AIMS Mathematics
http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(3): 593–612.
DOI:10.3934/math.2019.3.593
Received: 11 March 2019
Accepted: 09 May 2019
Published: 03 June 2019

Research article

Theory of discrete fractional Sturm–Liouville equations and visual results

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Abstract: In this article, we study discrete fractional Sturm-Liouville (DFSL) operators within Riemann-Liouville and Grünwald-Letnikov fractional operators with both delta and nabla operators. Self-adjointness of the DFSL operator is analyzed and fundamental spectral properties are proved. Besides, we get sum representation of solutions for DFSL problem by means of Laplace transform for nabla fractional difference equations and find the analytical solutions of the problem. Moreover, the results for DFSL problem, discrete Sturm-Liouville (DSL) problem with the second order, and fractional Sturm-Liouville (FSL) problem are compared with the second order classical Sturm-Liouville (CSL) problem. We display the results comparatively by tables and figures.

Keywords: fractional difference equations; Laplace transform; fractional differential equations; Sturm-Liouville; eigenfunctions, eigenvalues
Mathematics Subject Classification: Primary: 34A08, 39A70, 34B24, 28A75; Secondary: 44A10

1. Introduction

Fractional calculus is a notably attractive subject owing to having wide-ranging application areas of theoretical and applied sciences. Despite the fact that there are a large number of worthwhile mathematical works on the fractional differential calculus, there is no noteworthy parallel improvement of fractional difference calculus up to lately. This statement has shown that discrete fractional calculus has certain unforeseen hardship.

Fractional sums and differences were obtained firstly in Diaz-Osler [1], Miller-Ross [2] and Gray and Zhang [3] and they found discrete types of fractional integrals and derivatives. Later, several authors began to touch upon discrete fractional calculus; Goodrich-Peterson [4], Baleanu et al. [5], Ahrendt et al. [6]. Nevertheless, discrete fractional calculus is a rather novel area. The first studies have been done by Atıcı et al. [7–11], Abdeljawad et al. [12–14], Mozyrska et al. [15–17], Anastassiou [18, 19], Hein et al. [20] and Cheng et al. [21] and so forth [22–26].

Self-adjoint operators have an important place in differential operators. Levitan and Sargsian [27] studied self-adjoint Sturm-Liouville differential operators and they obtained spectral properties based on self-adjointness. Also, they found representation of solutions and hence they obtained asymptotic formulas of eigenfunctions and eigenvalues. Similarly, Dehghan and Mingarelli [28, 29] obtained for the first time representation of solution of fractional Sturm-Liouville problem and they obtained asymptotic formulas of eigenfunctions and eigenvalues of the problem. In this study, firstly we obtain self-adjointness of DFSL operator within nabla fractional Riemann-Liouville and delta fractional Grünwald-Letnikov operators. From this point of view, we obtain orthogonality of distinct eigenfunctions, reality of eigenvalues. In addition, we open a new gate by obtaining representation of solution of DFSL problem for researchers study in this area.

Self-adjointness of fractional Sturm-Liouville differential operators have been proven by Bas et al. [30, 31], Klimek et al. [32, 33]. Variational properties of fractional Sturm-Liouville problem has been studied in [34, 35]. However, self-adjointness of conformable Sturm-Liouville and DFSL with Caputo-Fabrizio operator has been proven by [36, 37]. Nowadays, several studies related to Atangana-Baleanu fractional derivative and its discrete version are done [38–45].

In this study, we consider DFSL operators within Riemann-Liouville and Grünwald-Letnikov sense, and we prove the self-adjointness, orthogonality of distinct eigenfunctions, reality of eigenvalues of DFSL operator. However, we get sum representation of solutions for DFSL equation by means Laplace transform for nabla fractional difference equations. Finally, we compare the results for the solution of DFSL problem, discrete Sturm-Liouville (DSL) problem with the second order, fractional Sturm-Liouville (FSL) problem and classical Sturm-Liouville (CSL) problem with the second order. The aim of this paper is to contribute to the theory of DFSL operator.

We discuss DFSL equations in three different ways with;
i) Self-adjoint (nabla left and right) Riemann-Liouville (**R-L**) fractional operator,

$$L_1 x(t) = \nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} x(t) \right) + q(t) x(t) = \lambda r(t) x(t), \; 0 < \mu < 1,$$

ii) Self-adjoint (delta left and right) Grünwald-Letnikov (**G-L**) fractional operator,

$$L_2 x(t) = \Delta_-^{\mu} \left(p(t) \Delta_+^{\mu} x(t) \right) + q(t) x(t) = \lambda r(t) x(t), \; 0 < \mu < 1,$$

iii)(nabla left) DFSL operator is defined by **R-L** fractional operator,

$$L_3 x(t) = \nabla_a^{\mu} \left(\nabla_a^{\mu} x(t) \right) + q(t) x(t) = \lambda x(t), \; 0 < \mu < 1.$$

2. Preliminaries

Definition 2.1. [4] Delta and nabla difference operators are defined by respectively

$$\Delta x(t) = x(t+1) - x(t), \qquad \nabla x(t) = x(t) - x(t-1). \tag{1}$$

Definition 2.2. [46] Falling function is defined by, $\alpha \in \mathbb{R}$

$$t^{\underline{\alpha}} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}, \tag{2}$$

where Γ is Euler gamma function.

Definition 2.3. [46] Rising function is defined by,$\alpha \in \mathbb{R}$,

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}. \tag{3}$$

Remark 1. Delta and nabla operators have the following properties

$$\Delta t^{\underline{\alpha}} = \alpha t^{\underline{\alpha-1}}, \tag{4}$$

$$\nabla t^{\overline{\alpha}} = \alpha t^{\overline{\alpha-1}}.$$

Definition 2.4. [2, 7] Fractional sum operators are defined by,
(i) The left defined nabla fractional sum with order $\mu > 0$ is defined by

$$\nabla_a^{-\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\mu-1}} x(s),\ t \in \mathbb{N}_{a+1}, \tag{5}$$

(ii) The right defined nabla fractional sum with order $\mu > 0$ is defined by

$$_b\nabla^{-\mu} x(t) = \frac{1}{\Gamma(\mu)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{\mu-1}} x(s),\ t \in {}_{b-1}\mathbb{N}, \tag{6}$$

where $\rho(t) = t-1$ is called backward jump operators, $\mathbb{N}_a = \{a, a+1, ...\}$, $_b\mathbb{N} = \{b, b-1, ...\}$.

