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Research article

Representation of solution of initial value problem for fuzzy linear multi-term fractional differential equation with continuous variable coefficient

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Abstract: We consider the representation of solutions of the initial value problems of fuzzy linear multi-term in-homogeneous fractional differential equations with continuous variable coefficients.

Keywords: fuzzy fractional differential equation; initial value problem; H-difference
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1. Introduction

Nowadays the fractional differential equations (FDEs) are powerful tools representing many problems in various fields such as viscoelasticity, control engineering, diffusion processes, signal processing and electromagnetism.

Recently many researchers are interested in research on fuzzy fractional differential equations(FFDEs). In one of the earliest papers, Agarwal et al. [1] considered a fractional differential equation of order $\alpha \in$ with uncertainty and introduced the concept of solution for proposed equations. Arshad and Lupulescu [2] presented the existence and uniqueness of solution for fractional differential equations with uncertainty using the concept of solution in means of [1]. Next, Khastan et al. [3] proved the existence result of solution for nonlinear fuzzy fractional differential equations with the Riemann-Liouville derivative by Schauder fixed-point theorem. Allahviranloo et al. [4] proved that uncertain fractional differential equation is equivalent to integral equation under

Riemann-Liouville H-differentiability and obtained the explicit solution in case of the fuzzy linear fractional differential equations. Salahshour et al. [5] studied the analytical methods to solve the FFDE under the fuzzy Caputo fractional differentiability.

Abdollahi et al. introduced the linear fuzzy Caputo-Fabrizio fractional differential equation with initial value problems and presented the general form of their solutions under generalized Hukuhara differentiability in [6] and H. V. Ngo et al. proved that the fractional fuzzy differential equation is not equal to the fractional fuzzy integral equation in general in [7]. Salahshour et al. [8] introduced that the solutions of FFDEs of order $0 < \beta < 1$ are obtained by fuzzy Laplace transforms under Riemann-Liouville H-differentiability. Chehlabi and Allahviranloo [9] investigated the concreted solutions for fuzzy linear fractional differential equations of order $0 < \alpha < 1$ under Riemann-Liouville H-differentiability by using fractional hyperbolic functions and extended the results obtained in [8]. Also Arqub et al. introduced the exact and the numerical solutions of various fuzzy differential equations based on the reproducing kernel Hilbert space method under strongly generalized differentiability [10–12].

Generally, the most of the FFDEs have no known exact solution. Thus the researches of approximate and numerical solutions of the FFDEs are important. Mazandarani and Vahidian Kamyad [13] obtained a fuzzy approximate solution to solve FFDEs with the Caputo-type fuzzy fractional derivative based on Hukuhara difference and strongly generalized fuzzy differentiability by using modified fractional Euler method. Also the operational matrix methods based on orthogonal polynomials have been proposed to solve FFDEs [14–18].

Based on above considerations, we are going to obtain the representation of solutions of the initial value problems of fuzzy linear multi-term in-homogeneous fractional differential equations with continuous variable coefficients.

The paper is organized as follows. In Section 2, we introduced some definitions and properties for fuzzy calculus and proposed basic results of our paper. The representation of solution of proposed problem is described in Section 3. The examples are presented to illustrate our result in Section 4. Finally, the conclusion is summarized in Section 5.

2. Preliminaries and basic results

We denote the set of all fuzzy numbers on $\mathbf{R}$ by $\mathbf{R}_{\mathrm{F}}$. A fuzzy number is a mapping $u : \mathbf{R} \to [0, 1]$ with the following properties:
(i) u is normal, i.e., $\exists x_0 \in \mathbf{R}\;\; u(x_0) = 1$,
(ii) u is a convex fuzzy subset, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}, \forall x, y \in \mathbf{R}\;\; \forall \lambda \in [0, 1]$,
(iii) u is upper semi-continuous on $\mathbf{R}$,
(iv) The set $\overline{\mathrm{supp}(u)}$ is compact in $\mathbf{R}$ (where $\mathrm{supp}(u) := \{x \in \mathbf{R} | u(x) > 0\}$).
Then $\mathbf{R}_{\mathrm{F}}$ is called the space of fuzzy numbers.
We denote the r-level form of the fuzzy number $u \in \mathbf{R}_{\mathrm{F}}$ by $[u]^r := [u_1(r),\;\; u_2(r)], 0 \leq r \leq 1$.
Let's $u, v \in \mathbf{R}_{\mathrm{F}}$. The distance $d : \mathbf{R}_{\mathrm{F}} \times \mathbf{R}_{\mathrm{F}} \to \mathbf{R}_{+}$ is defined by

$$d(u, v) := \sup_{r \in [0,1]} \max\{|u_1(r) - v_1(r)|, |u_2(r) - v_2(r)|\}.$$

We introduce following notations to understand our paper.
$C(I)$ is the set of all continuous real functions on I.

