AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(3): 527–533.
DOI:10.3934/math.2019.3.527
Received: 10 February 2019
Accepted: 14 May 2019
Published: 29 May 2019

Research article

$\mathcal{A}$-valued norm parallelism in Hilbert $\mathcal{A}$-modules

Ali Khalili and Maryam Amyari*

Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad 91735, Iran

* **Correspondence:** Email: maryam_amyari@yahoo.com; Tel: +985138817375;
Fax: +985138817375.

Abstract: We define the concept of $\mathcal{A}$-valued norm parallelism in a Hilbert $\mathcal{A}$-module, and then we investigate some properties of this notion and present some characterizations of $\mathcal{A}$-valued norm parallelism in a Hilbert $\mathcal{A}$-module. We also show that if X and Y are two inner product $\mathcal{A}$-modules and $T : X \to Y$ is a linear map such that $|Tx| = |x|$, then T preserves $\mathcal{A}$-valued norm parallelism in both directions.

Keywords: Hilbert $\mathcal{A}$-modules; preserving map; parallelism; $\mathcal{A}$-valued norm parallelism
Mathematics Subject Classification: 46L08, 46L05

1. Introduction and preliminary

Let $\mathcal{A}$ be a C^*-algebra. An inner product $\mathcal{A}$-module is a linear space X, which is a right $\mathcal{A}$-module with a map $\langle .,. \rangle : X \times X \to \mathcal{A}$ that satisfies the following properties:

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$;

(iii) $\langle x, ya\rangle = \langle x, y\rangle a$;

(vi) $\langle x, y\rangle^* = \langle y, x\rangle$, where $x, y, z \in X$, $a \in \mathcal{A}$, and $\lambda \in \mathbb{C}$.

If X is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ for each $x \in X$, then it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. Complex Hilbert spaces can be regarded as left Hilbert $\mathbb{C}$-modules. Any C^*-algebra $\mathcal{A}$ can be regarded as a Hilbert C^*-module over itself via $\langle a, b\rangle = a^*b$. For more details in Hilbert C^*-modules, we refer the reader to [1, 2, 5, 6].

Let $(X, \|\cdot\|)$ be a normed space and let $x, y \in X$. We say that x is norm-parallel to y, denoted by $x\|y$, if

$$\|x + \lambda y\| = \|x\| + \|y\| \quad \text{for} \quad \text{some} \quad \lambda \in \mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\} \tag{1.1}$$

It is obvious that the norm-parallelism is symmetric and $\mathbb{R}$-homogeneous, that is, $x\|y \Leftrightarrow \alpha x\|\beta y$ for all $\alpha, \beta \in \mathbb{R}$, but not transitive, that is, $x\|y$ and $y\|z$ do not imply $x\|z$ in general; see [8, Example

2.7]. It is known that in the setting of inner product spaces, the norm parallelism is equivalent to the linear dependence. In the case of normed linear spaces, two linearly dependent vectors are norm-parallel, but the converse is false in general. For example, consider the space $(\mathbb{R}^2, \|(\cdot,\cdot)\|)$, where $\|(x,y)\| = \max\{\frac{|x+y|}{2}, \frac{|x-y|}{2}\}$ for $x, y \in \mathbb{R}^2$. Let $x = (1,2), y = (-1,0)$, and $\lambda = -1$. Then $\|x + \lambda y\| = \|(2,2)\| = 2 = \dfrac{3}{2} + \dfrac{1}{2} = \|x\| + \|y\|$, so that $x\|y$ but they are not linearly dependent. The interested reader is referred to references [4, 7–9] for more details in norm-parallelism.

Theorem 1.1. *Let X be an inner product space and let $x, y \in X$. The following statements are equivalent:*
(i) $x\|y$;
(ii) $|\langle x, y\rangle| = \|x\|\|y\|$.

