) x d\kappa, \quad t \in \mathcal{J}_1.
\end{aligned}$$

For more details on the resolvent operator, we refer to [40, 41].

Definition 2.3. *A $Z-$valued stochastic process $\{z(t), t \in (-\infty, b]\}$ is called a mild solution of the stochastic system* (1.1) *if $z_0 = \Omega$, $z_{\rho(s,z_s)} \in \mathfrak{B}$, $z|_{[0,b]} \in \mathcal{PC}(Z)$ and*

1. *$z(t)$ is measurable and adapted to $\mathcal{F}_t$, $t \geq 0$.*
2. *$z(t) \in Z$ has càdlàg paths on $[0, b]$ almost everywhere and for every $t \in [0, b]$, $z(t)$ satisfies $z(t)= \mathcal{E}_j(t, z_t)$ for all $t \in (t_j, p_j]$, $j = 1, 2, \ldots, \mathcal{M}$, and*

$$\begin{aligned}
z(t) &= \mathfrak{R}(t)[\Omega(0) - \mathcal{F}_1(0, \Omega)] + \mathcal{F}_1(t, z_t) \\
&\quad + \int_0^t \mathfrak{R}(t-s)\mathcal{C}(s)v(s)ds + \int_0^t \mathfrak{R}(t-s)\mathcal{F}_2(s, z_{\rho(s,z_s)})dB^{\mathcal{H}}(s)
\end{aligned}$$

for all $t \in [0, t_1]$ and

$$\begin{aligned}
z(t) &= \mathfrak{R}(t-p_j)[\mathcal{E}_j(p_j, z_{p_j}) - \mathcal{F}_1(p_j, z_{p_j})] + \mathcal{F}_1(t, z_t) \\
&\quad + \int_{p_j}^t \mathfrak{R}(t-s)\mathcal{C}(s)v(s)ds + \int_{p_j}^t \mathfrak{R}(t-s)\mathcal{F}_2(s, z_{\rho(s,z_s)})dB^{\mathcal{H}}(s)
\end{aligned}$$

for all $t \in (p_j, t_{j+1}]$, $j = 1, 2, \ldots, \mathcal{M}$.

3. Solvability for stochastic system

In this section, we prove the existence of mild solutions for the stochastic system (1.1). Let $\rho : \mathcal{J}_1 \times \mathfrak{B} \to (-\infty, b]$ be a continuous function. To prove our main results, we need the following hypotheses:

[H1]: $\mathfrak{R}(t),\ t > 0$ is compact and there exists a constant $N > 0$ such that $\|\mathfrak{R}(t)\| \le N$ for every $t \in \mathcal{J}_1$.

[H2]: The function $t \to \Omega_t$ is continuous from the set $\mathcal{S}(\rho^-) = \{\rho(t,\psi) \le 0 : (t,\psi) \in \mathcal{J}_1 \times \mathfrak{B}\}$ into $\mathfrak{B}$ and there exists a bounded and continuous function $\mathcal{L}_\Omega : \mathcal{S}(\rho^-) \to (0,\infty)$ to ensure that $\|\Omega_t\|_{\mathfrak{B}} \le \mathcal{L}_\Omega(t)\|\Omega\|_{\mathfrak{B}}$ for all $t \in \mathcal{S}(\rho^-)$.

[H3]: There exists a constant $L_{\mathcal{F}_1} > 0$ such that the function $\mathcal{F}_1 : \mathcal{J}_1 \times \mathfrak{B} \to Z$ satisfies the following conditions

$$\mathbb{E}\|\mathcal{F}_1(t,\psi)\|^2 \le L_{\mathcal{F}_1}(\|\psi\|^2_{\mathfrak{B}} + 1),\ \forall\, \psi \in \mathfrak{B},\ t \in \mathcal{J}_1,$$
$$\mathbb{E}\|\mathcal{F}_1(t,\psi_1) - \mathcal{F}_1(t,\psi_2)\|^2 \le L_{\mathcal{F}_1}\ \|\psi_1 - \psi_2\|^2_{\mathfrak{B}},\ \forall\, \psi_1,\psi_2 \in \mathfrak{B},\ t \in \mathcal{J}_1.$$

[H4]: There exist constants $L_{\mathcal{E}_j} > 0,\ j = 1,2,\dots,\mathcal{M}$, such that the functions $\mathcal{E}_j : (t_j, p_j] \times \mathfrak{B} \to Z$, $j = 1,2,\dots,\mathcal{M}$, satisfies the following conditions

$$\mathbb{E}\|\mathcal{E}_j(t,\psi)\|^2 \le L_{\mathcal{E}_j}(\|\psi\|^2_{\mathfrak{B}} + 1),\ \forall\, \psi \in \mathfrak{B},$$
$$\mathbb{E}\|\mathcal{E}_j(t,\psi_1) - \mathcal{E}_j(t,\psi_2)\|^2 \le L_{\mathcal{E}_j}\|\psi_1 - \psi_2\|^2_{\mathfrak{B}},\ \forall\, \psi_1,\psi_2 \in \mathfrak{B}.$$

[H5]: The function $\mathcal{F}_2 : \mathcal{J}_1 \times \mathfrak{B} \to L^0_2(Y,Z)$ satisfies the conditions

(a) The function $\mathcal{F}_2(t,\cdot) : \mathfrak{B} \to L^0_2(Y,Z)$ is continuous for a.e $t \in \mathcal{J}_1$, and $t \to \mathcal{F}_2(t,\psi)$ is measurable for all $\psi \in \mathfrak{B}$.

(b) There exists a continuous function $\eta : \mathcal{J}_1 \to [0,\infty)$ and a continuous nondecreasing function $\Theta : [0,\infty) \to (0,\infty)$ to ensure that for all $(t,\psi) \in \mathcal{J}_1 \times \mathfrak{B}$

$$\mathbb{E}\|\mathcal{F}_2(t,\psi)\|^2_{L^0_2} \le \eta(t)\Theta_{\mathcal{F}_2}(\|\psi\|^2_{\mathfrak{B}}), \quad \liminf_{w\to\infty} \frac{\Theta_{\mathcal{F}_2}(w)}{w} = \Theta_1.$$

[H6]: The following inequality holds

$$\max_{1\le j\le \mathcal{M}} 2[\mathcal{K}^*_2]^2[L_{\mathcal{E}_j} + 8N^2(L_{\mathcal{E}_j} + L_{\mathcal{F}_1}) + 4L_{\mathcal{F}_1} + 8\mathcal{H}N^2 b^{2\mathcal{H}-1}\Theta_1 \int_0^b \eta(s)ds] < 1.$$

Lemma 3.1. *[28] Let $z : (-\infty, b] \to Z$ such that $z_0 = \Omega$ and $z|_{\mathcal{J}_1} \in \mathcal{PC}(Z)$. If [H2] be hold, then*

$$\|z_t\|_{\mathfrak{B}} \le (\mathcal{K}^*_3 + \mathcal{L}^*_\Omega)\|\,\Omega\,\|_{\mathfrak{B}} + \mathcal{K}^*_2\ \sup\{\mathbb{E}\|z(\omega)\| : \omega \in [0, \max\{0,t\}]\},\ t \in \mathcal{S}(\rho^-) \cup \mathcal{J}_1,$$

where $\mathcal{K}^*_2 = \sup_{t\in\mathcal{J}_1} \mathcal{K}_2(t)$, $\mathcal{K}^*_3 = \sup_{t\in\mathcal{J}_1} \mathcal{K}_3(t)$, and $\mathcal{L}^*_\Omega = \sup_{t\in\mathcal{S}(\rho^-)} \mathcal{L}_\Omega(t)$.