$C^F(I)$ is the set of all continuous fuzzy-valued functions on I.
$AC^F(I)$ is the set of all absolutely continuous fuzzy-valued functions on I.
$L^F(I)$ is the space of all Lebesque integrable fuzzy-valued functions on I, where $I = [0, L]$ and without losing generality, we promise that $L = 1$.
Definition 2.1. [4] Let $x, y \in \mathbf{R}_{\mathrm{F}}$. If there exists $z \in \mathbf{R}_{\mathrm{F}}$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $z = x \ominus_H y$.
Definition 2.2. [8] Let $f : I \to \mathbf{R}_{\mathrm{F}}$ and fix $x_0 \in (0, 1)$.
We say that f is (1)-differentiable at x_0, if for all $h > 0$ sufficiently near to 0, $f(x_0 + h) \ominus_H f(x_0)$, $f(x_0) \ominus_H f(x_0 - h)$ and the relation

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h}$$

exists. Then the limit is defined by $D^{(1)} f(x_0)$.
Definition 2.3. [4] Let $f \in L^F(I)$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as follow

$$I_{0+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x - s)^{\beta - 1} f(s) ds, x \in (0, 1],$$

where $0 < \nu \leq 1$.
Definition 2.4. [4, 7] Let $f : I \to \mathbf{R}_{\mathrm{F}}$, $0 < \beta < 1$. We say that f is fuzzy Riemann-Liouville H-differentiable of order β if

$$I_{0+}^{1-\beta} f(x) = \int_0^x \frac{(x - s)^{-\beta}}{\Gamma(1 - \beta)} f(s) ds, x \in (0, 1]$$

is (1)-differentiable. Then fuzzy Riemann-Liouville H-derivative of order β of function f is denoted ${}^{RL}D_{0+}^{\beta} f(x) := D^{(1)} I_{0+}^{1-\beta} f(x)$.
Definition 2.5. Let $f : I \to \mathbf{R}_{\mathrm{F}}$, $0 < \beta < 1$. We say that f is fuzzy Caputo-type differentiable of order β if H-difference $f(x) \ominus_H f(0)$ exists and $I_{0+}^{1-\beta}(f(x) \ominus_H f(0)) \in AC^F(I)$ satisfies. Then fuzzy Caputo-type derivative of order β of function f is denoted by

$$({}^{c}D_{0+}^{\beta} f)(x) := {}^{RL}D_{0+}^{\beta}(f(x) \ominus_H f(0)), x \in (0, 1].$$

We consider initial value problem of fuzzy multi-term fractional differential equation as

$$\begin{aligned} &{}^{c}D_{0+}^{\alpha} y(x) = f(x, y(x), {}^{c}D_{0+}^{\beta} y(x)), x \in I, \\ &y(0) = y_0, y_0 \in \mathbf{R}_{\mathrm{F}}. \end{aligned} \tag{2.1}$$

Definition 2.6. Let $y : I \to \mathbf{R}_{\mathrm{F}}$. y is called the solution of Eq. (2.1) if y is to be ${}^{c}D_{0+}^{\alpha} y(x), {}^{c}D_{0+}^{\beta} y(x) \in C^F(I)$ and satisfies Eq. (2.1).
Now let consider as follows

$$\begin{aligned} &{}^{c}D_{0+}^{\alpha} y(x) = z(x), x \in I, \\ &y(0) = y_0, y_0 \in \mathbf{R}_{\mathrm{F}}. \end{aligned} \tag{2.2}$$

Theorem 2.1. Let f of Eq. (2.1) be a continuous function with respect to every variable. If $y(x)$ is the solution of Eq. (2.1), the fuzzy-valued function $z(x)$ which is constructed by $z(x) := {}^cD^{\alpha}_{0+}y(x)$ is the solution in $C^F(I)$ of fuzzy integral equation as follows

$$z(x) = f(x, y_0 \oplus \frac{1}{\Gamma(\alpha)} \int_0^x \frac{z(s)}{(x-s)^{1-\alpha}} ds, I^{\alpha-\beta}_{0+} z(x)), x \in I. \tag{2.3}$$

Conversely if $z(x)$ is the solution in $C^F(I)$ of fuzzy integral equation (2.3), $y(x)$ which is constructed by

$$y(x) = y_0 \oplus \frac{1}{\Gamma(\alpha)} \int_0^x \frac{z(s)}{(x-s)^{1-\alpha}} ds$$

is the solution of Eq. (2.1).
Let use the following distance structure in $C^F(I)$

$$\forall u, v \in C^F(I), d^*(u, v) := \max_{t \in I} d(u(t), v(t)).$$

$(C^F(I), d^*)$ is a complete metric space. For any k, let consider distance as

$$\forall u, v \in C^F(I), d^*_k(u, v) := \max_{t \in I} e^{-kt} d(u(t), v(t)).$$

Then the distance d^*_k is equal to the distance d^*. Namely

$$\exists M, m > 0; \forall u, v \in C^F(I), m d^*_k(u, v) \le d^*(u, v) \le M d^*_k(u, v).$$

Theorem 2.2. Assume that the function in Eq. (2.3) is continuous in its all variables and especially, for $\forall y_1, y_2, z_1, z_2 \in \mathbf{R}_{\mathrm{F}}$ and f satisfies the following condition

$$d(f(x, y_1, z_1), f(x, y_2, z_2)) \le L_1 \cdot d(y_1, y_2) + L_2 \cdot d(z_1, z_2).$$

Then the fuzzy integral equation (2.3) has the unique solution.