Proof. (i) $\Rightarrow$ (ii): Let $x\|y$. Then $\|x + \lambda y\| = \|x\| + \|y\|$ for some $\lambda \in \mathbb{T}$.

$$\begin{aligned}(\|x\| + \|y\|)^2 = \|x + \lambda y\|^2 = \langle x + \lambda y, x + \lambda y\rangle &= \|x\|^2 + 2\mathrm{Re}\lambda\langle x, y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

Thus $|\langle x, y\rangle| = \|x\|\|y\|$.
(ii) $\Rightarrow$ (i): Suppose $|\langle x, y\rangle| = \|x\|\|y\|$. There exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and

$$|\langle x, y\rangle| = \lambda\langle x, y\rangle = \langle \lambda x, y\rangle.$$

Then $\|x\|\|y\| = \langle \lambda x, y\rangle$. In fact, $\mathrm{Re}\langle \lambda x, y\rangle = \|x\|\|y\|$. Therefore

$$\|x + \lambda y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle \lambda x, y\rangle + \|y\|^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

Hence $\|x + \lambda y\| = \|x\| + \|y\|$ for some $\lambda \in \mathbb{T}$. □

Let X be an inner product space and let $x, y \in X$. We say that a map $f : X \to X$ preserves parallelism if $x\|y$ implies $f(x)\|f(y)$. It is known that parallelism preserving maps may be nonlinear and discontinuous.

Remark 1.1. Suppose that X and Y are two inner product spaces and that $f : X \to Y$ is a homogeneous map (i.e., $f(\alpha x) = \alpha f(x)$ for each $\alpha \in \mathbb{C}$). Then f preserves parallelism. In fact, if $x, y \in X$ with $x\|y$, then there is $\alpha \in \mathbb{C}$ such that $x = \alpha y$, so $f(x) = f(\alpha y) = \alpha f(y)$ (i.e., $f(x)$ and $f(y)$ are linearly dependent). Thus $f(x)\|f(y)$. On the other hand, if f is an injective homogeneous map, then f preserves parallelism in both directions. Therefor if $f(x)\|f(y)$, then $f(x) = \alpha f(y)$ or $f(x) = f(\alpha y)$ for some $\alpha \in \mathbb{C}$. Then $x = \alpha y$ for some $\alpha \in \mathbb{C}$, that is, $x\|y$.

Theorem 1.2. *Suppose that X and Y are two normed spaces and that $T : X \to Y$ is a linear isometry map. Then T preserves parallelism in both directions,*

$$x\|y \Leftrightarrow Tx\|Ty \quad (x, y \in X).$$

Proof. Let $x\|y$. Then $\|x + \lambda y\| = \|x\| + \|y\|$ for some $\lambda \in \mathbb{T}$. On the other hand,

$$\|x\| + \|y\| = \|x + \lambda y\| = \|T(x + \lambda y)\| \leq \|Tx\| + \|Ty\| = \|x\| + \|y\|.$$

Hence $\|Tx + \lambda Ty\| = \|Tx\| + \|Ty\|$ for some $\lambda \in \mathbb{T}$, that is, $Tx\|Ty$. The converse is trivial. □

In the following theorem, we reach a relation between two unknown functions that preserve parallelism.

Theorem 1.3. *Let X and Y be two inner product spaces and let $f, g : X \to Y$ be two homogeneous maps such that*

$$x\|y \Rightarrow f(x)\|g(y) \tag{1.2}$$

for $x, y \in X$. Then

$$|\langle f(x), g(x)\rangle| = \|x\|^2\|y\|^{-2}\|f(y)\|\|g(y)\|.$$

Proof. Take $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$. Then $\|u\| = \|v\| = 1$ and $|\langle u, v\rangle| = 1 = \|u\|\|v\|$, that is, $u\|v$. Thus, there exists $\alpha \in \mathbb{C}$ such that $u = \alpha v$ and

$$1 = \|u\|\|v\| = |\langle u, v\rangle| = |\langle \alpha v, v\rangle| = |\alpha|\|v\|.$$

Hence $\alpha \in \mathbb{T}$ and $v = \overline{\alpha} u$.