Theorem 3.1. *If the hypotheses [H1]–[H6] are fulfilled. Then for each $v \in \mathcal{U}_{ad}$, the stochastic system* (1.1) *has at least one mild solution on $\mathcal{J}_1$, provided that*

$$\max_{1\le j\le \mathcal{M}} 2[\mathcal{K}^*_2]^2(L_{\mathcal{E}_j} + 4N^2 L_{\mathcal{E}_j} + 2(2N^2+1)L_{\mathcal{F}_1}) < 1. \tag{3.1}$$

Proof. On the space $\mathcal{BPC} = \{z \in \mathcal{PC}(Z) : z(0) = \Omega(0)\}$ endowed with the uniform convergence topology. For each $l > 0$, let

$$\overline{\mathcal{B}}_l = \{z \in \mathcal{BPC} : \|z\|^2_{\mathcal{PC}} \leq l\}.$$

Let the operator $\mathfrak{F} : \overline{\mathcal{B}}_l \to \mathcal{BPC}$ be specified by

$$(\mathfrak{F}z)(t) = \begin{cases} \Re(t)[\Omega(0) - \mathcal{F}_1(0,\Omega)] + \mathcal{F}_1(t,\bar{z}_t) \\ + \int_0^t \Re(t-s)C(s)v(s)ds + \int_0^t \Re(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s), & t \in [0,t_1],\ j = 0, \\ \mathcal{E}_j(t,\bar{z}_t), & t \in (t_j,p_j],\ j \geq 1, \\ \Re(t-p_j)[\mathcal{E}_j(p_j,\bar{z}_{p_j}) - \mathcal{F}_1(p_j,\bar{z}_{p_j})] + \mathcal{F}_1(t,\bar{z}_t) \\ + \int_{p_j}^t \Re(t-s)C(s)v(s)ds + \int_{p_j}^t \Re(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s), & t \in (p_j,t_{j+1}],\ j \geq 1, \end{cases}$$

where $\bar{z} : (-\infty, b] \to Z$ is such that $\bar{z}_0 = \Omega$ and $\bar{z} = z$ on $\mathcal{J}_1$. For $z \in \overline{\mathcal{B}}_l$, from Lemma 3.1, we have

$$\|\bar{z}_{\rho(s,\bar{z}_s)}\|^2_{\mathfrak{B}} \leq 2(\mathcal{K}_3^* + \mathcal{L}_\Omega^*)^2 \|\,\Omega\,\|^2_{\mathfrak{B}} + 2[\mathcal{K}_2^*]^2 l = l^*.$$

From [H1] and Hölder's inequality, we have

$$\begin{aligned} \mathbb{E}\left\| \int_{p_j}^t \Re(t-s)C(s)v(s)ds \right\|^2 &\leq \mathbb{E}\left[\int_{p_j}^t \|\Re(t-s)\|\|C(s)v(s)\|ds \right]^2 \\ &\leq N^2\|C\|^2_\infty (t_{j+1} - p_j)\mathbb{E}\int_{p_j}^t \|v(s)\|^2_{\mathcal{T}} ds \\ &\leq N^2\|C\|^2_\infty (t_{j+1} - p_j)\|v\|^2_{L^2_{\mathcal{F}}}. \end{aligned}$$

By Bochner theorem, it follows that $\Re(t-s)C(s)v(s)$ are integrable on (p_j, t), $j = 0, 1, \ldots, \mathcal{M}$. Therefore $\mathfrak{F}$ is well defined on $\overline{\mathcal{B}}_l$. Now, we split $\mathfrak{F}$ as $\mathfrak{F}_1 + \mathfrak{F}_2$, where

$$(\mathfrak{F}_1 z)(t) = \begin{cases} \Re(t)[\Omega(0) - \mathcal{F}_1(0,\Omega)] + \mathcal{F}_1(t,\bar{z}_t), & t \in [0,t_1],\ j = 0, \\ \mathcal{E}_j(t,\bar{z}_t), & t \in (t_j,p_j],\ j \geq 1, \\ \Re(t-p_j)[\mathcal{E}_j(p_j,\bar{z}_{p_j}) - \mathcal{F}_1(p_j,\bar{z}_{p_j})] + \mathcal{F}_1(t,\bar{z}_t), & t \in (p_j,t_{j+1}],\ j \geq 1, \end{cases}$$

and

$$(\mathfrak{F}_2 z)(t) = \begin{cases} \int_0^t \Re(t-s)C(s)v(s)ds + \int_0^t \Re(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s), & t \in [0,t_1],\ j = 0, \\ 0, & t \in (t_j,p_j],\ j \geq 1, \\ \int_{p_j}^t \Re(t-s)C(s)v(s)ds + \int_{p_j}^t \Re(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s), & t \in (p_j,t_{j+1}],\ j \geq 1. \end{cases}$$

For the sake of convenience, we break the proof into a sequence of steps.
Step 1. There exists $l > 0$ such that $\mathfrak{F}(\overline{\mathcal{B}}_l) \subset \overline{\mathcal{B}}_l$.
If we assume that this assertion is false, then for any $l > 0$, we can choose $z^l \in \overline{\mathcal{B}}_l$ and $t \in \mathcal{J}_1$ such that $\mathbb{E}\|\mathfrak{F}(z^l)(t)\|^2 > l$. By [H1], [H3]–[H6] and Hölder's inequality, we have for $t \in [0, t_1]$,

$$l < \mathbb{E}\|\mathfrak{F}(z^l)(t)\|^2 \leq 4\mathbb{E}\|\Re(t)[\Omega(0) - \mathcal{F}_1(0,\Omega)]\|^2 + 4\mathbb{E}\|\mathcal{F}_1(t,\overline{z^l}_t)\|^2$$

$$+4\mathbb{E}\Big\|\int_0^t \mathfrak{R}(t-s)C(s)v(s)ds\Big\|^2+4\mathbb{E}\Big\|\int_0^t \mathfrak{R}(t-s)\mathcal{F}_2(s,\overline{z^l}_{\rho(s,\overline{z^l}_s)})dB^{\mathcal{H}}(s)\Big\|^2$$

$$\begin{aligned}\leq\ & 8N^2[\mathcal{K}_1^2\|\Omega\|_{\mathfrak{B}}^2+\mathbb{E}\|\mathcal{F}_1(0,\Omega)\|^2]+4\mathbb{E}\|\mathcal{F}_1(t,\overline{z^l}_t)\|^2\\ &+4\mathbb{E}\Big[\int_0^t\|\mathfrak{R}(t-s)\|\|C(s)v(s)\|ds\Big]^2+8\mathcal{H}N^2t_1^{2\mathcal{H}-1}\int_0^t\mathbb{E}\|\mathcal{F}_2(s,\overline{z^l}_{\rho(s,\overline{z^l}_s)})\|_{L_2^0}^2ds\\ \leq\ & 8N^2[\mathcal{K}_1^2\|\Omega\|_{\mathfrak{B}}^2+L_{\mathcal{F}_1}(\|\Omega\|_{\mathfrak{B}}^2+1)]+4L_{\mathcal{F}_1}(\|\overline{z^l}_t\|_{\mathfrak{B}}^2+1)\\ &+4N^2\|C\|_\infty^2t_1\|v\|_{L^2_{\mathcal{F}}}^2+8\mathcal{H}N^2t_1^{2\mathcal{H}-1}\int_0^t\eta(s)\Theta_{\mathcal{F}_2}(\|\overline{z^l}_{\rho(s,\overline{z^l}_s)}\|_{\mathfrak{B}}^2)ds.\end{aligned}$$

For any $t\in(t_j,p_j],\ j=1,2,\ldots,\mathcal{M}$, we have

$$l<\mathbb{E}\|\mathfrak{F}(z^l)(t)\|^2\leq L_{\mathcal{E}_j}(\|\overline{z^l}_t\|_{\mathfrak{B}}^2+1).$$

Similarly, for any $t\in(p_j,t_{j+1}],\ j=1,2,\ldots,\mathcal{M}$, we have

$$\begin{aligned}l<\mathbb{E}\|\mathfrak{F}(z^l)(t)\|^2\ \leq\ & 4\mathbb{E}\|\mathfrak{R}(t-p_j)[\mathcal{E}_j(p_j,\overline{z^l}_{p_j})-\mathcal{F}_1(p_j,\overline{z^l}_{p_j})]\|^2+4\mathbb{E}\|\mathcal{F}_1(t,\overline{z^l}_t)\|^2\\ &+4\mathbb{E}\Big\|\int_{p_j}^t\mathfrak{R}(t-s)C(s)v(s)ds\Big\|^2+4\mathbb{E}\Big\|\int_{p_j}^t\mathfrak{R}(t-s)\mathcal{F}_2(s,\overline{z^l}_{\rho(s,\overline{z^l}_s)})dB^{\mathcal{H}}(s)\Big\|^2\\ \leq\ & 8N^2[\mathbb{E}\|\mathcal{E}_j(p_j,\overline{z^l}_{p_j})\|^2+\mathbb{E}\|\mathcal{F}_1(p_j,\overline{z^l}_{p_j})\|^2]+4\mathbb{E}\|\mathcal{F}_1(t,\overline{z^l}_t)\|^2\\ &+4\mathbb{E}\Big[\int_{p_j}^t\|\mathfrak{R}(t-s)\|\|C(s)v(s)\|ds\Big]^2+8\mathcal{H}N^2t_{j+1}^{2\mathcal{H}-1}\int_{p_j}^t\mathbb{E}\|\mathcal{F}_2(s,\overline{z^l}_{\rho(s,\overline{z^l}_s)})\|_{L_2^0}^2ds\\ \leq\ & 8N^2(L_{\mathcal{E}_j}+L_{\mathcal{F}_1})\|\overline{z^l}_{p_j}\|_{\mathfrak{B}}^2+8N^2(L_{\mathcal{E}_j}+L_{\mathcal{F}_1})+4L_{\mathcal{F}_1}(\|\overline{z^l}_t\|_{\mathfrak{B}}^2+1)\\ &+4N^2\|C\|_\infty^2t_{j+1}\|v\|_{L^2_{\mathcal{F}}}^2+8\mathcal{H}N^2t_{j+1}^{2\mathcal{H}-1}\int_{p_j}^t\eta(s)\Theta_{\mathcal{F}_2}(\|\overline{z^l}_{\rho(s,\overline{z^l}_s)}\|_{\mathfrak{B}}^2)ds.\end{aligned}$$

