



*Research article*

## On a generalized Lyapunov inequality for a mixed fractional boundary value problem

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**Abstract:** In this paper, we establish a new Lyapunov-type inequality for a differential equation involving left Riemann-Liouville and right Caputo fractional derivatives subject to Dirichlet-type boundary conditions.

**Keywords:** fractional derivative; Lyapunov inequality; Green’s function; eigenvalue problem

**Mathematics Subject Classification:** 34A08, 26A33

### 1. Introduction

The well-known classical Lyapunov inequality [15] states that, if  $u$  is a nontrivial solution of the Hill’s equation

$$u''(t) + q(t)u(t) = 0, \quad a < t < b, \tag{1.1}$$

subject to Dirichlet-type boundary conditions:

$$u(a) = u(b) = 0, \tag{1.2}$$

then

$$\int_a^b |q(t)| dt > \frac{4}{b-a}, \tag{1.3}$$

where  $q : [a, b] \rightarrow \mathbb{R}$  is a real and continuous function.

Later, in 1951, Wintner [24], obtained the following inequality:

$$\int_a^b q^+(t) dt > \frac{4}{b-a}, \tag{1.4}$$

where  $q^+(t) = \max\{q(t), 0\}$ .

A more general inequality was given by Hartman and Wintner in [12], that is known as Hartman Wintner-type inequality:

$$\int_a^b (t-a)(b-t)q^+(t)dt > b-a, \quad (1.5)$$

Since  $\max_{t \in [a,b]} (t-a)(b-t) = \frac{(b-a)^2}{4}$ , then, (1.5) implies (1.4).

The Lyapunov inequality and its generalizations have many applications in different fields such in oscillation theory, asymptotic theory, disconjugacy, eigenvalue problems.

Recently, many authors have extended the Lyapunov inequality (1.3) for fractional differential equations [1–13,15,18,20,22–24]. For this end, they substituted the ordinary second order derivative in (1.1) by a fractional derivative or a conformable derivative. The first result in which a fractional derivative is used instead of the ordinary derivative in equation (1.1), is the work of Ferreira [6]. He considered the following two-point Riemann-Liouville fractional boundary value problem

$$\begin{aligned} D_{a^+}^\alpha u(t) + q(t)u(t) &= 0, \quad a < t < b, \quad 1 < \alpha \leq 2 \\ u(a) &= u(b) = 0. \end{aligned}$$

And obtained the Lyapunov inequality:

$$\int_a^b |q(t)|dt > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}.$$

Then, he studied in [7], the Caputo fractional differential equation

$${}^C D_{a^+}^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2$$

under Dirichlet boundary conditions (1.2). In this case, the corresponding Lyapunov inequality has the form

$$\int_a^b |q(t)|dt > \frac{\alpha^\alpha \Gamma(\alpha)}{((\alpha-1)(b-a))^{\alpha-1}}.$$

Later Agarwal and Özbekler in [1], complimented and improved the work of Ferreira [6]. More precisely, they proved that if  $u$  is a nontrivial solution of the Riemann-Liouville fractional forced nonlinear differential equations of order  $\alpha \in (0, 2]$ :

$$D_{a^+}^\alpha u(t) + p(t)|u(t)|^{\mu-1}u(t) + q(t)|u(t)|^{\gamma-1}u(t) = f(t), \quad a < t < b,$$

satisfying the Dirichlet boundary conditions (1.2), then the following Lyapunov type inequality

$$\left( \int_a^b [p^+(t) + q^+(t)]dt \right) \left( \int_a^b [\mu_0 p^+(t) + \gamma_0 q^+(t) + |f(t)|]dt \right) > \frac{4^{2\alpha-3} \Gamma^2(\alpha)}{(b-a)^{2\alpha-2}}.$$

holds, where  $p, q, f$  are real-valued functions,  $0 < \gamma < 1 < \mu < 2$ ,  $\mu_0 = (2-\mu)\mu^{\mu/(2-\mu)}2^{2/(\mu-2)}$  and  $\gamma_0 = (2-\gamma)\gamma^{\gamma/(2-\gamma)}2^{2/(\gamma-2)}$ .