Definition 2.5. [47] Fractional difference operators are defined by,
(i) The nabla left fractional difference of order $\mu > 0$ is defined

$$\nabla_a^{\mu} x(t) = \nabla^n \nabla_a^{-(n-\mu)} x(t) = \frac{\nabla^n}{\Gamma(n-\mu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{n-\mu-1}} x(s),\ t \in \mathbb{N}_{a+1}, \tag{7}$$

(ii) The nabla right fractional difference of order $\mu > 0$ is defined

$$_b\nabla^{\mu} x(t) = (-1)^n \Delta^n\ _b\nabla^{-(n-\mu)} x(t) = \frac{(-1)^n \Delta^n}{\Gamma(n-\mu)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{n-\mu-1}} x(s),\ t \in {}_{b-1}\mathbb{N}. \tag{8}$$

Fractional differences in $(7-8)$ are called the **Riemann-Liouville (R-L)** definition of the μ-th order nabla fractional difference.

Definition 2.6. [1, 21, 48] Fractional difference operators are defined by,
(i) The left defined delta fractional difference of order μ, $0 < \mu \leq 1$, is defined by

$$\Delta_-^{\mu} x(t) = \frac{1}{h^{\mu}} \sum_{s=0}^{t} (-1)^s \frac{\mu(\mu-1)...(\mu-s+1)}{s!} x(t-s),\ t = 1, ..., N. \tag{9}$$

(ii) The right defined delta fractional difference of order μ, $0 < \mu \leq 1$, is defined by

$$\Delta_+^{\mu} x(t) = \frac{1}{h^{\mu}} \sum_{s=0}^{N-t} (-1)^s \frac{\mu(\mu-1)...(\mu-s+1)}{s!} x(t+s),\ t = 0, .., N-1. \tag{10}$$

Fractional differences in $(9-10)$ are called the **Grünwald-Letnikov (G-L)** definition of the μ-th order delta fractional difference.

Theorem 2.7. *[47] We define the summation by parts formula for **R-L** fractional nabla difference operator, u is defined on ${}_b\mathbb{N}$ and v is defined on $\mathbb{N}_a$, then*

$$\sum_{s=a+1}^{b-1} u(s)\,\nabla_a^{\mu} v(s) = \sum_{s=a+1}^{b-1} v(s)_b\,\nabla^{\mu} u(s)\,. \tag{11}$$

Theorem 2.8. *[26, 48] We define the summation by parts formula for **G-L** delta fractional difference operator, u, v is defined on $\{0, 1, ..., n\}$, then*

$$\sum_{s=0}^{n} u(s)\,\Delta_-^{\mu} v(s) = \sum_{s=0}^{n} v(s)\,\Delta_+^{\mu} u(s)\,. \tag{12}$$

Definition 2.9. [20] $f : \mathbb{N}_a \to \mathbb{R}$, $s \in \mathfrak{R}$, Laplace transform is defined as follows,

$$\mathfrak{L}_a\{f\}(s) = \sum_{k=1}^{\infty} (1-s)^{k-1} f(a+k)\,,$$

where $\mathfrak{R} = \mathbb{C}\backslash\{1\}$ and $\mathfrak{R}$ is called the set of regressive (complex) functions.

Definition 2.10. [20] Let $f, g : \mathbb{N}_a \to \mathbb{R}$, all $t \in \mathbb{N}_{a+1}$, convolution property of f and g is given by

$$(f * g)(t) = \sum_{s=a+1}^{t} f(t - \rho(s) + a)\,g(s)\,,$$

where $\rho(s)$ is the backward jump function defined in [46] as

$$\rho(s) = s - 1.$$

Theorem 2.11. *[20] $f, g : \mathbb{N}_a \to \mathbb{R}$, convolution theorem is expressed as follows,*

$$\mathfrak{L}_a\{f * g\}(s) = \mathfrak{L}_a\{f\}\,\mathfrak{L}_a\{g\}(s)\,.$$

Lemma 2.12. *[20] $f : \mathbb{N}_a \to \mathbb{R}$, the following property is valid,*

$$\mathfrak{L}_{a+1}\{f\}(s) = \frac{1}{1-s}\mathfrak{L}_a\{f\}(s) - \frac{1}{1-s} f(a+1)\,.$$

Theorem 2.13. *[20] $f : \mathbb{N}_a \to \mathbb{R}$, $0 < \mu < 1$, Laplace transform of nabla fractional difference*

$$\mathfrak{L}_{a+1}\{\nabla_a^{\mu} f\}(s) = s^{\mu}\mathfrak{L}_{a+1}\{f\}(s) - \frac{1-s^{\mu}}{1-s} f(a+1)\,,\ t \in \mathbb{N}_{a+1}.$$

Definition 2.14. [20] For $|p| < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $t \in \mathbb{N}_a$, discrete Mittag-Leffler function is defined by

$$E_{p,\alpha,\beta}(t,a) = \sum_{k=0}^{\infty} p^k \frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k + \beta + 1)},$$

where $t^{\overline{n}} = \begin{cases} t(t+1)\cdots(t+n-1)\,, & n \in \mathbb{Z} \\ \frac{\Gamma(t+n)}{\Gamma(t)}, & n \in \mathbb{R} \end{cases}$ is rising factorial function.

Theorem 2.15. *[20] For $|p| < 1$, $\alpha > 0, \beta \in \mathbb{R}$, $|1 - s| < 1$, and $|s|^{\alpha} > p$, Laplace transform of discrete Mittag-Leffler function is as follows,*

$$\mathfrak{L}_a \left\{ E_{p,\alpha,\beta} (., a) \right\} (s) = \frac{s^{\alpha-\beta-1}}{s^{\alpha} - p}.$$

Definition 2.16. Laplace transform of $f(t) \in \mathbb{R}^+$, $t \geq 0$ is defined as follows,

$$\mathfrak{L} \{f\} (s) = \int_0^{\infty} e^{-st} f(t) \, dt.$$

Theorem 2.17. *For $z, \theta \in \mathbb{C}, Re(\delta) > 0$, Mittag-Leffler function with two parameters is defined as follows*

$$E_{\delta,\theta} (z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\delta k + \theta)}.$$

Theorem 2.18. *Laplace transform of Mittag-Leffler function is as follows*

$$\mathfrak{L} \left\{ t^{\theta-1} E_{\delta,\theta} \left(\lambda t^{\delta} \right) \right\} (s) = \frac{s^{\delta-\theta}}{s^{\delta} - \lambda}.$$

Property 2.19. [28] $f : \mathbb{N}_a \to \mathbb{R}$, $0 < \mu < 1$, Laplace transform of fractional derivative in Caputo sense is as follows, $0 < \alpha < 1$,

$$\mathfrak{L} \left\{ {}^{C}D^{\alpha}_{0^+} f \right\} (s) = s^{\alpha} \mathfrak{L} \{f\} (s) - s^{\alpha-1} f(0).$$

Property 2.20. [28] $f : \mathbb{N}_a \to \mathbb{R}$, $0 < \mu < 1$, Laplace transform of left fractional derivative in Riemann-Liouville sense is as follows, $0 < \alpha < 1$,

$$\mathfrak{L} \{D^{\alpha}_{0^+} f\} (s) = s^{\alpha} \mathfrak{L} \{f\} (s) - I^{1-\alpha}_{0^+} f(t)\big|_{t=0},$$

here $I^{\alpha}_{0^+}$ is left fractional integral in Riemann-Liouville sense.