For any $t\in[0,b]$, we have

$$l<\mathbb{E}\|\mathfrak{F}(z^l)(t)\|^2\ \leq\ W^*+L_{\mathcal{E}_j}l^*+8N^2(L_{\mathcal{E}_j}+L_{\mathcal{F}_1})l^*+4L_{\mathcal{F}_1}l^*+8\mathcal{H}N^2b^{2\mathcal{H}-1}\Theta_{\mathcal{F}_2}(l^*)\int_0^t\eta(s)ds,$$

and hence,

$$\begin{aligned}l^*\ <\ & 2(\mathcal{K}_3^*+\mathcal{L}_\Omega^*)^2\|\,\Omega\,\|_{\mathfrak{B}}^2+2[\mathcal{K}_2^*]^2[W^*\\ &+L_{\mathcal{E}_j}l^*+8N^2(L_{\mathcal{E}_j}+L_{\mathcal{F}_1})l^*+4L_{\mathcal{F}_1}l^*+8\mathcal{H}N^2b^{2\mathcal{H}-1}\Theta_{\mathcal{F}_2}(l^*)\int_0^b\eta(s)ds],\end{aligned}$$

where

$$W^*\ =\ \max_{1\leq j\leq\mathcal{M}}\Big\{8N^2[\mathcal{K}_1^2\|\Omega\|_{\mathfrak{B}}^2+L_{\mathcal{F}_1}(\|\Omega\|_{\mathfrak{B}}^2+1)]+L_{\mathcal{E}_j}$$

$$+8N^2(L_{\mathcal{E}_j}+L_{\mathcal{F}_1})+4L_{\mathcal{F}_1}+4N^2\|C\|_\infty^2 b\|v\|^2_{L^2_{\mathcal{F}}}\Big\}.$$

Dividing both sides by l^* and taking the limit as $l^* \to \infty$, we have

$$1<2[\mathcal{K}_2^*]^2[L_{\mathcal{E}_j}+8N^2(L_{\mathcal{E}_j}+L_{\mathcal{F}_1})+4L_{\mathcal{F}_1}+8\mathcal{H}N^2b^{2\mathcal{H}-1}\Theta_1\int_0^b \eta(s)ds],$$

which is contrary to our assumption [H6]. Hence, for some $l>0$, $\mathfrak{F}(\overline{\mathcal{B}}_l)\subset\overline{\mathcal{B}}_l$.
Step 2. $\mathfrak{F}_1$ is a contraction map on $\overline{\mathcal{B}}_l$.
For any $y,z\in\overline{\mathcal{B}}_l$, if $t\in[0,t_1]$, then we have

$$\begin{aligned}
\mathbb{E}\|(\mathfrak{F}_1y)(t)-(\mathfrak{F}_1z)(t)\|^2 &\le L_{\mathcal{F}_1}\|\overline{y}_t-\overline{z}_t\|^2_{\mathfrak{B}}\\
&\le 2[\mathcal{K}_2^*]^2L_{\mathcal{F}_1}\sup\{\mathbb{E}\|\overline{y}(s)-\overline{z}(s)\|^2:0<s<t\}\\
&\le 2[\mathcal{K}_2^*]^2L_{\mathcal{F}_1}\sup_{s\in[0,b]}\mathbb{E}\|\overline{y}(s)-\overline{z}(s)\|^2\\
&= 2[\mathcal{K}_2^*]^2L_{\mathcal{F}_1}\sup_{s\in[0,b]}\mathbb{E}\|y(s)-z(s)\|^2,\quad (\text{since } \overline{z}=z \text{ in } [0,b])\\
&= 2[\mathcal{K}_2^*]^2L_{\mathcal{F}_1}\|y-z\|^2_{\mathcal{PC}}.
\end{aligned}$$

If $t\in(t_j,p_j],\ j=1,2,\ldots,\mathcal{M}$, then we have

$$\begin{aligned}
\mathbb{E}\|(\mathfrak{F}_1y)(t)-(\mathfrak{F}_1z)(t)\|^2 &\le L_{\mathcal{E}_j}\|\overline{y}_t-\overline{z}_t\|^2_{\mathfrak{B}}\\
&\le 2[\mathcal{K}_2^*]^2L_{\mathcal{E}_j}\sup_{s\in[0,b]}\mathbb{E}\|\overline{y}(s)-\overline{z}(s)\|^2\\
&= 2[\mathcal{K}_2^*]^2L_{\mathcal{E}_j}\|y-z\|^2_{\mathcal{PC}}.
\end{aligned}$$

Similarly, if $t\in(p_j,t_{j+1}],\ j=1,2,\ldots,\mathcal{M}$, then we have

$$\begin{aligned}
\mathbb{E}\|(\mathfrak{F}_1y)(t)-(\mathfrak{F}_1z)(t)\|^2 &\le 2N^2[2\mathbb{E}\|\mathcal{E}_j(p_j,\overline{y}_{p_j})-\mathcal{E}_j(p_j,\overline{z}_{p_j})\|^2+2\mathbb{E}\|\mathcal{F}_1(p_j,\overline{y}_{p_j})\\
&\quad -\mathcal{F}_1(p_j,\overline{z}_{p_j})\|^2]+2\mathbb{E}\|\mathcal{F}_1(t,\overline{y}_t)-\mathcal{F}_1(t,\overline{z}_t)\|^2\\
&\le 8N^2[\mathcal{K}_2^*]^2L_{\mathcal{E}_j}\sup\{\mathbb{E}\|\overline{y}(s)-\overline{z}(s)\|^2:0<s<t\}\\
&\quad +4[\mathcal{K}_2^*]^2L_{\mathcal{F}_1}(2N^2+1)\sup\{\mathbb{E}\|\overline{y}(s)-\overline{z}(s)\|^2:0<s<t\}\\
&\le 4[\mathcal{K}_2^*]^2[2N^2L_{\mathcal{E}_j}+(2N^2+1)L_{\mathcal{F}_1}]\sup_{s\in[0,b]}\mathbb{E}\|\overline{y}(s)-\overline{z}(s)\|^2\\
&= 4[\mathcal{K}_2^*]^2[2N^2L_{\mathcal{E}_j}+(2N^2+1)L_{\mathcal{F}_1}]\sup_{s\in[0,b]}\mathbb{E}\|y(s)-z(s)\|^2\\
&= 4[\mathcal{K}_2^*]^2[2N^2L_{\mathcal{E}_j}+(2N^2+1)L_{\mathcal{F}_1}]\|y-z\|^2_{\mathcal{PC}}.
\end{aligned}$$

For any $t\in[0,b]$, we have

$$\mathbb{E}\|(\mathfrak{F}_1y)(t)-(\mathfrak{F}_1z)(t)\|^2\le L_{\mathfrak{F}_1}\|y-z\|^2_{\mathcal{PC}}.$$

Taking supremum over t

$$\|\mathfrak{F}_1y-\mathfrak{F}_1z\|^2_{\mathcal{PC}}\le L_{\mathfrak{F}_1}\|y-z\|^2_{\mathcal{PC}},$$

where $L_{\mathfrak{F}_1} = 2[\mathcal{K}_2^*]^2(L_{\mathcal{E}_j} + 4N^2 L_{\mathcal{E}_j} + 2(2N^2+1)L_{\mathcal{F}_1})$. By Eq. (3.1), we see that $L_{\mathfrak{F}_1} < 1$. Hence, $\mathfrak{F}_1$ is a contraction map on $\overline{\mathcal{B}}_l$.