In 2017, Guezane-Lakoud et al. [11], derived a new Lyapunov type inequality for a boundary value problem involving both left Riemann-Liouville and right Caputo fractional derivatives in presence of natural conditions

$$\begin{aligned}
 -{}^C D_{b^-}^\alpha D_{a^+}^\beta u(t) + q(t)u(t) &= 0, \quad a < t < b, \quad 0 < \alpha, \beta \leq 1 \\
 u(a) = D_{a^+}^\beta u(b) &= 0,
 \end{aligned}$$

then, they obtained the following Lyapunov inequality:

$$\int_a^b |q(t)| dt > \frac{(\alpha + \beta - 1)\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha+\beta-1}}.$$

Recently, Ferreira in [9], derived a Lyapunov-type inequality for a sequential fractional right-focal boundary value problem

$$\begin{aligned}
 {}^C D_{a^+}^\alpha D_{a^+}^\beta u(t) + q(t)u(t) &= 0, \quad a < t < b \\
 u(a) = D_{a^+}^\gamma u(b) &= 0,
 \end{aligned}$$

where  $0 < \alpha, \beta, \gamma \leq 1$ ,  $1 < \alpha + \beta \leq 2$ , then, they obtained the following Lyapunov inequality:

$$\int_a^b (b-s)^{\alpha+\beta-\gamma-1} |q(t)| dt > \frac{1}{C},$$

where

$$\begin{aligned}
 C = (b-a)^\gamma \max \left\{ \frac{\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\beta+1)}, \right. \\
 \left. \frac{1-\alpha}{\beta\Gamma(\alpha+\beta)} \left( \frac{\Gamma(\beta-\gamma+1)\Gamma(\alpha+\beta-1)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\beta)} \right)^{\frac{\alpha+\beta-1}{\alpha-1}}, \text{ with } \alpha < 1 \right\}
 \end{aligned}$$

Note that more generalized Lyapunov type inequalities have been obtained for conformable derivative differential equations in [13]. For more results on Lyapunov-type inequalities for fractional differential equations, we refer to the recent survey of Ntouyas et al. [18].

In this work, we obtain Lyapunov type inequality for the following mixed fractional differential equation involving both right Caputo and left Riemann-Liouville fractional derivatives

$$-{}^C D_{b^-}^\alpha D_{a^+}^\beta u(t) + q(t)u(t) = 0, \quad a < t < b, \quad (1.6)$$

satisfying the Dirichlet boundary conditions (1.2), here  $0 < \beta \leq \alpha \leq 1$ ,  $1 < \alpha + \beta \leq 2$ ,  ${}^C D_{b^-}^\alpha$  denotes right Caputo derivative,  $D_{a^+}^\beta$  denotes the left Riemann-Liouville and  $q$  is a continuous function on  $[a, b]$ .

So far, few authors have considered sequential fractional derivatives, and some Lyapunov type inequalities have been obtained. In this study, we place ourselves in a very general context, in that in each fractional operator, the order of the derivative can be different. Such problems, with both left and right fractional derivatives arise in the study of Euler-Lagrange equations for fractional problems of the calculus of variations [2,16,17]. However, the presence of a mixed left and right Caputo or Riemann-Liouville derivatives of order  $0 < \alpha < 1$  leads to great difficulties in the study of the properties of the Green function since in this case it's given as a fractional integral operator.

We recall the concept of fractional integral and derivative of order  $p > 0$ . For details, we refer the reader to [14,19,21]

The left and right Riemann-Liouville fractional integral of a function  $g$  are defined respectively by

$$I_{a^+}^p g(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{g(s)}{(t-s)^{1-p}} ds,$$

$$I_{b^-}^p g(t) = \frac{1}{\Gamma(p)} \int_t^b \frac{g(s)}{(s-t)^{1-p}} ds.$$

The left and right Caputo derivatives of order  $p > 0$ , of a function  $g$  are respectively defined as follows:

$${}^C D_{a^+}^p g(t) = I_{a^+}^{n-p} g^{(n)}(t),$$

$${}^C D_{b^-}^p g(t) = (-1)^n I_{b^-}^{n-p} g^{(n)}(t),$$

and the left and right Riemann-Liouville fractional derivatives of order  $p > 0$ , of a function  $g$  are respectively defined as follows:

$$D_{a^+}^p g(t) = \frac{d^n}{dt^n} (I_{a^+}^{n-p} g)(t),$$

$$D_{b^-}^p g(t) = (-1)^n \frac{d^n}{dt^n} I_{b^-}^{n-p} g(t),$$

where  $n$  is the smallest integer greater or equal than  $p$ .