Therefore $|\langle f(u), g(\overline{\alpha} u\rangle| = \|f(\alpha v)\|\|g(v)\|$ and then $|\langle f(u), g(u)\rangle| = \|f(v)\|\|g(v)\|$. In other words,

$$|\langle f(\frac{x}{\|x\|}), g(\frac{x}{\|x\|})\rangle| = \|f(\frac{y}{\|y\|})\|\|g(\frac{y}{\|y\|})\|.$$

We get

$$|\langle f(x), g(x)\rangle| = \|x\|^2\|y\|^{-2}\|f(y)\|\|g(y)\|.$$

Moreover, fix $y_0 \in X$. For each $x \in X$, if $x\|y_0$, then

$$|\langle f(x), g(x)\rangle| = k\|x\|^2,$$

where $k = \|y_0\|^{-2}\|f(y_0)\|\|g(y_0)\|$. □

2. $\mathcal{A}$-valued norm parallelism in Hilbert $\mathcal{A}$-modules

Let X be an inner product $\mathcal{A}$-module. For $x \in X$, the unique square root of positive element $\langle x, x\rangle$ is denoted by $|x|$ and is called the $\mathcal{A}$-valued norm, where $|x|^2 = \langle x, x\rangle$.

In fact, the $\mathcal{A}$-valued norm is not a norm, for example, the triangle inequality $|x + y| \leq |x| + |y|$ does not hold in general; see [5].

In this section, we define a new definition of parallelism via $\mathcal{A}$-valued norm. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$. If $a \in \mathcal{A}$, then the absolute value of a is defined by $|a| = (a^*a)^{\frac{1}{2}}$. Let $\mathcal{A}$ be a unital C^*-algebra. Then $a \in \mathcal{A}$ is unitary if $aa^* = a^*a = 1$. The center of $\mathcal{A}$ is defined by $Z(\mathcal{A}) = \{a \in \mathcal{A} \ ab = ba \text{ for each } b \in \mathcal{A}\}$. The set of all positive elements of $\mathcal{A}$ that contained in $Z(\mathcal{A})$ is denoted by $Z(\mathcal{A})^+$.

Definition 2.1. Let X be a Hilbert $\mathcal{A}$-module and let $x, y \in X$. We say that x is $\mathcal{A}$-parallel to y, denoted by $x \uparrow y$, if

$$|x + \lambda y| = |x| + |y| \quad for \;\; some \quad \lambda \in \mathbb{T}.$$

Let X be a Hilbert $\mathcal{A}$-module and let $x, y \in X$ be linearly dependent.

Then there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $y = \alpha x$. By letting $\lambda = \frac{\overline{\alpha}}{|\alpha|}$, we have

$$|x + \lambda y| = |x + |\alpha| x| = (1 + |\alpha|)|x| = |x| + |y|.$$

Hence $x \uparrow y$.

The converse is not true in general. For example, if $R = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are considered as two elements of Hilbert $\mathbb{C}$-module $M_2(\mathbb{C})$, then for $\lambda = 1$, we reach $|R + I| = |R| + |I|$, that is, $R \uparrow I$. However, R and I are linearly independent.

Theorem 2.2. *[2, Theorem 2.3] Let X be a Hilbert $\mathcal{A}$-module and let $x, y \in X$. Then the following statements are equivalent:*

(i) $x \uparrow y$;

(ii) $\langle x, \lambda y \rangle = |x||y| \quad for \;\; some \quad \lambda \in \mathbb{T}$.

Recall that, in an inner product space, two elements x and y are orthogonal, if $\langle x, y \rangle = 0$.

Theorem 2.3. *Let X be a Hilbert $\mathcal{A}$-module and let $x, y \in X$ such that $x \uparrow y$. Then the following properties hold:*

(i) $xa \uparrow yb$ for $a, b \in Z(\mathcal{A})^+$.

(ii) $x \uparrow (x + \lambda y)$ for some $\lambda \in \mathbb{T}$.

(iii) If $|x| \in Z(\mathcal{A})$, then $x \perp (x|y| - \lambda y|x|)$ for some $\lambda \in \mathbb{T}$.