3. Main result

We consider the representation of solution of initial value problem for fuzzy linear multi-term fractional differential equation with continuous variable coefficient as

$$\begin{aligned} &{}^cD^{\alpha}_{0+}y(x) = a(x){}^cD^{\beta}_{0+}y(x) \oplus b(x)y(x) \oplus g(x), x \in I, 0 < \beta < \alpha < 1, \\ &y(0) = y_0, y_0 \in \mathbf{R}_{\mathrm{F}}, \end{aligned} \tag{3.1}$$

where $a, b \in C(I), g \in C^F(I)$.
First we will obtain the representation of solution for corresponding fuzzy integral equation (2.3).
If $z(x) := {}^cD^{\alpha}_{0+}y(x)$, the following relations satisfy

$$y(x) = y_0 \oplus I^{\alpha}_{0+} z(x).$$

Then we have

$$
\begin{aligned}
& {}^{c}D^{\beta}_{0+}y(x) = {}^{c}D^{\beta}_{0+}I^{\alpha}_{0+}z(x) = I^{\alpha-\beta}_{0+}z(x). \\
z(x) &= a(x)I^{\alpha-\beta}_{0+}z(x) \oplus b(x)(y_0 \oplus I^{\alpha}_{0+}z(x)) \oplus g(x) \\
&= a(x)I^{\alpha-\beta}_{0+}z(x) \oplus b(x)I^{\alpha}_{0+}z(x) \oplus g(x) \oplus b(x)y_0.
\end{aligned} \tag{3.2}
$$

Now let define the operator L by

$$(Lz)(x) := a(x)I^{\alpha-\beta}_{0+}z(x) \oplus b(x)I^{\alpha}_{0+}z(x), \quad x \in I.$$

From $a, b \in C(I)$, it is obvious that $L : C^F(I) \to C^F(I)$. By using the operator L, the integral equation (3.2) can be expressed as

$$z(x) = (Lz)(x) \oplus g(x) \oplus b(x)y_0.$$

Let $\hat{g}(x) := g(x) \oplus b(x)y_0$. Then the integral equation (3.2) is denoted by

$$z(x) = (Lz)(x) \oplus \hat{g}(x). \tag{3.3}$$

Also let denote the operator $\mathcal{I}\ominus_H L$ as

$$(\mathcal{I}\ominus_H L)z := z\ominus_H Lz,$$

where the operator $\mathcal{I}$ is identity operator. From Eq. (3.3), we have

$$
\begin{aligned}
& z(x)\ominus_H(Lz)(x) = \hat{g}(x), \\
& (\mathcal{I} \ominus_H L)z(x) = \hat{g}(x).
\end{aligned} \tag{3.4}
$$

Lemma 3.1. Let $a, b, c, d \in \mathbf{R}_{\mathrm{F}}$. The following relations are satisfied
(1) If $(a \oplus b)\ominus_H b$ exists, then $(a \oplus b)\ominus_H b = a$.
(2) If $(a \oplus b)\ominus_H(c \oplus d), (a\ominus_H c)$ and $(b\ominus_H d)$ exist, then

$$(a \oplus b)\ominus_H(c \oplus d) = (a\ominus_H c) \oplus (b\ominus_H d).$$

Proof. First we prove (1). Now let $E := (a \oplus b)\ominus_H b$. Then we get

$$E \oplus b = a \oplus b.$$

Also for $\forall r \in [0, 1]$, we have $[E]^r = [E^r_-, E^r_+], [a]^r = [a^r_-, a^r_+]$ and $[b]^r = [b^r_-, b^r_+]$. Therefore following relations holds

$$
\begin{aligned}
& [E \oplus b]^r = [a \oplus b]^r, \\
& [E^r_- + b^r_-, E^r_+ + b^r_+] = [a^r_- + b^r_-, a^r_+ + b^r_+], \\
& \begin{cases} E^r_- + b^r_- = a^r_- + b^r_-, \\ E^r_+ + b^r_+ = a^r_+ + b^r_+, \end{cases} \\
& \begin{cases} E^r_- = a^r_-, \\ E^r_+ = a^r_+. \end{cases}
\end{aligned}
$$

Namely $E = a$.
We prove (2). Let $E := (a \oplus b)\ominus_H(c \oplus d)$, $F := (a\ominus_H c) \oplus (b\ominus_H d)$. As $E \oplus (c \oplus d) = a \oplus b$ stands, for $\forall r \in [0,1]$, we get

$$[E^r_- + c^r_- + d^r_-, E^r_+ + c^r_+ + d^r_+] = [a^r_- + b^r_-, a^r_+ + b^r_+],$$
$$\begin{cases} E^r_- + c^r_- + d^r_- = a^r_- + b^r_-, \\ E^r_+ + c^r_+ + d^r_+ = a^r_+ + b^r_+. \end{cases}$$