We also recall the following properties of fractional operators. Let  $0 < p < 1$ , then:

- 1-  $I_{a^+}^{pC} D_{a^+}^p f(t) = f(t) - f(a)$ .
- 2-  $I_{b^-}^{pC} D_{b^-}^p f(t) = f(t) - f(b)$ .
- 3-  $(I_{a^+}^p c)(t) = \frac{c(t-a)^p}{\Gamma(p+1)}$ ,  $c \in \mathbb{R}$
- 4-  $D_{a^+}^p u(t) = {}^C D_{a^+}^p u(t)$ , when  $u(a) = 0$ .
- 5-  $D_{b^-}^p u(t) = {}^C D_{b^-}^p u(t)$ , when  $u(b) = 0$ .

## 2. Lyapunov inequality

Next we transform the problem (1.6) with (1.2) to an equivalent integral equation.

**Lemma 1.** Assume that  $0 < \alpha, \beta \leq 1$ . The function  $u$  is a solution to the boundary value problem (1.6) with (1.2) if and only if  $u$  satisfies the integral equation

$$u(t) = \int_a^b G(t, r) q(r) u(r) dr, \quad (2.1)$$

where

$$G(t, r) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^{\inf\{r,t\}} (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right. \\ \left. - \frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) \quad (2.2)$$

is the Green's function of problem (1.6) with (1.2).

*Proof.* Firstly, we apply the right side fractional integral  $I_{b^-}^\alpha$  to equation (1.6), then the left side fractional integral  $I_{a^+}^\beta$  to the resulting equation and taking into account the properties of Caputo and Riemann-Liouville fractional derivatives and the fact that  $D_{a^+}^\beta u(t) = {}^C D_{a^+}^\beta u(t)$ , we get

$$u(t) = I_{a^+}^\beta I_{b^-}^\alpha q(t)u(t) + \frac{c(t-a)^\beta}{\Gamma(\beta+1)}. \quad (2.3)$$

In view of the boundary condition  $u(b) = 0$ , we get

$$c = \frac{-\Gamma(\beta+1)}{(b-a)^\beta} I_{a^+}^\beta I_{b^-}^\alpha q(t)u(t) \Big|_{t=b}.$$

Substituting  $c$  in (2.3), it yields

$$\begin{aligned} u(t) &= I_{a^+}^\beta I_{b^-}^\alpha q(t)u(t) - \frac{(t-a)^\beta}{(b-a)^\beta} I_{a^+}^\beta I_{b^-}^\alpha q(t)u(t) \Big|_{t=b} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} \left( \int_s^b (r-s)^{\alpha-1} q(r)u(r)dr \right) ds \\ &\quad - \frac{(t-a)^\beta}{\Gamma(\alpha)\Gamma(\beta)(b-a)^\beta} \int_a^b (b-s)^{\beta-1} \left( \int_s^b (r-s)^{\alpha-1} q(r)u(r)dr \right) ds. \end{aligned}$$

Finally, by exchanging the order of integration, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left( \int_a^r (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) q(r)u(r)dr \\ &\quad + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_t^b \left( \int_a^t (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) q(r)u(r)dr \\ &\quad - \frac{(t-a)^\beta}{\Gamma(\alpha)\Gamma(\beta)(b-a)^\beta} \int_a^b \left( \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) q(r)u(r)dr, \end{aligned}$$

thus

$$u(t) = \int_a^b G(t,r)q(r)u(r)dr,$$

with

$$G(t,r) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \begin{cases} \int_a^r (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \\ -\frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds, a \leq r \leq t \leq b, \\ \int_a^t (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \\ -\frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds, a \leq t \leq r \leq b. \end{cases}$$

that can be written as

$$\begin{aligned} G(t,r) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^{\inf\{r,t\}} (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right. \\ &\quad \left. - \frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right). \end{aligned}$$

Conversely, we can verify that if  $u$  satisfies the integral equation (2.1), then  $u$  is a solution to the boundary value problem (1.6) with (1.2). The proof is completed.  $\square$

In the next Lemma we give the property of the Green function  $G$  that will be needed in the sequel.