3. Main results

3.1. Discrete fractional Sturm-Liouville equations

We consider discrete fractional Sturm-Liouville equations in three different ways as follows:
First Case : Self-adjoint L_1 DFSL operator is defined by (nabla right and left) **R-L** fractional operator,

$$L_1 x(t) = \nabla^{\mu}_{a} (p(t)_b \nabla^{\mu} x(t)) + q(t) x(t) = \lambda r(t) x(t), \; 0 < \mu < 1, \tag{13}$$

where $p(t) > 0$, $r(t) > 0$, $q(t)$ is a real valued function on $[a+1, b-1]$ and real valued, λ is the spectral parameter, $t \in [a+1, b-1]$, $x(t) \in l^2 [a+1, b-1]$. In $\ell^2 (a+1, b-1)$, the Hilbert space of sequences of complex numbers $u(a+1), ..., u(b-1)$ with the inner product is given by,

$$\langle u(n), v(n) \rangle = \sum_{n=a+1}^{b-1} u(n) v(n),$$

for every $u \in D_{L_1}$, let's define as follows

$$D_{L_1} = \left\{ u(n),\ v(n) \in \ell^2 (a+1, b-1) : L_1 u(n),\ L_1 v(n) \in \ell^2 (a+1, b-1) \right\}.$$

Second Case : Self-adjoint L_2 DFSL operator is defined by(delta left and right) **G-L** fractional operator,

$$L_2 x(t) = \Delta_-^{\mu} \left(p(t) \Delta_+^{\mu} x(t) \right) + q(t) x(t) = \lambda r(t) x(t),\ 0 < \mu < 1, \tag{14}$$

where p, r, λ is as defined above, $q(t)$ is a real valued function on $[0, n]$, $t \in [0, n]$, $x(t) \in l^2 [0, n]$. In $\ell^2 (0, n)$, the Hilbert space of sequences of complex numbers $u(0), ..., u(n)$ with the inner product is given by, n is a finite integer,

$$\langle u(i), r(i) \rangle = \sum_{i=0}^{n} u(i) r(i),$$

for every $u \in D_{L_2}$, let's define as follows

$$D_{L_2} = \left\{ u(i),\ v(i) \in \ell^2 (0, n) : L_2 u(n),\ L_2 r(n) \in \ell^2 (0, n) \right\}.$$

Third Case :L_3 DFSL operator is defined by (nabla left) **R-L** fractional operator,

$$L_3 x(t) = \nabla_a^{\mu} \left(\nabla_a^{\mu} x(t) \right) + q(t) x(t) = \lambda x(t),\ 0 < \mu < 1, \tag{15}$$

p, r, λ is as defined above, $q(t)$ is a real valued function on $[a+1, b-1]$, $t \in [a+1, b-1]$.

Firstly, we consider the first case and give the following theorems and proofs;

Theorem 3.1. *DFSL operator L_1 is self-adjoint.*

Proof.

$$u(t) L_1 v(t) = u(t) \nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} v(t) \right) + u(t) q(t) v(t), \tag{16}$$

$$v(t) L_1 u(t) = v(t) \nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} u(t) \right) + v(t) q(t) u(t). \tag{17}$$

If $(16 - 17)$ is subtracted from each other

$$u(t) L_1 v(t) - v(t) L_1 u(t) = u(t) \nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} v(t) \right) - v(t) \nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} u(t) \right)$$

and sum operator from $a+1$ to $b-1$ to both side of the last equality is applied, we get

$$\sum_{s=a+1}^{b-1} \left(u(s) L_1 v(s) - v(s) L_1 u(s) \right) = \sum_{s=a+1}^{b-1} u(s) \nabla_a^{\mu} \left(p(s)_b \nabla^{\mu} v(s) \right) \tag{18}$$

$$- \sum_{s=a+1}^{b-1} v(s) \nabla_a^{\mu} \left(p(s)_b \nabla^{\mu} u(s) \right).$$

If we apply the summation by parts formula in (11) to right hand side of (18), we have

$$\sum_{s=a+1}^{b-1} \left(u(s) L_1 v(s) - v(s) L_1 u(s) \right) = \sum_{s=a+1}^{b-1} p(s)_b \nabla^{\mu} v(s)_b \nabla^{\mu} u(s)$$

$$
\begin{aligned}
&- \sum_{s=a+1}^{b-1} p(s)_b \nabla^{\mu} u(s)_b \nabla^{\mu} v(s) \\
= \;& 0,
\end{aligned}
$$

$$\langle L_1 u, v\rangle = \langle u, L_1 v\rangle .$$

Hence, the proof completes. □

Theorem 3.2. *Two eigenfunctions, $u(t, \lambda_\alpha)$ and $v(t, \lambda_\beta)$, of the equation* (13) *are orthogonal as $\lambda_\alpha \neq \lambda_\beta$.*

Proof. Let λ_α and λ_β are two different eigenvalues corresponds to eigenfunctions $u(t)$ and $v(t)$ respectively for the the equation (13),

$$
\begin{aligned}
\nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} u(t)\right) + q(t) u(t) - \lambda_\alpha r(t) u(t) &= 0, \\
\nabla_a^{\mu} \left(p(t)_b \nabla^{\mu} v(t)\right) + q(t) v(t) - \lambda_\beta r(t) v(t) &= 0.
\end{aligned}
$$

If we multiply last two equations by $v(t)$ and $u(t)$ respectively, subtract from each other and apply definite sum operator, owing to the self-adjointness of the operator L_1, we have

$$\left(\lambda_\alpha - \lambda_\beta\right) \sum_{s=a+1}^{b-1} r(s) u(s) v(s) = 0,$$

since $\lambda_\alpha \neq \lambda_{\beta,}$

$$
\begin{aligned}
\sum_{s=a+1}^{b-1} r(s) u(s) v(s) &= 0, \\
\langle u(t), v(t)\rangle &= 0.
\end{aligned}
$$