Step 3. We show that $\mathfrak{F}_2$ is continuous on $\overline{\mathcal{B}}_l$.

Let $\{z^m\}_{m=1}^{\infty} \subseteq \overline{\mathcal{B}}_l$ be a sequence such that $z^m \to z$ in $\overline{\mathcal{B}}_l$ as $m \to \infty$. From axiom [A1], we have that $(\overline{z^m})_s \to \bar{z}_s$ uniformly for $s \in (-\infty, b]$ as $m \to \infty$. By hypothese [H5] and [42, Theorem 2.2, Step-3], we have

$$\mathcal{F}_2(s, \overline{z^m}_{\rho(s,(\overline{z^m})_s)}) \to \mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)}),$$

for any $s \in [0,t]$, and since

$$\mathbb{E}\|\mathcal{F}_2(s, \overline{z^m}_{\rho(s,(\overline{z^m})_s)}) - \mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)})\|^2_{L_2^0} \leq 2\Theta_{\mathcal{F}_2}(l^*)\eta(s).$$

For any $t \in (p_j, t_{j+1}]$, $j = 0, 1, \ldots, \mathcal{M}$, we have

$$\begin{aligned}
\mathbb{E}\|(\mathfrak{F}_2 z^m)(t) - (\mathfrak{F}_2 z)(t)\|^2 &= \mathbb{E}\left\| \int_{p_j}^t \mathfrak{R}(t-s)[\mathcal{F}_2(s, \overline{z^m}_{\rho(s,\overline{z^m}_s)}) - \mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)})]dB^{\mathcal{H}}(s) \right\|^2 \\
&\leq 2\mathcal{H}N^2 t_{j+1}^{2\mathcal{H}-1} \int_{p_j}^t \mathbb{E}\|\mathcal{F}_2(s, \overline{z^m}_{\rho(s,(\overline{z^m})_s)}) - \mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)})\|^2_{L_2^0} ds \\
&\leq 2\mathcal{H}N^2 b^{2\mathcal{H}-1} \int_0^t \mathbb{E}\|\mathcal{F}_2(s, \overline{z^m}_{\rho(s,(\overline{z^m})_s)}) - \mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)})\|^2_{L_2^0} ds.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\|\mathfrak{F}_2 z^m - \mathfrak{F}_2 z\|^2_{\mathcal{PC}} \to 0 \ \ as \ \ m \to \infty.$$

Thus, $\mathfrak{F}_2$ is continuous.

Step 4. We show that $\{\mathfrak{F}_2 z : z \in \overline{\mathcal{B}}_l\}$ is equicontinuous.

Since $\mathfrak{R}(t)$ is compact, which implies that the continuity of $\mathfrak{R}(t)$ in $(0,b]$. Let $p_j < \epsilon < t \leq t_{j+1}$, $j = 0, 1, \ldots, \mathcal{M}$, and $\omega > 0$ such that $\|\mathfrak{R}(\xi_1 - s) - \mathfrak{R}(\xi_2 - s)\|^2 < \epsilon$ for every $\xi_1, \xi_2 \in (p_j, t_{j+1}]$ with $|\xi_1 - \xi_2| < \omega$. For each $z \in \overline{\mathcal{B}}_l$, $0 < |\kappa| < \omega$ with $t, t+\kappa \in (p_j, t_{j+1}]$, $j = 0, 1, \ldots, \mathcal{M}$, we have

$$\begin{aligned}
\mathbb{E}\|(\mathfrak{F}_2 z)(t+\kappa) - (\mathfrak{F}_2 z)(t)\|^2 &\leq 4\mathbb{E}\left\| \int_t^{t+\kappa} \mathfrak{R}(t+\kappa-s)C(s)v(s)ds \right\|^2 \\
&\quad + 4\mathbb{E}\left\| \int_{p_j}^t [\mathfrak{R}(t+\kappa-s) - \mathfrak{R}(t-s)]C(s)v(s)ds \right\|^2 \\
&\quad + 4\mathbb{E}\left\| \int_t^{t+\kappa} \mathfrak{R}(t+\kappa-s)\mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s) \right\|^2 \\
&\quad + 4\mathbb{E}\left\| \int_{p_j}^t [\mathfrak{R}(t+\kappa-s) - \mathfrak{R}(t-s)]\mathcal{F}_2(s, \bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s) \right\|^2 \\
&= 4[\chi_1 + \chi_2],
\end{aligned}$$

where

$$\chi_1 \leq \mathbb{E}\Big[\int_t^{t+\kappa} \|\mathfrak{R}(t+\kappa-s)\|\|C(s)v(s)\|ds \Big]^2$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\int_{p_j}^{t}\|\mathfrak{R}(t+\kappa-s)-\mathfrak{R}(t-s)\|\|C(s)v(s)\|ds\Big]^2\\
\leq\ & N^2\kappa\|C\|_\infty^2\mathbb{E}\int_t^{t+\kappa}\|v(s)\|_{\mathcal{T}}^2ds\\
&+\|C\|_\infty^2 t_{j+1}\mathbb{E}\int_{p_j}^{t}\|\mathfrak{R}(t+\kappa-s)-\mathfrak{R}(t-s)\|^2\|v(s)\|_{\mathcal{T}}^2ds\\
\leq\ & N^2\kappa\|C\|_\infty^2\mathbb{E}\int_t^{t+\kappa}\|v(s)\|_{\mathcal{T}}^2ds+\epsilon\|C\|_\infty^2 t_{j+1}\mathbb{E}\int_{p_j}^{t}\|v(s)\|_{\mathcal{T}}^2ds,
\end{aligned}
$$

$$
\begin{aligned}
\chi_2\ \leq\ & 2\mathcal{H}N^2t_{j+1}^{2\mathcal{H}-1}\int_t^{t+\kappa}\mathbb{E}\|\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})\|_{L_2^0}^2ds\\
&+2\mathcal{H}t_{j+1}^{2\mathcal{H}-1}\int_{p_j}^{t}\|\mathfrak{R}(t+\kappa-s)-\mathfrak{R}(t-s)\|^2\mathbb{E}\|\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})\|_{L_2^0}^2ds\\
\leq\ & 2\mathcal{H}N^2t_{j+1}^{2\mathcal{H}-1}\Theta_{\mathcal{F}_2}(l^*)\int_t^{t+\kappa}\eta(s)ds+2\epsilon\mathcal{H}t_{j+1}^{2\mathcal{H}-1}\Theta_{\mathcal{F}_2}(l^*)\int_{p_j}^{t}\eta(s)ds.
\end{aligned}
$$

We conclude that $\mathbb{E}\|(\mathfrak{F}_2z)(t+\kappa)-(\mathfrak{F}_2z)(t)\|^2\ \to\ 0$ as $\kappa\to 0$ and ϵ is sufficiently small. Hence, $\{\mathfrak{F}_2z : z\in\overline{\mathcal{B}}_l\}$ is equicontinuous. Also, clearly $\{\mathfrak{F}_2z : z\in\overline{\mathcal{B}}_l\}$ is uniformly bounded.

Step 5. The set $Q(t)=\{(\mathfrak{F}_2z)(t) : z\in\overline{\mathcal{B}}_l\},\ t\in\mathcal{J}_1$ is relatively compact in Z.