Therefore

$$\begin{cases} E^r_- = a^r_- + b^r_- - c^r_- - d^r_-, \\ E^r_+ = a^r_+ + b^r_+ - c^r_+ - d^r_+. \end{cases}$$

In the one hand, let $F := (a\ominus_H c) \oplus (b\ominus_H d)$, $G := a\ominus_H c$. Because $G \oplus c = a$ stands, we have

$$[G^r_- + c^r_-, G^r_+ + c^r_+] = [a^r_-, a^r_+],$$
$$\begin{cases} G^r_- = a^r_- - c^r_-, \\ G^r_+ = a^r_+ - c^r_+, \end{cases}$$
$$\begin{cases} F^r_- = a^r_- - c^r_- + b^r_- - d^r_-, \\ F^r_+ = a^r_+ - c^r_+ + b^r_+ - d^r_+. \end{cases}$$

Therefore the proof is completed. □

If $z \in C^F(I)$ is the solution of the fuzzy integral equation (3.3) and $D(\mathcal{I}\ominus_H L)$ is the domain of the operator $\mathcal{I}\ominus_H L$, we can think the following Lemma.

Lemma 3.2. If $z \in D(L)$ is the solution of the integral equation (3.3) and

$$\forall k \in \{0, \cdots\}, L^k \hat{g} \in D(L).$$

Then the following relations are satisfied
(1) $\forall k \in N, L^k z \in D(\mathcal{I}\ominus_H L)$,
(2) $\sum\limits_{k=0}^{n} L^k z \in D(\mathcal{I}\ominus_H L)$,
where $D(L)$ is the domain of the operator L.

Proof. First we prove (1). Because $z \in D(L)$ is the solution of the integral equation (3.3), the following relations are satisfied

$$\begin{aligned} z(x) &= (Lz)(x) \oplus \hat{g}(x), \\ Lz(x) &= (L^2 z)(x) \oplus L\hat{g}(x), \\ Lz(x)\ominus_H(L^2 z)(x) &= L\hat{g}(x), \\ (\mathcal{I}\ominus_H L)Lz(x) &= L\hat{g}(x). \end{aligned} \tag{3.5}$$

Consequently we have $Lz \in D(\mathcal{I}\ominus_H L)$. Similarity we can prove

$$\forall k \in N, \quad L^k z \in D(\mathcal{I}\ominus_H L).$$

Next let prove (2). By the assumption of z, we get

$$z(x) \oplus Lz(x) = (Lz)(x) \oplus \hat{g}(x) \oplus (L^2 z)(x) \oplus L\hat{g}(x).$$

Also the following relation is satisfied

$$\begin{aligned} z(x) \oplus Lz(x) &= (Lz)(x) \oplus (L^2 z)(x) \oplus \hat{g}(x) \oplus L\hat{g}(x) \\ &= L(z(x) \oplus Lz(x)) \oplus \hat{g}(x) \oplus L\hat{g}(x). \end{aligned}$$

Therefore we have

$$(z(x) \oplus Lz(x))\ominus_H L(z(x) \oplus Lz(x)) = \hat{g}(x) \oplus L\hat{g}(x),$$

$$(\mathcal{I}\ominus_H L)(z(x) \oplus Lz(x)) = \hat{g}(x) \oplus L\hat{g}(x),$$

$$z(x) \oplus Lz(x) \in D(\mathcal{I}\ominus_H L).$$

In the same way, we can prove $\sum_{k=0}^{n} L^k z \in D(\mathcal{I}\ominus_H L)$. □

For $z \in D(\mathcal{I}\ominus_H L)$, we can know that

$$(\mathcal{I}\ominus_H L) \circ \sum_{k=0}^{n} L^k z(x) = \sum_{k=0}^{n} L^k z(x)\ominus_H L \circ \sum_{k=0}^{n} L^k z(x) = \sum_{k=0}^{n} L^k z(x)\ominus_H \sum_{k=0}^{n} L^{k+1} z(x)$$

$$= (z(x) \oplus Lz(x) \oplus \cdots \oplus L^n z(x))\ominus_H(Lz(x) \oplus \cdots \oplus L^{n+1} z(x)).$$

where the symbol $\circ$ means that the operator $(\mathcal{I}\ominus_H L)$ is applied to $\sum_{k=0}^{n} L^k z(x)$.