Hence, the proof completes. □

Theorem 3.3. *All eigenvalues of the equation* (13) *are real.*

Proof. Let $\lambda = \alpha + i\beta$, owing to the self-adjointness of the operator L_1, we can write

$$
\begin{aligned}
\langle L_1 u(t), u(t)\rangle &= \langle u(t), L_1 u(t)\rangle, \\
\langle \lambda r u(t), u(t)\rangle &= \langle u(t), \lambda r(t) u(t)\rangle,
\end{aligned}
$$

$$\left(\lambda - \overline{\lambda}\right) \langle u(t), u(t)\rangle_r = 0.$$

Since $\langle u(t), u(t)\rangle_r \neq 0$,

$$\lambda = \overline{\lambda}$$

and hence $\beta = 0$. The proof completes. □

Secondly, we consider the second case and give the following theorems and proofs;

Theorem 3.4. *DFSL operator L_2 is self-adjoint.*

Proof.

$$u(t)L_2v(t)=u(t)\Delta_-^{\mu}(p(t)\Delta_+^{\mu}v(t))+u(t)q(t)v(t), \tag{19}$$

$$v(t)L_2u(t)=v(t)\Delta_-^{\mu}(p(t)\Delta_+^{\mu}u(t))+v(t)q(t)u(t). \tag{20}$$

If $(19-20)$ is subtracted from each other

$$u(t)L_2v(t)-v(t)L_2u(t)=u(t)\Delta_-^{\mu}(p(t)\Delta_+^{\mu}v(t))-v(t)\Delta_-^{\mu}(p(t)\Delta_+^{\mu}u(t))$$

and definite sum operator from 0 to t to both side of the last equality is applied, we have

$$\sum_{s=0}^{t}(u(s)L_1v(s)-v(s)L_2u(s))=\sum_{s=0}^{t}u(s)\Delta_-^{\mu}(p(s)\Delta_+^{\mu}v(s))-\sum_{s=0}^{t}v(s)\Delta_-^{\mu}(p(s)\Delta_+^{\mu}u(s)). \tag{21}$$

If we apply the summation by parts formula in (12) to r.h.s. of (21), we get

$$\begin{aligned}\sum_{s=0}^{t}(u(s)L_2v(s)-v(s)L_2u(s)) &= \sum_{s=0}^{t}p(s)\Delta_+^{\mu}v(s)\Delta_+^{\mu}u(s)\\ &\quad -\sum_{s=0}^{t}p(s)\Delta_+^{\mu}u(s)\Delta_+^{\mu}v(s)\\ &= 0,\end{aligned}$$

$$\langle L_2u,v\rangle=\langle u,L_2v\rangle.$$

Hence, the proof completes. □

Theorem 3.5. *Two eigenfunctions, $u(t,\lambda_\alpha)$ and $v(t,\lambda_\beta)$, of the equation* (14) *are orthogonal as $\lambda_\alpha\neq\lambda_\beta$. orthogonal.*

Proof. Let λ_α and λ_β are two different eigenvalues corresponds to eigenfunctions $u(t)$ and $v(t)$ respectively for the the equation (14),

$$\begin{aligned}\Delta_-^{\mu}(p(t)\Delta_+^{\mu}u(t))+q(t)u(t)-\lambda_\alpha r(t)u(t) &= 0,\\ \Delta_-^{\mu}(p(t)\Delta_+^{\mu}v(t))+q(t)v(t)-\lambda_\beta r(t)v(t) &= 0.\end{aligned}$$

If we multiply last two equations to $v(t)$ and $u(t)$ respectively, subtract from each other and apply definite sum operator, owing to the self-adjointness of the operator L_2, we get

$$\left(\lambda_\alpha-\lambda_\beta\right)\sum_{s=0}^{t}r(s)u(s)v(s)=0,$$

since $\lambda_\alpha\neq\lambda_{\beta,}$

$$\begin{aligned}\sum_{s=0}^{t}r(s)u(s)v(s) &= 0\\ \langle u(t),v(t)\rangle &= 0.\end{aligned}$$

So, the eigenfunctions are orthogonal. The proof completes. □

Theorem 3.6. *All eigenvalues of the equation* (14) *are real.*

Proof. Let $\lambda = \alpha + i\beta$, owing to the self-adjointness of the operator L_2

$$\begin{aligned} \langle L_2 u(t), u(t)\rangle &= \langle u(t), L_2 u(t)\rangle, \\ \langle \lambda r(t) u(t), u(t)\rangle &= \langle u(t), \lambda r(t) u(t)\rangle, \end{aligned}$$

$$\left(\lambda - \overline{\lambda}\right)\langle u, u\rangle_r = 0.$$

Since $\langle u, u\rangle_r \neq 0$,

$$\lambda = \overline{\lambda},$$

and hence $\beta = 0$. The proof completes. □

3.2. Sum representation of solution of discrete fractional Sturm-Liouville problem

Now, we consider the third case and give the following theorem and proof;

Theorem 3.7.

$$L_3 x(t) = \nabla_a^\mu \left(\nabla_a^\mu x(t)\right) + q(t) x(t) = \lambda x(t), 0 < \mu < 1, \tag{22}$$

$$x(a+1) = c_1,\ \nabla_a^\mu x(a+1) = c_2, \tag{23}$$

where $p(t) > 0$, $r(t) > 0$, $q(t)$ *is defined and real valued,* λ *is the spectral parameter. The sum representation of solution of the problem* (22) − (23) *is found as follows,*

$$x(t) = c_1\left[(1 + q(a+1)) E_{\lambda,2\mu,\mu-1}(t,a) - \lambda E_{\lambda,2\mu,2\mu-1}(t,a)\right] \tag{24}$$

$$+ c_2\left[E_{\lambda,2\mu,2\mu-1}(t,a) - E_{\lambda,2\mu,\mu-1}(t,a)\right] - \sum_{s=a+1}^{t} E_{\lambda,2\mu,2\mu-1}(t - \rho(s) + a) q(s) x(s),$$

where $|\lambda| < 1$, $|1-s| < 1$, *and* $|s|^\alpha > \lambda$ *from Theorem 2.15.*