Proof. (i) Suppose $a, b \in Z(\mathcal{A})^+$. Then

$$|xa|^2 = \langle xa, xa \rangle = a^* \langle x, x \rangle a = a \langle x, x \rangle a = a|x|^2 a = (|x|a)^2.$$

So $|xa| = |x|a = a|x|$ and similarly $|yb| = |y|b = b|y|$. Now $\langle xa, \lambda yb \rangle = a^* \langle x, \lambda y \rangle b = a|x||y|b = |xa||yb|$ for some $\lambda \in \mathbb{T}$. Thus $xa \uparrow yb$.

(ii) Let $\langle x, \lambda y \rangle = |x||y|$ for some $\lambda \in \mathbb{T}$. Then $\langle x, x + \lambda y \rangle = \langle x, x \rangle + \langle x, \lambda y \rangle = |x|^2 + |x||y| = |x|(|x| + |y|) = |x||x + \lambda y|$. Hence $x \uparrow (x + \lambda y)$.

(iii) Since $x \uparrow y$, we have $\langle x, \lambda y \rangle = |x||y|$ for some $\lambda \in \mathbb{T}$.

Thus

$$\begin{aligned} \langle x, x|y| - \lambda y|x| \rangle &= \langle x, x \rangle |y| - \langle x, \lambda y \rangle |x| \\ &= |x|^2|y| - |x|^2|y| = 0, \end{aligned}$$

where $|x| \in Z(\mathcal{A})^+$. □

Remark 2.1. Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be unitary. If X is a Hilbert $\mathcal{A}$-module and $x, y \in X$, then $x \uparrow y$ if and only if $xa \uparrow ya$, where $a \in Z(\mathcal{A})^+$. If $x, y \in X$, then Theorem 2.3 implies that $xa \uparrow ya$.

Conversely, if $xa \uparrow ya$, then $\langle xa, \lambda ya \rangle = |xa||ya|$ for some $\lambda \in \mathbb{T}$.

Hence $\langle x, \lambda y \rangle = a^* \langle x, \lambda y \rangle a = \langle xa, \lambda ya \rangle = |xa||ya| = a^*|x||y|a = a^*a|x||y| = |x||y|$, so $x \uparrow y$.

In the following result, we state the relation between the $\mathcal{A}$-parallelism and $\mathcal{A}$-linear dependence in the setting of Hilbert $\mathcal{A}$-module.

Corollary 2.4. *Let X be a Hilbert $\mathcal{A}$-module and let $x, y \in X$. If $\mathcal{A}$ is unital and $|y| = 1$, then $x \uparrow y$ if and only if $x = \lambda y|x|$ for some $\lambda \in \mathbb{T}$.*

Proof. If $x \uparrow y$, then $\langle x, \lambda y\rangle = |x|$ for some $\lambda \in \mathbb{T}$. We get

$$\begin{aligned} \langle x - \lambda y|x|, x - \lambda y|x|\rangle &= \langle x, x\rangle - \langle x, \lambda y\rangle|x| - |x|\langle \lambda y, x\rangle + |x|^2 \\ &= |x|^2 - |x|^2 - |x|^2 + |x|^2 = 0. \end{aligned}$$

Hence $x = \lambda y|x|$.

Conversely, if $x = \lambda y|x|$, then $\langle x, \lambda y\rangle = \langle \lambda y|x|, \lambda y\rangle = |\lambda|^2|x||y|^2 = |x|$ for some $\lambda \in \mathbb{T}$, and so $x \uparrow y$. □

We state some facts about the $\mathcal{A}$-valued norm from [9, Lemma 4.1].

Remark 2.2. Let $x \perp y$. Then $x \uparrow y$ if and only if $\langle x, \lambda y\rangle = |x||y|$ for some $\lambda \in \mathbb{T}$, and this holds if and only if $|x||y| = 0$ and this occurs if and only if $|x + \lambda y| = |x - \lambda y|$ and finally the last equality holds if and only if $|x + \lambda ya| = |x - \lambda ya|$ for all $a \in Z(\mathcal{A})^+$.