Because $z \in D(\mathcal{I}\ominus_H L)$, $z(x)\ominus_H Lz(x)$ exists. Also applying the operator L to the both of Eq. (3.4), we get

$$L(\mathcal{I}\ominus_H L)z(x) = L\hat{g}(x).$$

Similarity, we can obtain

$$L^n(z(x)\ominus_H Lz(x)) = L^n z(x)\ominus_H L^{n+1} z(x).$$

On the other hand, from (2) of Lemma 3.1, we get

$$\begin{aligned}
z(x) &= (z(x)\ominus_H 0) \oplus (Lz(x)\ominus_H Lz(x)) = (z(x) \oplus Lz(x))\ominus_H(0 \oplus Lz(x)) \\
&= (z(x) \oplus Lz(x))\ominus_H(Lz(x) \oplus 0) = (\mathcal{I} \ominus_H L)z(x) \oplus Lz(x) \\
&= (\mathcal{I} \ominus_H L)z(x) \oplus (\mathcal{I} \ominus_H L)Lz(x) \oplus L^2 z(x) \\
&= (\mathcal{I} \ominus_H L)(z(x) \oplus Lz(x)) \oplus L^2 z(x) \\
&\cdots \\
&= (\mathcal{I}\ominus_H L) \circ \sum_{k=0}^{n} L^k z(x) \oplus L^{n+1} z(x).
\end{aligned}$$

Namely

$$(\mathcal{I}\ominus_H L) \circ \sum_{k=0}^{n} L^k z(x) = z(x)\ominus_H L^{n+1} z(x). \tag{3.6}$$

Let estimate $d_k^*(\hat{0}, Lz)$ for $\forall k > 0$. Since

$$\begin{aligned}
d_k^*(\hat{0}, Lz) &= \max_{t\in I} e^{-kt} d(\hat{0}, Lz(t)) = \max_{t\in I} e^{-kt} d(\hat{0}, a(t)I_{0+}^{\alpha-\beta} z(t) \oplus b(t)I_{0+}^{\alpha} z(t)) \\
&\leq \max_{t\in I} e^{-kt} \left\{ d(\hat{0}, a(t)I_{0+}^{\alpha-\beta} z(t)) + d(\hat{0}, b(t)I_{0+}^{\alpha} z(t)) \right\} \\
&= \max_{t\in I} e^{-kt} \left\{ |a(t)| \cdot d(\hat{0}, I_{0+}^{\alpha-\beta} z(t)) + |b(t)| \cdot d(\hat{0}, I_{0+}^{\alpha} z(t)) \right\},
\end{aligned}$$

$$d(\hat{0}, I_{0+}^{\alpha} z(t)) \leq I_{0+}^{\alpha} d(\hat{0}, z(t)) \leq \frac{e^{kt}}{k^{\alpha}} d_k^*(\hat{0}, z)$$

and

$$d(\hat{0}, I_{0+}^{\alpha-\beta} z(t)) \le \frac{e^{kt}}{k^{\alpha-\beta}} d_k^*(\hat{0}, z),$$

we have

$$\begin{aligned} d_k^*(\hat{0}, Lz) &\le \max_{t\in I} e^{-kt} \left\{ |a(t)| \cdot d(\hat{0}, I_{0+}^{\alpha-\beta} z(t)) + |b(t)| \cdot d(\hat{0}, I_{0+}^{\alpha} z(t)) \right\} \\ &\le \max_{t\in I} e^{-kt} \left\{ |a(t)| \cdot \frac{e^{kt}}{k^{\alpha-\beta}} d_k^*(\hat{0}, z) + |b(t)| \cdot \frac{e^{kt}}{k^{\alpha}} d_k^*(\hat{0}, z) \right\} \\ &= \left\{ \|a(t)\|_{C(I)} \cdot \frac{1}{k^{\alpha-\beta}} + \|b(t)\|_{C(I)} \cdot \frac{1}{k^{\alpha}} \right\} d_k^*(\hat{0}, z). \end{aligned}$$

If there is a positive number k_* satisfying

$$w := \|a(t)\|_{C(I)} \cdot \frac{1}{k_*^{\alpha-\beta}} + \|b(t)\|_{C(I)} \cdot \frac{1}{k_*^{\alpha}} < 1,$$

then we have

$$d_{k_*}^*(\hat{0}, Lz) \le w d_{k_*}^*(\hat{0}, z).$$

Therefore

$$\begin{aligned} &d_{k_*}^*(\hat{0}, L^2 z) \le w d_{k_*}^*(\hat{0}, Lz) \le w^2 d_{k_*}^*(\hat{0}, z), \\ &d_{k_*}^*(\hat{0}, L^n z) \le w^n d_{k_*}^*(\hat{0}, z). \end{aligned} \tag{3.7}$$

Now let denote as

$$S(x) := z(x) \ominus_H L^{n+1} z(x).$$

Then we get

$$S(x) \oplus L^{n+1} z(x) = z(x).$$

Therefore

$$d(z(x) \ominus_H L^{n+1} z(x), z(x)) = d(S(x), S(x) \oplus L^{n+1} z(x)) = d(\hat{0}, L^{n+1} z(x)).$$