Proof. Let's use the Laplace transform of both side of the equation (22) by Theorem 2.13, and let $q(t) x(t) = g(t)$,

$$\begin{aligned} \mathfrak{L}_{a+1}\left\{\nabla_a^\mu \left(\nabla_a^\mu x\right)\right\}(s) + \mathfrak{L}_{a+1}\{g\}(s) &= \lambda \mathfrak{L}_{a+1}\{x\}(s), \\ = s^\mu \mathfrak{L}_{a+1}\left\{\nabla_a^\mu x\right\}(s) - \frac{1-s^\mu}{1-s} c_2 &= \lambda \mathfrak{L}_{a+1}\{x\}(s) - \mathfrak{L}_{a+1}\{g\}(s), \\ = s^\mu\left(s^\mu \mathfrak{L}_{a+1}\{x\}(s) - \frac{1-s^\mu}{1-s} c_1\right) - \frac{1-s^\mu}{1-s} c_2 &= \lambda \mathfrak{L}_{a+1}\{x\}(s) - \mathfrak{L}_{a+1}\{g\}(s), \end{aligned}$$

$$= \mathfrak{L}_{a+1}\{x\}(s) = \frac{1-s^\mu}{1-s}\frac{1}{s^{2\mu}-\lambda}\left(s^\mu c_1 + c_2\right) - \frac{1}{s^{2\mu}-\lambda}\mathfrak{L}_{a+1}\{g\}(s),$$

from Lemma 2.12, we get

$$\mathfrak{L}_a\{x\}(s) = c_1\left(\frac{s^\mu - \lambda}{s^{2\mu}-\lambda}\right) - \frac{1-s}{s^{2\mu}-\lambda}\left(\frac{1}{1-s}\mathfrak{L}_a\{g\}(s) - \frac{1}{1-s} g(a+1)\right) + c_2\left(\frac{1-s^\mu}{s^{2\mu}-\lambda}\right). \tag{25}$$

Applying inverse Laplace transform to the equation (25), then we get representation of solution of the problem $(22)-(23)$,

$$\begin{aligned} x(t) &= c_1\left((1+q(a+1))E_{\lambda,2\mu,\mu-1}(t,a)-\lambda E_{\lambda,2\mu,2\mu-1}(t,a)\right) \\ &\quad +c_2\left(E_{\lambda,2\mu,2\mu-1}(t,a)-E_{\lambda,2\mu,\mu-1}(t,a)\right) \\ &\quad -\sum_{s=a+1}^{t} E_{\lambda,2\mu,2\mu-1}(t-\rho(s)+a)\,q(s)\,x(s). \end{aligned}$$

□

4. General discussions

Now, let us consider comparatively discrete fractional Sturm-Liouville (DFSL) problem, discrete Sturm-Liouville (DSL) problem, fractional Sturm-Liouville (FSL) problem and classical Sturm-Liouville (CSL) problem respectively as follows by taking $q(t)=0$,

DFSL problem:

$$\nabla_0^\mu\left(\nabla_0^\mu x(t)\right)=\lambda x(t), \tag{26}$$

$$x(1)=1,\ \nabla_a^\mu x(1)=0, \tag{27}$$

and its analytic solution is as follows by the help of Laplace transform in Lemma 2.12

$$x(t)=E_{\lambda,2\mu,\mu-1}(t,0)-\lambda E_{\lambda,2\mu,2\mu-1}(t,0), \tag{28}$$

DSL problem:

$$\nabla^2 x(t)=\lambda x(t), \tag{29}$$

$$x(1)=1,\ \nabla x(1)=0, \tag{30}$$

and its analytic solution is as follows

$$x(t)=\frac{1}{2}(1-\lambda)^{-t}\left[\left(1-\sqrt{\lambda}\right)^t\left(1+\sqrt{\lambda}\right)-\left(-1+\sqrt{\lambda}\right)\left(1+\sqrt{\lambda}\right)^t\right], \tag{31}$$

FSL problem:

$${}^{C}D_{0^+}^\mu\left(D_{0^+}^\mu x(t)\right)=\lambda x(t), \tag{32}$$

$$I_{0^+}^{1-\mu}x(t)\Big|_{t=0}=1,\ \ D_{0^+}^\mu x(t)\Big|_{t=0}=0, \tag{33}$$

and its analytic solution is as follows by the help of Laplace transform in Property 2.19 and 2.20

$$x(t)=t^{\mu-1}E_{2\mu,\mu}\left(\lambda t^{2\mu}\right), \tag{34}$$

CSL problem:

$$x''(t)=\lambda x(t), \tag{35}$$

$$x(0)=1,\ x'(0)=0, \tag{36}$$

and its analytic solution is as follows

$$x(t)=\cosh t\sqrt{\lambda}, \tag{37}$$

where the domain and range of function $x(t)$ and Mittag-Leffler functions must be well defined. Note that we may show the solution of CSL problem can be obtained by taking $\mu \to 1$ in the solution of FSL problem and similarly, the solution of DSL problem can be obtained by taking $\mu \to 1$ in the solution of DFSL problem.

Firstly, we compare the solutions of DFSL and DSL problems and from here we show that the solutions of DFSL problem converge to the solutions of DSL problem as $\mu \to 1$ in Figure 1 for discrete Mittag-Leffler function $E_{p,\alpha,\beta}(t,a) = \sum_{k=0}^{1000} p^k \frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}$; let $\lambda = 0.01$,