Proposition 2.5. *Let X and Y be two Hilbert $\mathcal{A}$-modules and let $T : X \to Y$ be a linear map such that $|Tx| = |x|$. Then T preserves $\uparrow$ in both directions.*

Proof. Suppose $x \uparrow y$. Then $|x + \lambda y| = |x| + |y|$ for some $\lambda \in \mathbb{T}$. Therefor $|T(x) + \lambda T(y)| = |T(x + \lambda y)| = |x + \lambda y| = |x| + |y| = |T(x)| + |T(y)|$. So T preserves $\uparrow$ in both directions . □

Definition 2.6. Let $\mathcal{A}$ be a C^*-algebra and let $a, b \in \mathcal{A}$. We say that $a \uparrow b$ if

$$|a + \lambda b| = |a| + |b| \quad for\ \ some \quad \lambda \in \mathbb{T}.$$

Theorem 2.7. *[2, Theorem 2.1] Let $\mathcal{A}$ be a C^*-algebra and let $a, b \in \mathcal{A}$. Then the following statements are equivalent:*

(i) $a \uparrow b$;

(ii) $\lambda a^* b = |a||b| \quad for\ \ some\ \ \lambda \in \mathbb{T}$.

Moreover if $\mathcal{A}$ is unital and $a + \lambda b$ is invertible in $\mathcal{A}$, then $a \uparrow b$ if and only if there is a unitary $u \in \mathcal{A}$ such that $a = u|a|$ and $\lambda b = u|b|$. In fact, $u = (a + \lambda b)|a + \lambda b|^{-1}$.

Recall that, if a and b are two positive elements of $\mathcal{A}$ such that $a \in Z(\mathcal{A})$, then $|a + b| = |a| + |b|$.

Theorem 2.8. *Let X be a Hilbert $\mathcal{A}$-module and let $x, y \in X$ such that $|x| \in Z(\mathcal{A})$. If $x \uparrow y$, then $\langle x, x\rangle \uparrow \langle x, y\rangle$ in $\mathcal{A}$.*

Proof. Suppose $x \uparrow y$. Then $\langle x, \lambda y\rangle = |x||y| \quad for\ \ some\ \ \lambda \in \mathbb{T}$. Thus

$$\begin{aligned} |\langle x, x\rangle + \lambda\langle x, y\rangle| = |\langle x, x + \lambda y\rangle| &= \big||x||x + \lambda y|\big| \quad (\text{since } \ x \uparrow (x + \lambda y)) \\ &= \big||x|(|x| + |y|)\big| \quad (\text{since } \ x \uparrow y) \\ &= \big||x|^2 + |x||y|\big| \\ &= \big||x|^2\big| + \big||x||y|\big| \\ &= \big|\langle x, x\rangle\big| + \big|\langle x, \lambda y\rangle\big| = |\langle x, x\rangle| + |\langle x, y\rangle|. \end{aligned}$$

Hence $\langle x, x\rangle \uparrow \langle x, y\rangle$.

□

Example 2.9. Let $X = \mathcal{A} = M_2(\mathbb{R})$. Then $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module. Let $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For $\lambda = 1$, we have $\langle S, I\rangle = S^*I = SI = |S||I|$, that is, $S \uparrow I$. On the other hand, $\langle S, S\rangle \uparrow \langle I, I\rangle$ in $\mathcal{A}$, since for $\lambda = 1$, we have $\langle S, S\rangle^*\langle I, I\rangle = SS^*I = |\langle S, S\rangle||\langle I, I\rangle|$.

Now, we borrow the following definition from [3].

Definition 2.10. Suppose that X and Y are two Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras. A map $\psi : X \to Y$ is a φ-morphism of Hilbert C^*-modules if $\langle \psi(x), \psi(y)\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in X$.

It is also easy to see that each φ-morphism is a linear map and $\psi(xa) = \psi(x)\varphi(a)$ for each $x \in X$, and $a \in \mathcal{A}$.

Remark 2.3. Suppose that X and Y are two Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is the inclusion map, then Ψ preserves $\uparrow$ in both directions.
Since $|\psi(x)|^2 = \langle \psi(x), \psi(x)\rangle = \varphi(\langle x, x\rangle) = \varphi(|x|^2) = |x|^2$, Theorem 2.5 implies that ψ preserves $\uparrow$ in both directions.

Acknowledgments

We would like to thank the referees for several useful comments.

Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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