Clearly, $Q(0)=\{0\}$ is compact. Let ξ is real number and $t\in(p_j,t_{j+1}],\ j=0,1,\dots,\mathcal{M}$, be fixed with $0<\xi<t$. For $z\in\overline{\mathcal{B}}_l$, we define

$$
(\mathfrak{F}_2^{\xi}z)(t)=\begin{cases}\int_0^{t-\xi}\mathfrak{R}(t-s)C(s)v(s)ds+\int_0^{t-\xi}\mathfrak{R}(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s), & t\in[0,t_1],\ j=0,\\ 0, & t\in(t_j,p_j],\ j\geq 1,\\ \int_{p_j}^{t-\xi}\mathfrak{R}(t-s)C(s)v(s)ds+\int_{p_j}^{t-\xi}\mathfrak{R}(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s), & t\in(p_j,t_{j+1}],\ j\geq 1.\end{cases}
$$

Since $\mathfrak{R}(t)$ is compact, the set $Q^{\xi}(t)=\{(\mathfrak{F}_2^{\xi}z)(t) : z\in\overline{\mathcal{B}}_l\}$ is relatively compact in Z for every ξ. For $t\in(p_j,t_{j+1}],\ j=0,1,\dots,\mathcal{M}$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\mathfrak{F}_2z)(t)-(\mathfrak{F}_2^{\xi}z)(t)\|^2\\
&\leq 2\mathbb{E}\Big\|\int_{p_j}^{t}\mathfrak{R}(t-s)C(s)v(s)ds-\int_{p_j}^{t-\xi}\mathfrak{R}(t-s)C(s)v(s)ds\Big\|^2\\
&\quad+2\mathbb{E}\Big\|\int_{p_j}^{t}\mathfrak{R}(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s)-\int_{p_j}^{t-\xi}\mathfrak{R}(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s)\Big\|^2\\
&\leq 2\mathbb{E}\Big\|\int_{t-\xi}^{t}\mathfrak{R}(t-s)C(s)v(s)ds\Big\|^2+2\mathbb{E}\Big\|\int_{t-\xi}^{t}\mathfrak{R}(t-s)\mathcal{F}_2(s,\bar{z}_{\rho(s,\bar{z}_s)})dB^{\mathcal{H}}(s)\Big\|^2\\
&\leq 2N^2\xi\|C\|_\infty^2\mathbb{E}\int_{t-\xi}^{t}\|v(s)\|_{\mathcal{T}}^2ds+4\mathcal{H}N^2t_{j+1}^{2\mathcal{H}-1}\Theta_{\mathcal{F}_2}(l^*)\int_{t-\xi}^{t}\eta(s)ds\ \ \to\ \ 0\ \text{ as }\ \xi\to 0.
\end{aligned}
$$

The relatively compact set $Q^{\xi}(t)$ and set $Q(t)$ are arbitrarily close. Hence, $Q(t)=\{(\mathfrak{F}_2z)(t) : z\in\overline{\mathcal{B}}_l\}$ is relatively compact in Z. By using step 3–5 along with Arzela-Ascoli theorem, we obtain that the $\mathfrak{F}_2$ is a completely continuous operator. Hence, by Krasnoselskii's theorem [43], we realize that the operator $\mathfrak{F}_1+\mathfrak{F}_2$ has a fixed point, which is a mild solution of the stochastic system (1.1). □

4. Existence of stochastic optimal controls

In this section, we prove the existence of optimal controls for the stochastic system. Let z^v be the mild solution of the stochastic system (1.1) corresponding to the control $v \in \mathcal{U}_{ad}$. We consider the Lagrange problem $(\mathcal{LP})$: Find an optimal state-control pair $(z^*, v^*) \in \mathcal{BPC} \times \mathcal{U}_{ad}$ such that

$$\mathfrak{I}(z^*, v^*) \leq \mathfrak{I}(z^v, v) \ \text{ for all } v \in \mathcal{U}_{ad},$$

where

$$\mathfrak{I}(z^v, v) = \mathbb{E} \int_0^b \mathfrak{M}(t, z_t^v, z^v(t), v(t))dt.$$

For the existence of optimal controls, we shall introduce the following hypotheses

[H7]: The function $\mathfrak{M} : \mathcal{J}_1 \times \mathfrak{B} \times Z \times \mathcal{T} \to \mathbb{R} \cup \{\infty\}$ satisfies:

(a) $\mathfrak{M}$ is Borel measurable

(b) $\mathfrak{M}(t, z_1, z_2, \cdot)$ is convex function on $\mathcal{T}$ for each $z_1 \in \mathfrak{B}$, $z_2 \in Z$ and almost all $t \in \mathcal{J}_1$.

(c) $\mathfrak{M}(t, \cdot, \cdot, \cdot)$ is sequentially lower semi-continuous on $\mathfrak{B} \times Z \times \mathcal{T}$ for almost all $t \in \mathcal{J}_1$.

(d) There exist constants $\omega_1, \omega_2 \geq 0$, $\omega_3 > 0$ and Φ is non-negative function in $L^1(\mathcal{J}_1, \mathbb{R})$ such that

$$\mathfrak{M}(t, z_1, z_2, v) \geq \Phi(t) + \omega_1 \|z_1\|_{\mathfrak{B}} + \omega_2 \|z_2\| + \omega_3 \|v\|_{\mathcal{T}}^2.$$

[H8]: The operator C is strongly continuous.

Theorem 4.1. *Assume that the presumptions [H1]–[H8] are fulfilled. Then the problem $(\mathcal{LP})$ admits at least one optimal control pair on $\mathcal{BPC} \times \mathcal{U}_{ad}$.*

Proof. If $\inf\{\mathfrak{I}(z^v, v) : v \in \mathcal{U}_{ad}\} = +\infty$, there is nothing to prove. Next, we choose $\inf\{\mathfrak{I}(z^v, v) : v \in \mathcal{U}_{ad}\} = \epsilon < +\infty$ and using the hypotheses [H7], we obtain

$$\mathfrak{I}(z^v, v) \ \geq \ \int_0^b \Phi(t)dt + \omega_1 \int_0^b \|z_t^v(t)\|_{\mathfrak{B}} dt + \omega_2 \int_0^b \|z^v(t)\| dt + \omega_3 \int_0^b \|v(t)\|_{\mathcal{T}}^2 \ \geq \ \epsilon \ > \ -\infty.$$

By definition of infimum, there exists a minimizing sequence $\{(z^k, v^k)\} \subset \mathscr{R}_{ad}$, where $\mathscr{R}_{ad}$=$\{ (z, v) : z$ be the mild solution of the stochastic system (1.1) corresponding to the control $v \in \mathcal{U}_{ad}\}$ such that

$$\mathfrak{I}(z^k, v^k) \to \epsilon \text{ as } k \to +\infty.$$

Since $\{v^k\} \subseteq \mathcal{U}_{ad}$, $\{v^k\}$ is bounded in the space $L^2_{\mathcal{F}}(\mathcal{J}_1, \mathcal{T})$, then exists a subsequence, relabeled as $\{v^k\}$, and $v^* \in L^2_{\mathcal{F}}(\mathcal{J}_1, \mathcal{T})$ such that v^k converges weakly to v^* in $L^2_{\mathcal{F}}(\mathcal{J}_1, \mathcal{T})$ as $k \to \infty$. Since $\mathcal{U}_{ad}$ is convex and closed, then by Marzur Lemma, we have $v^* \in \mathcal{U}_{ad}$.

Let z^k be the sequence of mild solutions of the stochastic system (1.1) corresponding to v^k and z^k fulfills the consecutive integral equations

$$z^k(t) = \begin{cases} \mathfrak{R}(t)[\Omega(0) - \mathcal{F}_1(0, \Omega)] + \mathcal{F}_1(t, \overline{z^k}_t) & \\ + \int_0^t \mathfrak{R}(t-s)C(s)v^k(s)ds + \int_0^t \mathfrak{R}(t-s)\mathcal{F}_2(s, \overline{z^k}_{\rho(s, \overline{z^k}_s)})dB^{\mathcal{H}}(s), & t \in [0, t_1], \ j = 0, \\ \mathcal{E}_j(t, \overline{z^k}_t), & t \in (t_j, p_j], \ j \geq 1, \\ \mathfrak{R}(t - p_j)[\mathcal{E}_j(p_j, \overline{z^k}_{p_j}) - \mathcal{F}_1(p_j, \overline{z^k}_{p_j})] + \mathcal{F}_1(t, \overline{z^k}_t) & \\ + \int_{p_j}^t \mathfrak{R}(t-s)C(s)v^k(s)ds + \int_{p_j}^t \mathfrak{R}(t-s)\mathcal{F}_2(s, \overline{z^k}_{\rho(s, \overline{z^k}_s)})dB^{\mathcal{H}}(s), & t \in (p_j, t_{j+1}], \ j \geq 1. \end{cases}$$