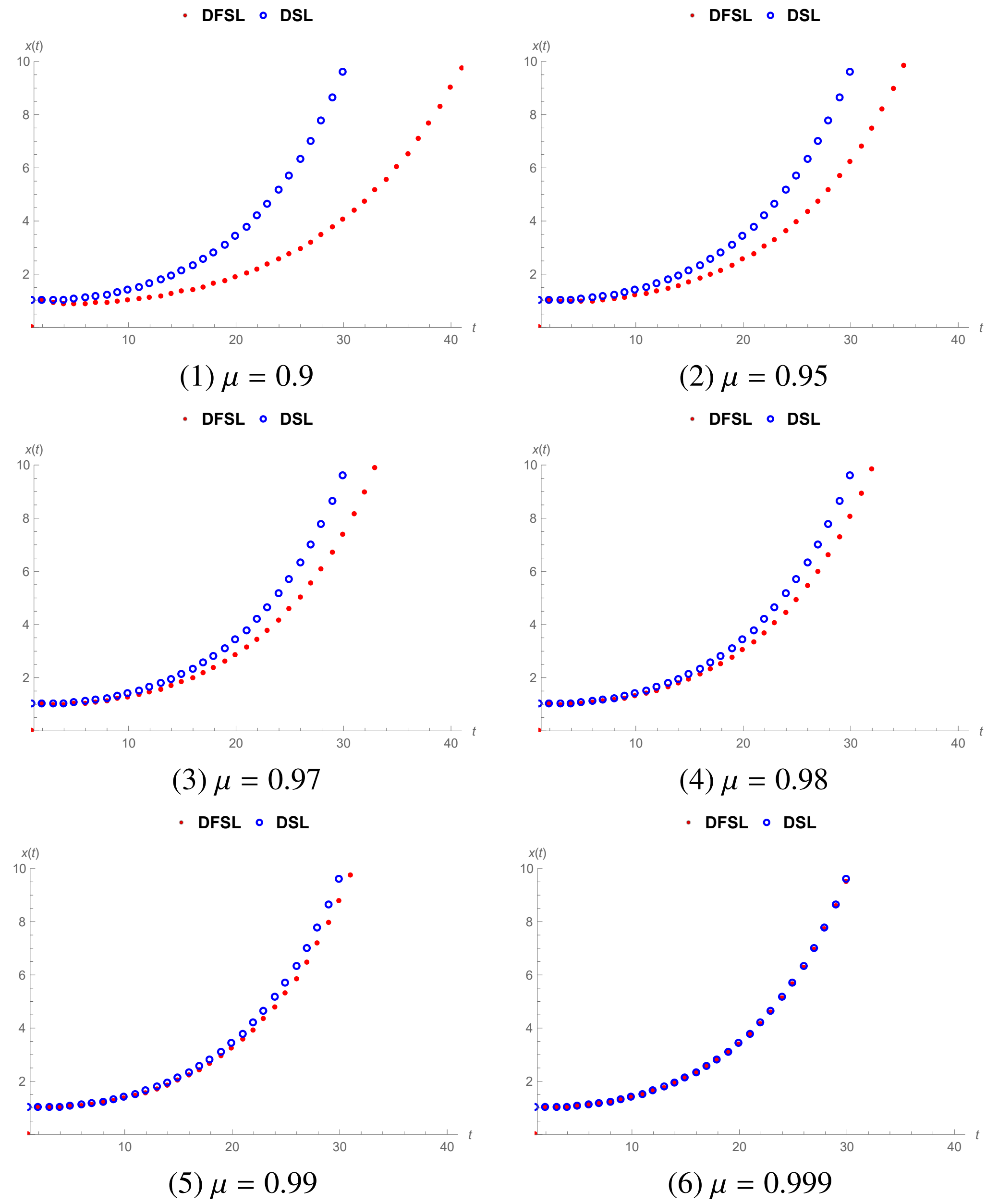

Figure 1. Comparison of solutions of DFSL–DSL problems.

Secondly, we compare the solutions of DFSL, DSL, FSL and CSL problems for discrete Mittag-Leffler function $E_{p,\alpha,\beta}(t,a) = \sum_{k=0}^{1000} p^k \frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}$. At first view, we observe the solution of DSL and CSL problems almost coincide in any order μ, and we observe the solutions of DFSL and FSL problem almost coincide in any order μ. However, we observe that all of the solutions of DFSL, DSL, FSL and CSL problems almost coincide to each other as $\mu \to 1$ in Figure 2. Let $\lambda = 0.01$,

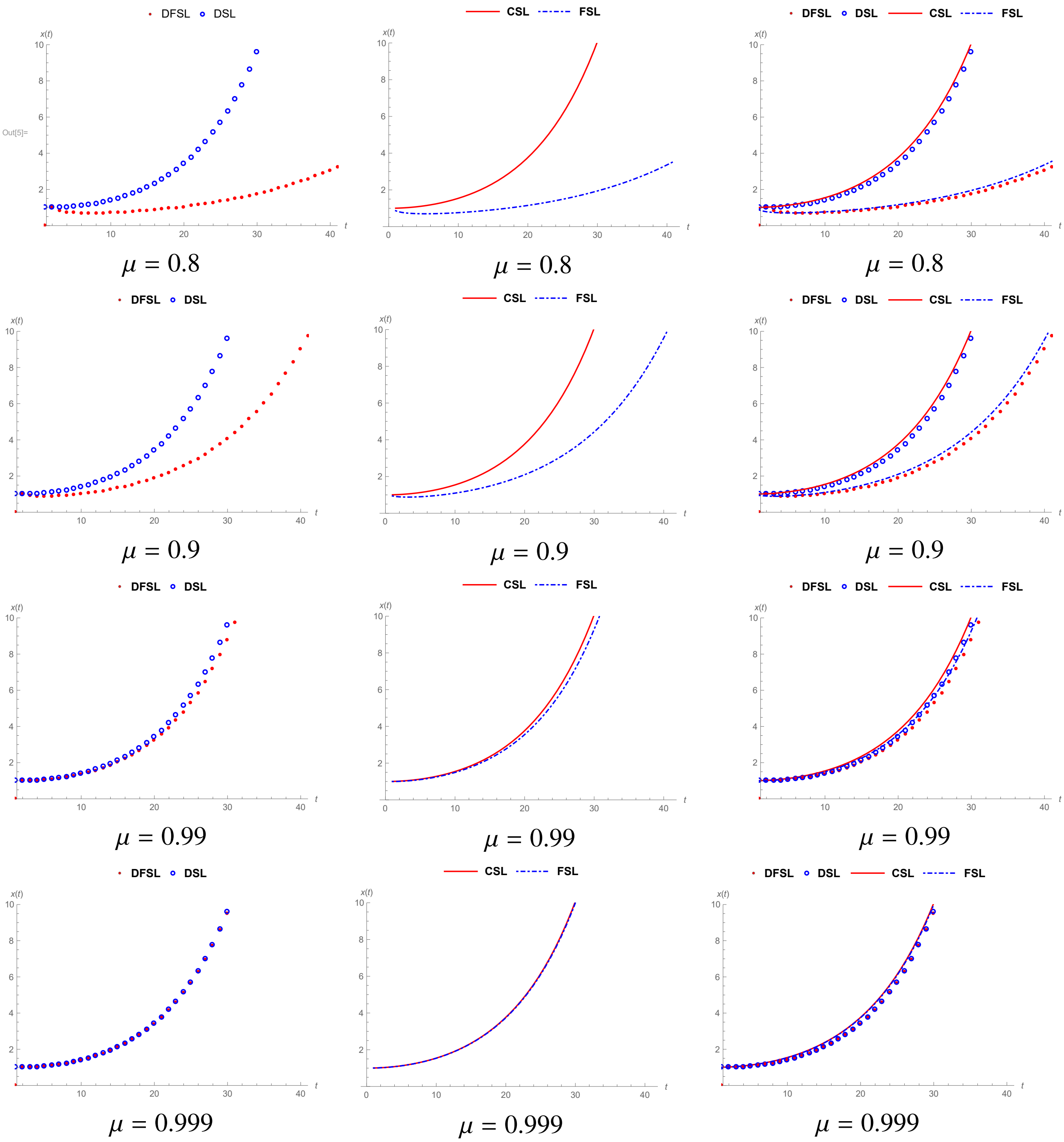

Figure 2. Comparison of solutions of DFSL–DSL–CSL–SL problems.