**Lemma 2.** Assume that  $0 < \beta \leq \alpha \leq 1$ ,  $1 < \alpha + \beta \leq 2$ , then the Green function  $G(t, r)$  given in (2.2) of problem (1.6) with (1.2) satisfies the following property:

$$|G(t, r)| \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)(\alpha + \beta - 1)(\alpha + \beta)} \left( \frac{\alpha(b-a)}{\beta + \alpha} \right)^{\alpha + \beta - 1},$$

for all  $a \leq r \leq t \leq b$ .

*Proof.* Firstly, for  $a \leq r \leq t \leq b$ , we have  $G(t, r) \geq 0$ . In fact, we have

$$\begin{aligned} G(t, r) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^r (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right. \\ &\quad \left. - \frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) \\ &\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right. \\ &\quad \left. - \frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( 1 - \frac{(t-a)^\beta}{(b-a)^\beta} \right) \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \geq 0 \end{aligned} \quad (2.4)$$

in addition,

$$\begin{aligned} G(t, r) &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^r (r-s)^{\beta-1} (r-s)^{\alpha-1} ds \right. \\ &\quad \left. - \frac{(r-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) \\ &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(r-a)^{\alpha+\beta-1}}{(\alpha + \beta - 1)} \right. \\ &\quad \left. - \frac{(r-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(r-a)^{\alpha+\beta-1}}{(\alpha + \beta - 1)} - \frac{(r-a)^{\beta+\alpha}}{\alpha(b-a)} \right). \end{aligned} \quad (2.5)$$

Thus, from (2.4) and (2.5), we get

$$0 \leq G(t, r) \leq h(r), \quad a \leq r \leq t \leq b, \quad (2.6)$$

where

$$h(s) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(s-a)^{\alpha+\beta-1}}{(\alpha + \beta - 1)} - \frac{(s-a)^{\beta+\alpha}}{\alpha(b-a)} \right),$$

it is clear that  $h(s) \geq 0$ , for all  $s \in [a, b]$ .

Now, for  $a \leq t \leq r \leq b$ , we have

$$\begin{aligned}
 G(t, r) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^t (t-s)^{\beta-1} (r-s)^{\alpha-1} ds \right. \\
 &\quad \left. - \frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (b-s)^{\beta-1} (r-s)^{\alpha-1} ds \right) \\
 &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \int_a^t (t-s)^{\beta-1} (t-s)^{\alpha-1} ds \right. \\
 &\quad \left. - \frac{(t-a)^\beta}{(b-a)} \int_a^r (r-s)^{\alpha-1} ds \right) \\
 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(t-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)} - \frac{(t-a)^\beta (r-a)^\alpha}{\alpha(b-a)} \right) \\
 &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(t-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)} - \frac{(t-a)^{\beta+\alpha}}{\alpha(b-a)} \right) = h(t). \tag{2.7}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 G(t, r) &\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (r-a)^{\alpha-1} \int_a^t (t-s)^{\beta-1} ds \\
 &\quad - \frac{(t-a)^\beta}{(b-a)^\beta} \int_a^r (r-s)^{\beta-1} (r-s)^{\alpha-1} ds \\
 &\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(t-a)^\alpha (t-a)^\beta}{\beta(b-a)} - \frac{(t-a)^\beta (r-a)^{\alpha+\beta-1}}{(b-a)^\beta (\alpha+\beta-1)} \right) \\
 &\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(t-a)^{\alpha+\beta}}{\beta(b-a)} - \frac{(t-a)^\beta (r-a)^{\alpha-1}}{(\alpha+\beta-1)} \right) \\
 &\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{(t-a)^{\alpha+\beta}}{\beta(b-a)} - \frac{(t-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)} \right),
 \end{aligned}$$

since  $\beta \leq \alpha$ , we get

$$G(t, r) \geq -h(t), \quad a \leq t \leq r \leq b. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$|G(t, r)| \leq h(t), \quad a \leq t \leq r \leq b. \tag{2.9}$$

Finally, by differentiating the function  $h$ , it yields

$$h'(s) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (s-a)^{\alpha+\beta-2} \left( 1 - \frac{(\beta+\alpha)(s-a)}{\alpha(b-a)} \right).$$