From Eq. (3.7), we have

$$d_{k_*}^*(z \ominus_H L^{n+1} z, z) = d_{k_*}^*(\hat{0}, L^{n+1} z) \le w^{n+1} d_{k_*}^*(\hat{0}, z).$$

Namely

$$\lim_{n\to\infty} d_{k_*}^*(z \ominus_H L^{n+1} z, z) = 0.$$

Consequently from Eq. (3.6), we hold

$$(\mathcal{I} \ominus_H L) \circ \sum_{k=0}^{\infty} L^k z(x) = z(x).$$

Similarity from Eq. (3.4), Since

$$\sum_{k=0}^{\infty} L^k (\mathcal{I} \ominus_H L) z(x) = \sum_{k=0}^{\infty} L^k \hat{g}(x),$$

we get

$$z(x) = \sum_{k=0}^{\infty} L^k \hat{g}(x). \tag{3.8}$$

The following Theorem gives the representation of solution for the proposed problem (3.1).

Theorem 3.1. If $\forall k \in \{0, 1, \cdots\}, L^k\hat{g} \in D(L)$ holds, the initial value problem (3.1) has the representation of the solution as follows

$$y(x) = y_0 \oplus I^{\alpha}_{0+}g(x) \oplus I^{\alpha}_{0+}(b(x)y_0) \oplus \sum_{k=1}^{\infty} I^{\alpha}_{0+}(a(x)I^{\alpha-\beta}_{0+} \oplus b(x)I^{\alpha}_{0+})^{k} \hat{g}(x).$$

Proof. Eq. (3.8) can be rewritten as

$$z(x) = \hat{g}(x) \oplus \sum_{k=1}^{\infty} L^k \hat{g}(x) = \hat{g}(x) \oplus \sum_{k=1}^{\infty} (a(x)I^{\alpha-\beta}_{0+} \oplus b(x)I^{\alpha}_{0+})^{k} \hat{g}(x).$$

Therefore we obtain

$$\begin{aligned} y(x) &= y_0 \oplus I^{\alpha}_{0+}z(x) = y_0 \oplus I^{\alpha}_{0+}(\hat{g}(x) \oplus \sum_{k=1}^{\infty} (a(x)I^{\alpha-\beta}_{0+} \oplus b(x)I^{\alpha}_{0+})^{k} \hat{g}(x)) \\ &= y_0 \oplus I^{\alpha}_{0+}\hat{g}(x) \oplus \sum_{k=1}^{\infty} I^{\alpha}_{0+}(a(x)I^{\alpha-\beta}_{0+} \oplus b(x)I^{\alpha}_{0+})^{k} \hat{g}(x) \\ &= y_0 \oplus I^{\alpha}_{0+}(g(x) \oplus b(x)y_0) \oplus \sum_{k=1}^{\infty} I^{\alpha}_{0+}(a(x)I^{\alpha-\beta}_{0+} \oplus b(x)I^{\alpha}_{0+})^{k} \hat{g}(x) \\ &= y_0 \oplus I^{\alpha}_{0+}g(x) \oplus I^{\alpha}_{0+}(b(x)y_0) \oplus \sum_{k=1}^{\infty} I^{\alpha}_{0+}(a(x)I^{\alpha-\beta}_{0+} \oplus b(x)I^{\alpha}_{0+})^{k} \hat{g}(x). \end{aligned}$$

The proof is completed. □

4. Examples

We consider the analytical representation of solution for the fuzzy fractional differential equation as following

$$\begin{aligned} &{}^cD^{\alpha}_{0+}y(x) = \lambda \odot y(x),\ x \in I, \\ &y(0) = y_0, y_0 \in \mathbf{R}_{\mathrm{F}}. \end{aligned} \tag{4.1}$$

By using Theorem 3.3, we get $\hat{g}(x) := \lambda y_0$, $(Lz)(x) := \lambda I^{\alpha}_{0+}z(x), \quad x \in I$. Therefore the conditions of Theorem 3.1 are satisfied.

Corollary 4.1. The initial value problem (4.1) have the solution as follows
in case of $\lambda > 0$,

$$y(x) = E_{\alpha}(\lambda x^{\alpha}) \odot y_0,$$

in case of $\lambda < 0$,

$$y(x) = E_{2\alpha,1}(\lambda^2 x^{2\alpha}) \odot y_0 \oplus \lambda x^{\alpha} E_{2\alpha,\alpha+1}(\lambda^2 x^{2\alpha}) \odot y_0.$$

Proof. In case of $\lambda > 0$, we get

$$\begin{aligned}
y(x) &= y_0 \oplus I_{0+}^{\alpha}(\lambda \odot y_0) \oplus \sum_{k=1}^{\infty} I_{0+}^{\alpha}(\lambda I_{0+}^{\alpha})^k \lambda \odot y_0 \\
&= y_0 \oplus I_{0+}^{\alpha}(\lambda \odot y_0) \oplus \sum_{k=1}^{\infty} \lambda^{k+1}(I_{0+}^{\alpha})^{k+1} \odot y_0 \\
&= y_0 \oplus \lambda \frac{x^{\alpha}}{\Gamma(\alpha+1)} \odot y_0 \oplus \sum_{k=1}^{\infty} \lambda^{k+1} \frac{x^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} \odot y_0 \\
&= y_0 \oplus \sum_{k=0}^{\infty} \lambda^{k+1} \frac{x^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} \odot y_0 \\
&= \sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^k}{\Gamma(k\alpha+1)} \odot y_0 = E_{\alpha}(\lambda x^{\alpha}) \odot y_0.
\end{aligned}$$