Thirdly, we compare the solutions of DFSL problem $(22-23)$ with different orders, different potential functions and different eigenvalues for discrete Mittag-Leffler function $E_{p,\alpha,\beta}(t,a)=\sum_{k=0}^{1000}p^{k}\frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}$ in the Figure 3;

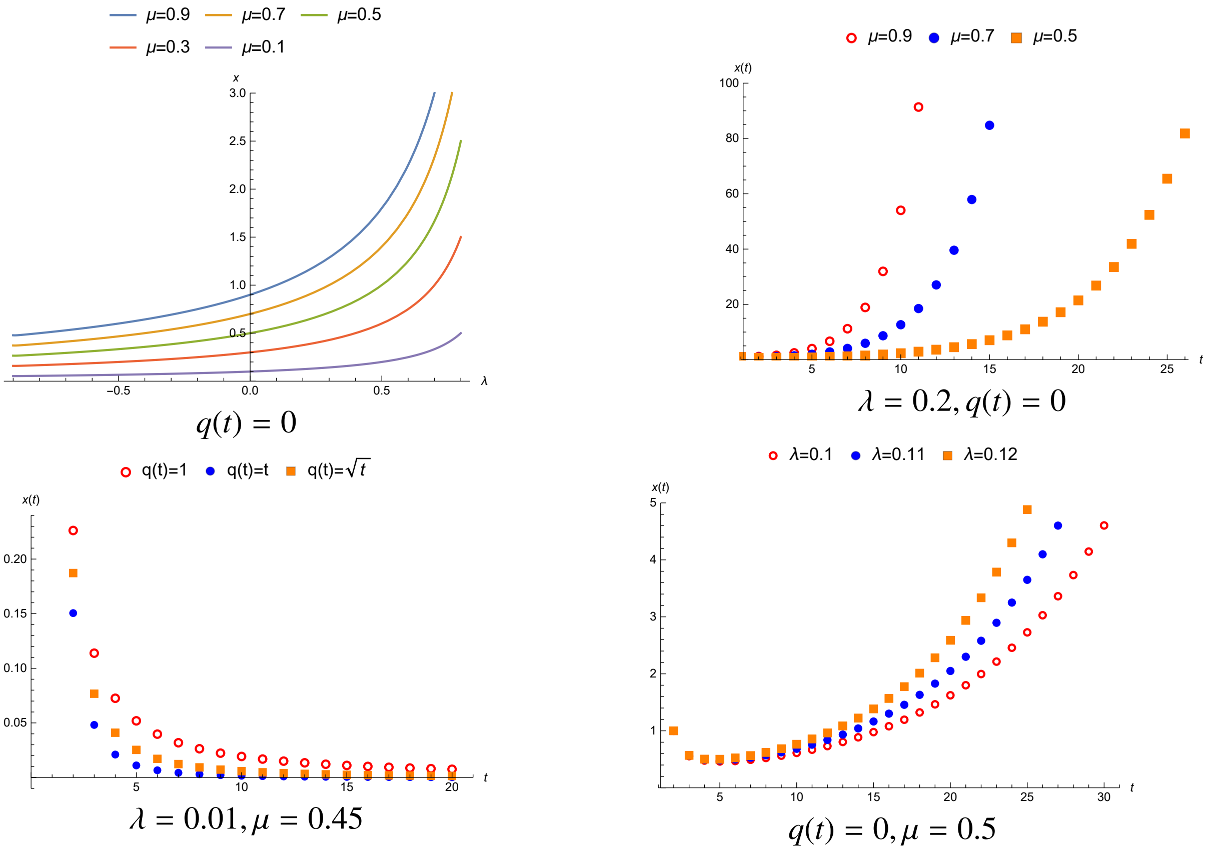

Figure 3. Analysis of solutions of DFSL problem.

Eigenvalues of DFSL problem $(22-23)$, correspond to some specific eigenfunctions for numerical values of discrete Mittag-Leffler function $E_{p,\alpha,\beta}(t,a)=\sum_{k=0}^{i}p^{k}\frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}$, is given with different orders while $q(t)=0$ in Table 1;

Finally, we give the solutions of DFSL problem $(22-23)$ with different orders, different potential functions and different eigenvalues for discrete Mittag-Leffler function $E_{p,\alpha,\beta}(t,a)=\sum_{k=0}^{100}p^{k}\frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}$ in Tables 2–4;

Table 1. Approximations to three eigenvalues of the problem (22–23).

i	$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
750	−0.992	−0.982	−0.057
1000	−0.989	−0.977	−0.057
2000	−0.996	−0.990	−0.057

$x(5), \mu = 0.5$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.986	−0.941	−0.027
−0.990	−0.954	−0.027
−0.995	−0.978	−0.027

$x(10), \mu = 0.9$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.483	−0.483	0
−0.559	−0.435	0
−0.654	−0.435	0

$x(2000), \mu = 0.1$

i	$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
750	−0.951	−0.004	0
1000	−0.963	−0.004	0
2000	−0.981	−0.004	0

$x(20), \mu = 0.5$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.868	−0.793	−0.0003
−0.898	−0.828	−0.0003
−0.947	−0.828	−0.0003

$x(100), \mu = 0.9$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.190	-3.290×10^{-6}	0
−0.394	-3.290×10^{-6}	0
−0.548	-3.290×10^{-6}	0

$x(1000), \mu = 0.7$

i	$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
750	−0.414	-9.59×10^{-7}	0
1000	−0.478	-9.59×10^{-7}	0
2000	−0.544	-9.59×10^{-7}	0

$x(1000), \mu = 0.3$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.853	−0.0003	0
−0.887	−0.0003	0
−0.940	−0.0003	0

$x(100), \mu = 0.8$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.330	-4.140×10^{-6}	0
−0.375	-4.140×10^{-6}	0
−0.361	-4.140×10^{-6}	0

$x(1000), \mu = 0.9$

i	$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
750	−0.303	-3.894×10^{-6}	0
1000	−0.335	-3.894×10^{-6}	0
2000	−0.399	-3.894×10^{-6}	0

$x(1000), \mu = 0.8$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.192	−0.066	0
−0.197	−0.066	0
−0.289	−0.066	0

$x(2000), \mu = 0.6$

$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
−0.985	−0.955	−0.026
−0.989	−0.941	−0.026
−0.994	−0.918	−0.026

$x(10), \mu = 0.83$

Table 2. $q(t) = 0, \lambda = 0.2$.