Let $\mathcal{F}_2^k(s) \equiv \mathcal{F}_2(s, \overline{z^k}_{\rho(s,\overline{z^k}_s)})$. Then, for each $z^k \in \overline{\mathcal{B}}_l \subset \mathcal{BPC}$, by hypotheses [H5], we obtain that $\mathcal{F}_2^k : \mathcal{J}_1 \to L_2^0(Y,Z)$ is bounded operator. Hence, $\mathcal{F}_2^k(\cdot) \in L^2(\mathcal{J}_1, L_2^0(Y,Z))$. Furthermore, $\{\mathcal{F}_2^k(\cdot)\}$ is bounded in $L^2(\mathcal{J}_1, L_2^0(Y,Z))$, there are subsequence, relabeled as $\{\mathcal{F}_2^k(\cdot)\}$ and $\mathcal{F}_2^*(\cdot) \in L^2(\mathcal{J}_1, L_2^0(Y,Z))$ such that $\mathcal{F}_2^k(\cdot) \xrightarrow{w} \mathcal{F}_2^*(\cdot)$ in $L^2(\mathcal{J}_1, L_2^0(Y,Z))$ as $k \to \infty$.
Next, we consider the following stochastic system

$$\begin{cases} d\mathfrak{D}(t,z_t) = \mathcal{A}[\mathfrak{D}(t,z_t) + \int_0^t \mathcal{G}(t-s)\mathfrak{D}(t,z_s)ds]dt + C(t)v^*(t)dt + \mathcal{F}_2^*(t)dB^{\mathcal{H}}(t) \\ \qquad\qquad t \in (p_j, t_{j+1}],\ j = 0,1,\dots,\mathcal{M}, \\ z(t) = \mathcal{E}_j(t,z_t),\quad t \in (t_j, p_j],\ j = 1,2,\dots,\mathcal{M}, \\ z_0 = \Omega \in \mathfrak{B}. \end{cases} \tag{4.1}$$

By Theorem 3.1, we know that the stochastic system (4.1) has a mild solution

$$z^*(t) = \begin{cases} \mathfrak{R}(t)[\Omega(0) - \mathcal{F}_1(0,\Omega)] + \mathcal{F}_1(t,\overline{z^*}_t) & \\ + \int_0^t \mathfrak{R}(t-s)C(s)v^*(s)ds + \int_0^t \mathfrak{R}(t-s)\mathcal{F}_2^*(s)dB^{\mathcal{H}}(s), & t \in [0,t_1],\ j = 0, \\ \mathcal{E}_j(t,\overline{z^*}_t), & t \in (t_j,p_j],\ j \geq 1, \\ \mathfrak{R}(t-p_j)[\mathcal{E}_j(p_j,\overline{z^*}_{p_j}) - \mathcal{F}_1(p_j,\overline{z^*}_{p_j})] + \mathcal{F}_1(t,\overline{z^*}_t) & \\ + \int_{p_j}^t \mathfrak{R}(t-s)C(s)v^*(s)ds + \int_{p_j}^t \mathfrak{R}(t-s)\mathcal{F}_2^*(s)dB^{\mathcal{H}}(s), & t \in (p_j,t_{j+1}],\ j \geq 1. \end{cases}$$

For any $t \in [0,t_1]$, we have

$$\mathbb{E}\|z^k(t) - z^*(t)\|^2 \leq 3[\Upsilon_1^k(t) + \Upsilon_2^k(t) + \Upsilon_3^k(t)],$$

where

$$\begin{aligned} \Upsilon_1^k(t) &= \mathbb{E}\|\mathcal{F}_1(t,\overline{z^k}_t) - \mathcal{F}_1(t,\overline{z^*}_t)\|^2 \\ &\leq L_{\mathcal{F}_1}\|\overline{z^k}_t - \overline{z^*}_t\|_{\mathfrak{B}}^2 \\ &\leq 2[\mathcal{K}_2^*]^2 L_{\mathcal{F}_1} \sup\{\mathbb{E}\|\overline{z^k}(s) - \overline{z^*}(s)\|^2 : 0 < s < t\} \\ &\leq 2[\mathcal{K}_2^*]^2 L_{\mathcal{F}_1} \sup_{s\in[0,b]} \mathbb{E}\|\overline{z^k}(s) - \overline{z^*}(s)\|^2 \\ &= 2[\mathcal{K}_2^*]^2 L_{\mathcal{F}_1} \sup_{s\in[0,b]} \mathbb{E}\|z^k(s) - z^k(s)\|^2, \quad (\text{since } \bar{z} = z \text{ in } [0,b]) \\ &= 2[\mathcal{K}_2^*]^2 L_{\mathcal{F}_1}\|z^k - z^*\|_{\mathcal{PC}}^2, \\ \Upsilon_2^k(t) &= \mathbb{E}\left\|\int_0^t \mathcal{R}(t-s)C(s)[v^k(s) - v^*(s)]ds\right\|^2 \\ &\leq N^2 t_1 \mathbb{E}\int_0^t \|C(s)v^k(s) - C(s)v^*(s)\|^2 ds, \\ \Upsilon_3^k(t) &= \mathbb{E}\left\|\int_0^t \mathcal{R}(t-s)[\mathcal{F}_2^k(s) - \mathcal{F}_2^*(s)]dB^H(s)\right\|^2 \\ &\leq 2\mathcal{H}t_1^{2\mathcal{H}-1}\int_0^t \mathbb{E}\|\mathfrak{R}(t-s)[\mathcal{F}_2^k(s) - \mathcal{F}_2^*(s)]\|_{L_2^0}^2 ds. \end{aligned}$$

For any $t \in (t_j, p_j],\ j = 1, 2, \ldots, \mathcal{M}$, we have

$$\mathbb{E}\|z^k(t) - z^*(t)\|^2 \leq 2[\mathcal{K}_2^*]^2 L_{\mathcal{E}_j}\|z^k - z^*\|^2_{\mathcal{PC}}.$$

For any $t \in (p_j, t_{j+1}],\ j = 1, 2, \ldots, \mathcal{M}$, we have

$$\mathbb{E}\|z^k(t) - z^*(t)\|^2 \leq 4[\Psi_1^k(t) + \Psi_2^k(t) + \Psi_3^k(t) + \Psi_4^k(t)],$$

where

$$\begin{aligned}
\Psi_1^k(t) &= 2N^2\mathbb{E}\|\mathcal{E}_j(p_j, \overline{z^k}_{p_j}) - \mathcal{E}_j(p_j, \overline{z^*}_{p_j})\|^2 \\
&\leq 4[\mathcal{K}_2^*]^2 N^2 L_{\mathcal{E}_j}\|z^k - z^*\|^2_{\mathcal{PC}}, \\
\Psi_2^k(t) &= 2N^2\mathbb{E}\|\mathcal{F}_1(p_j, \overline{z^k}_{p_j}) - \mathcal{F}_1(p_j, \overline{z^*}_{p_j})\|^2 \\
&\quad + \mathbb{E}\|\mathcal{F}_1(t, \overline{z^k}_t) - \mathcal{F}_1(t, \overline{z^*}_t)\|^2 \\
&\leq (2N^2 + 1)L_{\mathcal{F}_1}\|\overline{z^k}_t - \overline{z^*}_t\|^2_{\mathfrak{B}} \\
&\leq 2[\mathcal{K}_2^*]^2(2N^2 + 1)L_{\mathcal{F}_1}\|z^k - z^*\|^2_{\mathcal{PC}},
\end{aligned}$$

$$\begin{aligned}
\Psi_3^k(t) &= \mathbb{E}\left\| \int_{p_j}^t \mathcal{R}(t-s)\mathcal{C}(s)[v^k(s) - v^*(s)]ds \right\|^2 \\
&\leq N^2 t_{j+1}\mathbb{E}\int_{p_j}^t \|\mathcal{C}(s)v^k(s) - \mathcal{C}(s)v^*(s)\|^2 ds,
\end{aligned}$$

$$\begin{aligned}
\Psi_4^k(t) &= \mathbb{E}\left\| \int_{p_j}^t \mathcal{R}(t-s)[\mathcal{F}_2^k(s) - \mathcal{F}_2^*(s)]dB^H(s) \right\|^2 \\
&\leq 2\mathcal{H}t_{j+1}^{2\mathcal{H}-1}\int_{p_j}^t \mathbb{E}\|\mathfrak{R}(t-s)[\mathcal{F}_2^k(s) - \mathcal{F}_2^*(s)]\|^2_{L_2^0} ds.
\end{aligned}$$