Also in case of $\lambda < 0$, we have

$$\begin{aligned}
y(x) &= y_0 \oplus I_{0+}^{\alpha}(\lambda \odot y_0) \oplus \sum_{k=1}^{\infty} I_{0+}^{\alpha}(\lambda I_{0+}^{\alpha})^k (\lambda \odot y_0) \\
&= y_0 \oplus I_{0+}^{\alpha}(\lambda \odot y_0) \oplus \sum_{k=1}^{\infty} \lambda^{k+1}(I_{0+}^{\alpha})^{k+1} \odot y_0 \\
&= y_0 \oplus \lambda \frac{x^{\alpha}}{\Gamma(\alpha+1)} \odot y_0 \oplus \left(\sum_{k=1}^{\infty} \lambda^{2k}(I_{0+}^{\alpha})^{2k} 1\right) \odot y_0 \oplus \left(\sum_{k=1}^{\infty} \lambda^{2k+1}(I_{0+}^{\alpha})^{2k+1} 1\right) \odot y_0 \\
&= y_0 \oplus \left(\sum_{k=1}^{\infty} \lambda^{2k} \frac{x^{2k\alpha}}{\Gamma(2k\alpha+1)}\right) \odot y_0 \oplus \lambda \frac{x^{\alpha}}{\Gamma(\alpha+1)} \odot y_0 \oplus \left(\sum_{k=1}^{\infty} \lambda^{2k+1} \frac{x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}\right) \odot y_0 \\
&= y_0 \oplus \left(\sum_{k=1}^{\infty} \lambda^{2k} \frac{x^{2k\alpha}}{\Gamma(2k\alpha+1)}\right) \odot y_0 \oplus \left(\sum_{k=0}^{\infty} \lambda^{2k+1} \frac{x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}\right) \odot y_0 \\
&= \left(\sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^{2k}}{\Gamma(2k\alpha+1)}\right) \odot y_0 \oplus \lambda x^{\alpha} \left(\sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^{2k}}{\Gamma(2k\alpha+\alpha+1)}\right) \odot y_0 \\
&= E_{2\alpha,1}(\lambda^2 x^{2\alpha}) \odot y_0 \oplus \lambda x^{\alpha} E_{2\alpha,\alpha+1}(\lambda^2 x^{2\alpha}) \odot y_0.
\end{aligned}$$

□

Next we consider the representation of solution for initial value problem of the non-homogeneous fuzzy fractional differential equation with constant coefficient as following

$$\begin{aligned}
&{}^{c}D_{0+}^{\alpha} y(x) = \lambda \odot y(x) + g(x), x \in I, \\
&y(0) = y_0, y_0 \in \mathbf{R}_{\mathrm{F}}.
\end{aligned} \tag{4.2}$$

Let's denote $\hat{g}(x) := g(x) \oplus \lambda y_0$, $(Lz)(x) := \lambda I_{0+}^{\alpha} z(x), x \in I$ in order to use Theorem 3.1.

Corollary 4.2. If $\forall k \in \{0, 1, \cdots\}, L^k g \in D(L)$, then initial value problem (4.2) have the solution as follows

in case of $\lambda > 0$,

$$y(t) = E_{\alpha}(\lambda x^{\alpha}) \odot y_0 \oplus \int_0^x (x-t)^{\alpha-1} g(t) E_{\alpha,\alpha}(\lambda(x-t)^{\alpha}) dt,$$

in case of $\lambda < 0$,

$$y(x) = E_{2\alpha,1}(\lambda^2 x^{2\alpha}) \odot y_0 \oplus \lambda x^\alpha E_{2\alpha,\alpha+1}(\lambda^2 x^{2\alpha}) \odot y_0 \\ \oplus \int\limits_0^x (x-t)^{\alpha-1} E_{2\alpha,\alpha}(\lambda^2 (x-t)^{2\alpha}) g(t)dt \oplus \lambda \int\limits_0^x (x-t)^{2\alpha-1} E_{2\alpha,2\alpha}(\lambda^2 (x-t)^{2\alpha}) g(t)dt.$$