$x(t)$	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 0.7$	$\mu = 0.9$
$x(1)$	1	1	1	1	1
$x(2)$	0.125	0.25	0.625	0.875	1.125
$x(3)$	0.075	0.174	0.624	1.050	1.575
$x(5)$	0.045	0.128	0.830	1.968	4.000
$x(7)$	0.0336	0.111	1.228	4.079	11.203
$x(9)$	0.0274	0.103	1.878	8.657	31.941
$x(12)$	0.022	0.098	3.622	27.05	154.56
$x(15)$	0.0187	0.0962	7.045	84.75	748.56
$x(16)$	0.0178	0.0961	8.800	124.04	1266.5
$x(18)$	0.0164	0.0964	13.737	265.70	3625.6
$x(20)$	0.0152	0.0972	21.455	569.16	10378.8

Table 3. $\lambda = 0.01$, $\mu = 0.45$.

$x(t)$	$q(t) = 1$	$q(t) = t$	$q(t) = \sqrt{t}$
$x(1)$	1	1	1
$x(2)$	0.2261	0.1505	0.1871
$x(3)$	0.1138	0.0481	0.0767
$x(5)$	0.0518	0.0110	0.0252
$x(7)$	0.0318	0.0043	0.0123
$x(9)$	0.0223	0.0021	0.0072
$x(12)$	0.0150	0.0010	0.0039
$x(15)$	0.0110	0.0005	0.0025
$x(16)$	0.0101	0.0004	0.0022
$x(18)$	0.0086	0.0003	0.0017
$x(20)$	0.0075	0.0002	0.0014

Table 4. $\lambda = 0.01$, $\mu = 0.5$.

$x(t)$	$q(t) = 1$	$q(t) = t$	$q(t) = \sqrt{t}$
$x(1)$	1	1	1
$x(2)$	0.2261	0.1505	0.1871
$x(3)$	0.1138	0.0481	0.0767
$x(5)$	0.0518	0.0110	0.0252
$x(7)$	0.0318	0.0043	0.0123
$x(9)$	0.0223	0.0021	0.0072
$x(12)$	0.0150	0.0010	0.0039
$x(15)$	0.0110	0.0005	0.0025
$x(16)$	0.0101	0.0004	0.0022
$x(18)$	0.0086	0.0003	0.0017
$x(20)$	0.0075	0.0002	0.0014

4.1. Discussions on eigenvalues and eigenfunctions of DFSL, DSL, FSL, CSL problems

Now, let's consider the problems together DFSL (26) − (27) , DSL (29) − (30), FSL (32) − (33) and CSL (35) − (36) . Eigenvalues of these problems are the roots of the following equation

$$x(35) = 0.$$

Thus, if we apply the solutions (28) , (31) , (34) and (37) of these four problems to the equation above respectively, we can find the eigenvalues of these problems for the orders $\mu = 0.9$ and $\mu = 0.99$ respectively in *Table* 5, and *Table* 6,

Table 5. $\mu = 0.9$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
DFSL	−0.904	−0.859	−0.811	−0.262	−0.157	−0.079	−0.029	**−0.003**	0.982	
FSL	−0.497	−0.383	−0.283	−0.196	−0.124	−0.066	−0.026	**−0.003**	0	...
DSL	−1.450	−0.689	−0.469	−0.310	−0.194	−0.112	−0.055	−0.019	**−0.002**	
CSL	−0.163	−0.128	−0.098	−0.072	−0.050	−0.032	−0.008	**−0.002**	0	

Table 6. $\mu = 0.99$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
DFSL	−0.866	−0.813	−0.200	−0.115	−0.057	−0.020	**−0.002**	0	0.982	
FSL	−0.456	−0.343	−0.246	−0.165	−0.100	−0.051	−0.018	**−0.002**	0	...
DSL	−1.450	−0.689	−0.469	−0.310	−0.194	−0.112	−0.055	−0.019	**−0.002**	...
CSL	−0.163	−0.128	−0.098	−0.072	−0.050	−0.032	−0.008	**−0.002**	0	

In here, we observe that these four problems have real eigenvalues under different orders $\mu = 0.9$ and $\mu = 0.99$, hence we can find eigenfunctions putting these eigenvalues into the four solutions. Furthermore, as the order changes, we can see that eigenvalues change for DFSL problems.

5. Conclusion

We consider firstly discrete fractional Sturm-Liouville (DFSL) operators with nabla Riemann-Liouville and delta Grünwald-Letnikov fractional operators and we prove self-adjointness of the DFSL operator and fundamental spectral properties. However, we analyze DFSL problem, discrete Sturm-Liouville (DSL) problem, fractional Sturm-Liouville (FSL) problem and classical Sturm-Liouville (CSL) problem by taking $q(t) = 0$ in applications. Firstly, we compare the solutions of DFSL and DSL problems and we observe that the solutions of DFSL problem converge to the solutions of DSL problem when $\mu \to 1$ in *Fig*. 1. Secondly, we compare the solutions of DFSL, DSL, FSL and CSL problems in *Fig*. 2. At first view, we observe the solutions of DSL and CSL problems almost coincide with any order μ, and we observe the solutions of DFSL and FSL problem almost coincide with any order μ. However, we observe that all of solutions of DFSL, DSL, FSL and CSL problems almost coincide with each other as $\mu \to 1$. Thirdly, we compare the solutions of DFSL problem $(22-23)$ with different orders, different potential functions and different eigenvalues in $Fig.$ 3.

Eigenvalues of DFSL problem $(22-23)$ corresponded to some specific eigenfunctions is given with different orders in $Table$1. We give the eigenfunctions of DFSL problem $(22-23)$ with different orders, different potential functions and different eigenvalues in $Table$ 2, $Table$ 3 and $Table$ 4.

In Section 4.1, we consider DFSL, DSL, FSL and CSL problems together and thus, we can compare the eigenvalues of these four problems in $Table$ 5 and $Table$ 6 for different values of μ. We observe that these four problems have real eigenvalues under different values of μ, from here we can find eigenfunctions corresponding eigenvalues. Moreover, when the order change, eigenvalues change for DFSL problems.

Consequently, important results in spectral theory are given for discrete Sturm-Liouville problems. These results will lead to open gates for the researchers studied in this area. Especially, representation of solution will be practicable for future studies. It worths noting that visual results both will enable to be understood clearly by readers and verify the results to the integer order discrete case while the order approaches to one.

Acknowledgment

This paper includes a part of Ph.D. thesis data of Ramazan OZARSLAN.

Conflict of interest

The authors declare no conflict of interest.

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