For $t \in [0, b]$, we have

$$\mathbb{E}\|z^k(t) - z^*(t)\|^2 \leq L_0\|z^k - z^*\|^2_{\mathcal{PC}} + \Phi_1^k(t) + \Phi_2^k(t),$$

where

$$\begin{aligned}
L_0 &= \max_{1 \leq j \leq \mathcal{M}}\Big[16[\mathcal{K}_2^*]^2 N^2 L_{\mathcal{E}_j} + 8[\mathcal{K}_2^*]^2(2N^2 + 1)L_{\mathcal{F}_1} + 2[\mathcal{K}_2^*]^2 L_{\mathcal{E}_j}\Big] < 1, \\
\Phi_1^k(t) &= 4N^2 b\,\mathbb{E}\int_0^t \|\mathcal{C}(s)v^k(s) - \mathcal{C}(s)v^*(s)\|^2 ds, \\
\Phi_2^k(t) &= 8\mathcal{H}b^{2\mathcal{H}-1}\int_0^t \mathbb{E}\|\mathfrak{R}(t-s)[\mathcal{F}_2^k(s) - \mathcal{F}_2^*(s)]\|^2_{L_2^0} ds.
\end{aligned}$$

Thus, we have

$$\|z^k - z^*\|^2_{\mathcal{PC}} \leq \frac{\Phi_1^k(t) + \Phi_2^k(t)}{1 - L_0}.$$

By [H8], we have

$$\|Cv^k - Cv^*\|^2_{L^2(\mathcal{J}_1,Z)} \to 0 \text{ as } k \to \infty. \tag{4.2}$$

By the Lebesgue dominated convergence theorem and Eq. (4.2), we have

$$\Phi_1^k(t),\ \Phi_2^k(t) \to 0 \text{ as } k \to \infty.$$

Hence,

$$z^k \ \to\ z^* \ \text{ in } \ \mathcal{BPC} \ \text{ as } \ k \ \to \ \infty.$$

By [H5], we obtain

$$\mathcal{F}_2^k(\,\cdot\,) \to \mathcal{F}_2(\,\cdot\,,\overline{z^*}_{\rho(\,\cdot\,,\overline{z^*}_{\cdot})}) \ \text{ in } \ \mathcal{BPC} \ \text{ as } \ k \ \to \ \infty.$$

Limit is unique, so we obtain

$$\mathcal{F}_2^*(t) = \mathcal{F}_2(t,\overline{z^*}_{\rho(t,\overline{z^*}_t)}).$$

Thus, z^* can be given

$$z^*(t) = \begin{cases} \Re(t)[\Omega(0) - \mathcal{F}_1(0,\Omega)] + \mathcal{F}_1(t,\overline{z^*}_t) & \\ + \int_0^t \Re(t-s)C(s)v^*(s)ds + \int_0^t \Re(t-s)\mathcal{F}_2(s,\overline{z^*}_{\rho(s,\overline{z^*}_s)})dB^{\mathcal{H}}(s), & t \in [0,t_1],\ j=0, \\ \mathcal{E}_j(t,\overline{z^*}_t), & t \in (t_j,p_j],\ j \geq 1, \\ \Re(t-p_j)[\mathcal{E}_j(p_j,\overline{z^*}_{p_j}) - \mathcal{F}_1(p_j,\overline{z^*}_{p_j})] + \mathcal{F}_1(t,\overline{z^*}_t) & \\ + \int_{p_j}^t \Re(t-s)C(s)v^*(s)ds + \int_{p_j}^t \Re(t-s)\mathcal{F}_2(s,\overline{z^*}_{\rho(s,\overline{z^*}_s)})dB^{\mathcal{H}}(s), & t \in (p_j,t_{j+1}],\ j \geq 1. \end{cases}$$

Since $\mathcal{BPC} \hookrightarrow L^1(\mathcal{J}_1,Z)$, by using the [H7] and Balder's theorem [44], we have

$$\epsilon \leq \Im(z^*,v^*) = \mathbb{E}\int_0^b \mathfrak{M}(t,z_t^*,z^*,v^*)dt \leq \lim_{k\to\infty} \mathbb{E}\int_0^b \mathfrak{M}(t,z_t^k,z^k,v^k)dt = \epsilon,$$

which shows that $\Im$ attains its infimum at $(z^*,v^*) \in \mathcal{BPC} \times \mathcal{U}_{ad}$. □

5. Example

Consider the following non-instantaneous impulsive stochastic partial neutral integro-differential control system driven by fBm with state-dependent delay:

$$\begin{aligned} d\mathfrak{D}(t,\mu_t)(\varepsilon) &= \frac{\partial^2}{\partial\varepsilon^2}\Big[\mathfrak{D}(t,\mu_t)(\varepsilon) + \int_0^t \mathfrak{S}(t-s)\mathfrak{D}(s,\mu_s)(\varepsilon)ds\Big]dt \\ &\quad + \int_0^1 \mathfrak{K}(\varepsilon,s)v(s,t)ds\,dt + \int_{-\infty}^t \omega_3(t,s-t,\varepsilon,\mu(s-\rho_1(t)\rho_2(\|\mu(t)\|),\varepsilon))ds\,dB^{\mathcal{H}}(t), \\ &\quad v \in \mathcal{U}_{ad},\ (t,\varepsilon) \in \cup_{j=0}^{\mathcal{M}}(p_j,t_{j+1}] \times [0,\pi], \\ \mu(t,\varepsilon) &= \int_{-\infty}^t \overline{\omega_j}(s-t,\varepsilon)\mu(s,\varepsilon)ds, \quad (t,\varepsilon) \in \cup_{j=1}^{\mathcal{M}}(t_j,p_j] \times [0,\pi] \\ \mu(t,0) &= 0 \ \ = \mu(t,\pi), \end{aligned}$$

$$\mu(s,\varepsilon) = \Omega(s,\varepsilon),\ (s,\varepsilon) \in (-\infty,0] \times [0,\pi], \tag{5.1}$$

with cost functional as

$$\mathfrak{J}(\mu,v) = \mathbb{E}\int_0^1\int_0^\pi\int_{-\infty}^0 \|\mu(t+s,\varepsilon)\|^2 ds d\varepsilon dt + \mathbb{E}\int_0^1\int_0^\pi \|\mu(t,\varepsilon)\|^2 d\varepsilon dt + \mathbb{E}\int_0^1\int_0^\pi \|v(t,\varepsilon)\|^2_{\mathcal{T}} d\varepsilon dt,$$

where $0 = t_0 = p_0 < t_1 < p_1 < \cdots < t_{\mathcal{M}} < p_{\mathcal{M}} < t_{\mathcal{M}+1} = b = 1$, $\mathfrak{K} : [0,\pi]\times[0,1]$ is continuous and $B^{\mathcal{H}}$ is a fBm with the Hurst index $1/2 < \mathcal{H} < 1$. In this system

$$\mathfrak{D}(t,\mu_t)(\varepsilon) = \mu(t,\varepsilon) - \int_{-\infty}^t \omega_1(s-t)\mu(s,\varepsilon)ds.$$

Consider the space $Z = \mathcal{T} = L^2[0,\pi]$ and $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ by $\mathcal{A}\theta = \theta''$ and domain of $\mathcal{A}$ is defined as

$$D(\mathcal{A}) = \{\theta \in Z :\ \theta,\ \theta' \text{ are absolutely continuous},\ \theta'' \in Z,\ \theta(0) = \theta(\pi) = 0\}. \tag{5.2}$$

Then $\mathcal{A}$ generates a C_0-semigroup $\mathfrak{R}(t)$ which is compact, self-adjoint. And there exist normalized set $\theta_n(v) = \sqrt{2/\pi}\, sin\,(nv),\ n \in \mathbb{N}$ of eigenvectors of $\mathcal{A}$ corresponding to eigenvalues $n^2,\ n \in \mathbb{N}$. Since the resolvent operator $\mathfrak{R}(t)$ is compact, there exists a constant $N > 0$ such that $\|\mathfrak{R}(t)\| \le N$, then the hypotheses [H1] is fulfilled. Next, we define the admissible control set $\mathcal{U}_{ad} = \{v(\cdot,\varepsilon)|[0,1] \to \mathcal{T}$ is measurable, $\mathcal{F}_t$-adapted stochastic processes, and $\|v\|_{L^2_{\mathcal{F}}} \le \alpha, \alpha > 0\}$.