Proof. In case of $\lambda > 0$, we get

$$\begin{aligned} y(x) &= y_0 \oplus I_{0+}^\alpha g(x) \oplus I_{0+}^\alpha (\lambda \odot y_0) \oplus \sum_{k=1}^\infty I_{0+}^\alpha (\lambda I_{0+}^\alpha)^k (g(x) \oplus \lambda \odot y_0) \\ &= y_0 \oplus I_{0+}^\alpha (\lambda \odot y_0) \oplus \sum_{k=1}^\infty I_{0+}^\alpha (\lambda I_{0+}^\alpha)^k \lambda \odot y_0 \oplus I_{0+}^\alpha g(x) \oplus \sum_{k=1}^\infty I_{0+}^\alpha (\lambda I_{0+}^\alpha)^k g(x) \\ &= E_\alpha(\lambda x^\alpha) \odot y_0 \oplus \sum_{k=0}^\infty \lambda^k I_{0+}^{(k+1)\alpha} g(x) \\ &= E_\alpha(\lambda x^\alpha) \odot y_0 \oplus \sum_{k=0}^\infty \lambda^k \frac{1}{\Gamma((k+1)\alpha)} \int\limits_0^x \frac{g(t)}{(x-t)^{1-(k+1)\alpha}} dt \\ &= E_\alpha(\lambda x^\alpha) \odot y_0 \oplus \int\limits_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda (x-t)^\alpha) g(t)dt. \end{aligned}$$

And in case of $\lambda < 0$, we obtain

$$\begin{aligned} y(x) &= y_0 \oplus I_{0+}^\alpha g(x) \oplus I_{0+}^\alpha (\lambda y_0) \oplus \sum_{k=1}^\infty I_{0+}^\alpha (\lambda I_{0+}^\alpha)^k (g(x) \oplus \lambda y_0) \\ &= y_0 \oplus \sum_{k=0}^\infty \lambda \odot^{2k} I_{0+}^{(2k+1)\alpha} g(x) \oplus \sum_{k=0}^\infty \lambda^{2k+1} I_{0+}^{(2k+1)\alpha} y_0 \oplus \sum_{k=1}^\infty \lambda^{2k-1} I_{0+}^{2k\alpha} g(x) \oplus \sum_{k=1}^\infty \lambda^{2k-1} I_{0+}^{2k\alpha} \lambda \odot y_0 \\ &= \sum_{k=0}^\infty \lambda^{2k} I_{0+}^{2k\alpha} \odot y_0 \oplus \sum_{k=0}^\infty \lambda^{2k+1} I_{0+}^{(2k+1)\alpha} y_0 \oplus \sum_{k=0}^\infty \lambda^{2k} I_{0+}^{(2k+1)\alpha} g(x) \oplus \sum_{k=1}^\infty \lambda^{2k-1} I_{0+}^{2k\alpha} g(x) \\ &= \left(\sum_{k=0}^\infty \frac{(\lambda^2 x^{2\alpha})^k}{\Gamma(2k\alpha+1)} \right) \odot y_0 \oplus \lambda x^\alpha \left(\sum_{k=0}^\infty \frac{(\lambda^2 x^{2\alpha})^k}{\Gamma(2k\alpha+\alpha+1)} \right) \odot y_0 \\ &\quad \oplus \sum_{k=0}^\infty \lambda^{2k} \frac{1}{\Gamma((2k+1)\alpha)} \int\limits_0^x \frac{g(t)}{(x-t)^{1-(2k+1)\alpha}} dt \oplus \sum_{k=1}^\infty \lambda^{2k-1} \frac{1}{\Gamma(2k\alpha)} \int\limits_0^x \frac{g(t)}{(x-t)^{1-2k\alpha}} dt \\ &= E_{2\alpha,1}(\lambda^2 x^{2\alpha}) \odot y_0 \oplus \lambda x^\alpha E_{2\alpha,\alpha+1}(\lambda^2 x^{2\alpha}) \odot y_0 \\ &\quad \oplus \int\limits_0^x (x-t)^{\alpha-1} E_{2\alpha,\alpha}(\lambda^2 (x-t)^{2\alpha}) g(t)dt \oplus \int\limits_0^x (x-t)^{\alpha-1} \left(\sum_{k=0}^\infty \lambda^{2k+1} \frac{(x-t)^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\alpha)} \right) g(t)dt \\ &= E_{2\alpha,1}(\lambda^2 x^{2\alpha}) \odot y_0 \oplus \lambda x^\alpha E_{2\alpha,\alpha+1}(\lambda^2 x^{2\alpha}) \odot y_0 \\ &\quad \oplus \int\limits_0^x (x-t)^{\alpha-1} E_{2\alpha,\alpha}(\lambda^2 (x-t)^{2\alpha}) g(t)dt \oplus \lambda \int\limits_0^x (x-t)^{2\alpha-1} E_{2\alpha,2\alpha}(\lambda^2 (x-t)^{2\alpha}) g(t)dt. \end{aligned}$$

5. Conclusion

In this manuscript, we have studied the representation of the solution for initial value problem for fuzzy linear multi-term fractional differential equations with continuous variable coefficients. We obtained the representation of solutions for proposed problem by using the representation of solution of the corresponding fuzzy integral equations.

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Conflict of interest

The authors declare no conflict of interest.

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