We can see that  $h'(s) = 0$  for  $s_0 = a + \frac{\alpha(b-a)}{(\beta+\alpha)} \in (a, b)$ ,  $h'(s) < 0$  for  $s > s_0$  and  $h'(s) > 0$  for  $s < s_0$ . Hence, the function  $h(s)$  has a unique maximum given by

$$\max_{s \in [a, b]} h(s) = h(s_0)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left( \frac{\left(\frac{\alpha(b-a)}{\beta+\alpha}\right)^{\alpha+\beta-1}}{(\alpha+\beta-1)} - \frac{\left(\frac{\alpha(b-a)}{\beta+\alpha}\right)^{\beta+\alpha}}{\alpha(b-a)} \right) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)} \left( \frac{\alpha(b-a)}{\beta+\alpha} \right)^{\alpha+\beta-1}.
\end{aligned}$$

From (2.6) and (2.9), we get  $|G(t, r)| \leq h(s_0)$ , from which the intended result follows.  $\square$

Next, we state and prove the Lyapunov type inequality for problem (1.6) with (1.2).

**Theorem 3.** *Assume that  $0 < \beta \leq \alpha \leq 1$  and  $1 < \alpha + \beta \leq 2$ . If the fractional boundary value problem (1.6) with (1.2) has a nontrivial continuous solution, then*

$$\int_a^b |q(r)| dr \geq \frac{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}}{(\alpha(b-a))^{\alpha+\beta-1}}. \quad (2.10)$$

*Proof.* Let  $X = C[a, b]$  be the Banach space endowed with norm  $\|u\| = \max_{t \in [a, b]} |u(t)|$ . It follows from Lemma 1 that a solution  $u \in X$  to the boundary value problem (1.6) with (1.2) satisfies

$$\begin{aligned}
|u(t)| &\leq \int_a^b |G(t, r)| |q(r)| |u(r)| dr \\
&\leq \|u\| \int_a^b |G(t, r)| q(r) dr,
\end{aligned}$$

Now, applying Lemma 2 to equation (2.1), it yields

$$|u(t)| \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)} \left( \frac{\alpha(b-a)}{\beta+\alpha} \right)^{\alpha+\beta-1} \|u\| \int_a^b |q(r)| dr$$

Hence,

$$\|u\| \leq \frac{(\alpha(b-a))^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta-1)(\alpha+\beta)^{\alpha+\beta}} \|u\| \int_a^b |q(r)| dr,$$

from which the inequality (2.10) follows. Note that the constant in (2.10) is not sharp. The proof is completed.  $\square$

**Remark 4.** *Note that, according to boundary conditions (1.2), the Caputo derivatives  ${}^C D_{b^-}^\alpha$  and  ${}^C D_{a^+}^\beta$  coincide respectively with the Riemann-Liouville derivatives  $D_{b^-}^\alpha$  and  $D_{a^+}^\beta$ . So, equation (1.6) is reduced to the one containing only Caputo derivatives or only Riemann-Liouville derivatives, i.e.,*

$$-{}^C D_{b^-}^\alpha {}^C D_{a^+}^\beta u(t) + q(t)u(t) = 0, \quad a < t < b$$

or

$$-{}^C D_{b^-}^\alpha D_{a^+}^\beta u(t) + q(t)u(t) = 0, \quad a < t < b$$

Furthermore, by applying the reflection operator  $(Qf)(t) = f(a+b-t)$  and taking into account that  $Q^C D_{a^+}^\alpha = {}^C D_{b^-}^\alpha Q$  and  $Q^C D_{b^-}^\beta = {}^C D_{a^+}^\beta Q$  (see [21]), we can see that, the boundary value problem (1.6) with (1.2) is equivalent to the following problem

$$\begin{aligned}
-{}^C D_{a^+}^\alpha D_{b^-}^\beta u(t) + q(t)u(t) &= 0, \quad a < t < b, \\
u(a) &= u(b) = 0.
\end{aligned}$$



**Remark 5.** If we take  $\alpha = \beta = 1$ , then the Lyapunov type inequality (2.3) is reduced to

$$\int_a^b |q(t)| dt \geq \frac{4}{b-a}.$$

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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