Let $l \ge 0$, $1 \le q < \infty$, $\Lambda : (-\infty,-l] \to \mathbb{R}$, be a measurable and non-negative function. We denote by $\mathcal{PC}_l \times L^q(\Lambda, Z)$ the set consists of all classes of functions $\Omega : (-\infty,0] \to Z$ such that $\Omega|_{[-l,0]} \in \mathcal{PC}([-l,0],Z)$, $\Lambda\|\Omega\|^q$ is Lebesgue integrable on $(-\infty,-l)$ and $\Omega(\cdot)$ is Lebesgue measurable on $(-\infty,-l)$ with norm

$$\|\Omega\|_{\mathfrak{B}} = \sup\{\|\Omega(\kappa)\| \ :\ -l \le \kappa \le 0\} + \Big(\int_{-\infty}^{-l} \Lambda(\kappa)\ \|\Omega(\kappa)\|^q d\kappa\Big)^{1/q}.$$

The space $\mathcal{PC}_0 \times L^2(\Lambda, Z)$ satisfies the axioms [A1]–[A3] with choice $\mathcal{K}_1 = 1$, $\mathcal{K}_3(t) = \gamma(-t)^{1/2}$, $\mathcal{K}_2(t) = 1 + \Big(\int_{-t}^0 \Lambda(\kappa)d\kappa\Big)^{1/2}$, for $t \ge 0$. To get points of interest about the phase space, see [21,38].
Let $\eta(\kappa)(\varepsilon) = \eta(\kappa,\varepsilon)$, $(\kappa,\varepsilon) \in (-\infty,0]\times[0,\pi]$. Set

$$\mu(t)(\varepsilon) = \mu(t,\varepsilon),\quad \rho(t,\eta) = \rho_1(t)\rho_2(\|\eta(0)\|),$$

we have

$$\begin{aligned}
\mathcal{F}_1(t,\eta)(\varepsilon) &= \int_{-\infty}^0 \omega_1(\kappa)\eta(\kappa)(\varepsilon)d\kappa,\\
\mathcal{F}_2(t,\eta)(\varepsilon) &= \int_{-\infty}^0 \omega_2(t,\kappa,\varepsilon,\eta(\kappa)(\varepsilon))d\kappa,\\
C(t)v(t)(\varepsilon) &= \int_0^1 \mathfrak{K}(\varepsilon,s)v(s,t)ds,\\
\mathcal{E}_j(t,\eta)(\varepsilon) &= \int_{-\infty}^0 \overline{\omega_j}(\kappa,\varepsilon)\eta(\kappa)(\varepsilon)d\kappa,\ \ j = 1,2,\ldots,\mathcal{M}.
\end{aligned}$$

Moreover, we assume that

1. $\rho_i : [0,\infty) \to [0,\infty),\ i = 1,2,$ are continuous functions.
2. $\omega_1 : \mathbb{R} \to \mathbb{R}$ is continuous function, and

$$l_{\mathcal{F}_1} = \Big(\int_{-\infty}^{0} \frac{(\omega_1(s))^2}{\Lambda(s)} ds \Big)^{1/2} < \infty.$$

3. There exist continuous functions $a_{31}, a_{32} : \mathbb{R} \to \mathbb{R}$ such that continuous function $\omega_3 : \mathbb{R}^4 \to \mathbb{R}$ satisfies the conditions

$$|\omega_3(t,s,\varepsilon,y)| \le a_{31}(t)a_{32}(s)|y|,\ \ (t,s,\varepsilon,y) \in \mathbb{R}^4,\ \ \text{with}\ \ l_{\mathcal{F}_2} = \Big(\int_{-\infty}^{0} \frac{(a_{32}(s))^2}{\Lambda(s)} ds \Big)^{1/2} < \infty.$$

4. There exist continuous functions $c_j : \mathbb{R} \to \mathbb{R}$ such that continuous functions $\overline{\omega_j} : \mathbb{R}^2 \to \mathbb{R},\ j = 1,2,\ldots,\mathcal{M}$ satisfies the conditions

$$|\overline{\omega_j}(s,\varepsilon)| \le c_j(s),\ (s,\varepsilon) \in \mathbb{R}^2,\ \ \text{with}\ \ l_{\mathcal{E}_j} = \Big(\int_{-\infty}^{0} \frac{(c_j(s))^2}{\Lambda(s)} ds \Big)^{1/2} < \infty.$$

From the above facts, we obtain

$$\begin{aligned}
\mathbb{E}\|\mathcal{F}_1(t,\eta)\|^2 &= \mathbb{E}\Big[\Big(\int_0^{\pi} \Big(\int_{-\infty}^{0} \omega_1(\kappa)\eta(\kappa)(\varepsilon) d\kappa \Big)^2 d\varepsilon \Big)^{1/2}\Big]^2 \\
&\le \mathbb{E}\Big[\Big(\int_{-\infty}^{0} \frac{(\omega_1(\kappa))^2}{\Lambda(\kappa)} d\kappa \Big)^{1/2} \Big(\int_{-\infty}^{0} \Lambda(\kappa)\|\eta(\kappa)\|^2 d\kappa \Big)^{1/2}\Big]^2 \\
&\le \Big[l_{\mathcal{F}_1}\Big(\|\eta(0)\| + \Big(\int_{-\infty}^{0} \Lambda(\kappa)\|\eta(\kappa)\|^2 d\kappa \Big)^{1/2} \Big)\Big]^2 \\
&= L_{\mathcal{F}_1}\|\eta\|_{\mathfrak{B}}^2,
\end{aligned}$$

where $L_{\mathcal{F}_1} = [l_{\mathcal{F}_1}]^2$.

$$\begin{aligned}
&\mathbb{E}\|\mathcal{F}_1(t,\eta_1) - \mathcal{F}_1(t,\eta_2)\|^2 \\
&\qquad = \mathbb{E}\Big[\Big(\int_0^{\pi} \Big(\int_{-\infty}^{0} \omega_1(\kappa)[\eta_1(\kappa)(\varepsilon) - \eta_2(\kappa)(\varepsilon)] d\kappa \Big)^2 d\varepsilon \Big)^{1/2}\Big]^2 \\
&\qquad \le \mathbb{E}\Big[\Big(\int_{-\infty}^{0} \frac{(\omega_1(\kappa))^2}{\Lambda(\kappa)} d\kappa \Big)^{1/2} \Big(\int_{-\infty}^{0} \Lambda(\kappa)\|\eta_1(\kappa) - \eta_2(\kappa)\|^2 d\kappa \Big)^{1/2}\Big]^2 \\
&\qquad \le \Big[l_{\mathcal{F}_1}\Big(\|\eta_1(0) - \eta_2(0)\| + \Big(\int_{-\infty}^{0} \Lambda(\kappa)\|\eta_1(\kappa) - \eta_2(\kappa)\|^2 d\kappa \Big)^{1/2} \Big)\Big]^2 \\
&\qquad = L_{\mathcal{F}_1}\|\eta_1 - \eta_2\|_{\mathfrak{B}}^2,
\end{aligned}$$

where $L_{\mathcal{F}_1} = [l_{\mathcal{F}_1}]^2$. Similarly, we have $\mathbb{E}\|\mathcal{F}_2(t,\eta)\|^2 \le L_{\mathcal{F}_2}\|\eta\|_{\mathfrak{B}}^2$, $\mathbb{E}\|\mathcal{E}_j(t,\eta_1) - \mathcal{E}_j(t,\eta_2)\|^2 \le L_{\mathcal{E}_j}\|\eta_1 - \eta_2\|_{\mathfrak{B}}^2$, and $\mathbb{E}\|\mathcal{E}_j(t,\eta)\|^2 \le L_{\mathcal{E}_j}\|\eta\|_{\mathfrak{B}}^2$, where $L_{\mathcal{E}_j} = [l_{\mathcal{E}_j}]^2,\ L_{\mathcal{F}_2} = [\|a_{31}\|_\infty l_{\mathcal{F}_2}]^2$. Further, we can impose some suitable conditions on the above-defined functions to verify the hypotheses of the Theorems 3.1 and 4.1. Therefore, the problem ($\mathcal{LP}$) corresponding to the stochastic system (5.1) has at least one optimal control pair.

6. Conclusion

In this manuscript, we studied the stochastic optimal control problem for a class of non-instantaneous impulsive stochastic neutral integro-differential equation driven by fBm. We define a concept of the piecewise continuous mild solutions for the proposed system, which is used to construct a suitable operator and apply fixed point technique to derive the existence result. Also, we prove the existence of optimal controls for the proposed system, which is used to derive optimization conditions. Finally, the obtained results have been verified through an example. There are two direct issues which require further study. First, we will investigate the optimal control problems for the non-instantaneous impulsive stochastic delay differential equations driven by Lévy processes [45]. Second, we will be devoted to studying the approximate controllability for the Markov and semi-Markov switched stochastic system [46, 47].

Acknowledgments

We are very thankful to the anonymous reviewers and editor for their constructive comments and suggestions which help us to improve the manuscript.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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