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Research article

Hamilton's gradient estimate for fast diffusion equations under geometric flow

Ghodratallah Fasihi-Ramandi*

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

* **Correspondence:** Email: fasihi@sci.ikiu.ac.ir; Tel: +982833901321; Fax: +982833901321.

Abstract: Suppose that M is a complete noncompact Riemannian manifold of dimension n. In the present paper, we obtain a Hamilton's gradient estimate for positive solutions of the fast diffusion equations

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad 1 - \frac{4}{n+8} < m < 1$$

on $M \times (-\infty, 0]$ under the geometric flow.

Keywords: fast diffusion equation; Ricci flow; Hamilton inequality; gradient estimates
Mathematics Subject Classification: 53C21, 53C44, 58J35

1. Introduction

Starting with the pioneering work of P. Li and S. T. Yau in the seminal paper [6], gradient estimates are also called differential Harnack inequalities, because one can obtain the classical Harnack inequality after integrating the gradient estimate along paths in space-time. These concepts are very powerful tools in geometric analysis. For example, R. Hamilton established differential Harnack inequalities for the mean curvature along the mean curvature flow and for the scalar curvature along the Ricci flow. Both have important applications in the analysis of singularities.

In Perelman's work on the Poincar conjecture and the geometrization conjecture, differential Harnack inequality played an important role. Since then, there have been many works on gradient estimates along the Ricci flow or the conjugate Ricci flow for the solution of the heat equation or the conjugate heat equation; examples include ([3], [7]). Later, Sun [8] extended these results to general geometric flow.

Under some curvature constraints, in [4] the authors have established a Hamilton's gradient estimates for the fast diffusion equations under Ricci flow on a complete noncompact Riemannian manifold. We can strengthen the assumption of their results by considering the general geometric flow. In this paper, we will study the interesting Li-Yau type estimate for positive solutions of fast diffusion equations (FDE for short)

$$\frac{\partial u}{\partial t} = \Delta u^m, m < 1 \tag{1.1}$$

on complete noncompact Riemannian manifold M with evolving metric under the general geometric flow.

Before presenting our main results about the equation, it seems necessary to support our idea of considering this equation. FDE describes physical processes of diffusion in plasma, gas kinetics, thin liquid film dynamics and so on. This equation also arises in many geometric phenomena, and we refer the reader to the book [10] for more details. The exact solutions have obtained for anomalous diffusions in the context of the Tsallis statistics [9]. Also, fractional diffusion equation and diffusion equation associated with non-extensive statistical mechanics have been studied (for instance, see [2] and [5]).

Let $t \in [0, T]$ and $(M, g(t))$ be a complete solution to the general geometric flow

$$\frac{\partial g_{ij}}{\partial t} = 2h_{ij}. \tag{1.2}$$

To study the positive solution of FDE, we use the following transformation,

$$f = \frac{m}{1-m}(u^{1-m} - 1) \tag{1.3}$$

which is known as Hopf transformation of u, and it is very useful in forgoing because

$$\lim_{m\to 1} f = \log u.$$

By above assumption (3.1) can be rewritten as

$$f_t = \frac{m^2}{(1-m)f + m}\Big(\Delta f + \frac{2m-1}{(1-m)f + m}|\nabla f|^2\Big). \tag{1.4}$$

Now we can present our main result for the system (3.1) and (1.4) in the following theorem.

Theorem 1.1. *Suppose $(M^n, g(t))_{t\in[0,T]}$ is a complete solution to (1.2) and*

$$-\frac{k}{2}g_{ij} \le R_{ij} \le \frac{k}{2}g_{ij}, \quad -\frac{k}{2}g_{ij} \le h_{ij} \le \frac{k}{2}g_{ij}$$

on $B_{\rho,T}$ for some positive constant k. Assume that f is any positive solution to (1.4) and $1 - \frac{4}{n+8} < m < 1$. If $0 \le f \le 1 - \frac{1}{m}$ in $B_{\rho,T}$ for each m, then there exists a constant $C = C(n)$ such that

$$\frac{|\nabla f|}{1-f} \le C\Big[(\frac{1}{\sqrt{m}} + 1)\sqrt{k} + \frac{1}{\rho} + \frac{1}{\sqrt{mt}}\Big]$$

in $B_{\frac{\rho}{2},T}$ with $t \ne 0$. Together with the transformation (1.3), we have

$$m\frac{|\nabla u|}{u^m} \le C\Big[(\frac{1}{\sqrt{m}} + 1)\sqrt{k} + \frac{1}{\rho} + \frac{1}{\sqrt{mt}}\Big]\frac{1-mu^{1-m}}{1-m}$$

in $B_{\frac{\rho}{2},T}$ with $t \ne 0$.

Bailesteanu, Cao and Pulemotov in [1] proved a gradient estimate for positive solutions to the heat equation $u_t = \Delta u$ under the Ricci flow. Now, as a corollary we obtain the same inequality when the Riemannain metric is evolved by the general geometric flow (1.2).

Corollary 1.2. *When $m \to 1$ and $0 < u \leq A$, we have*

$$\frac{|\nabla u|}{u} \leq C\Big(\sqrt{k} + \frac{1}{\rho} + \frac{1}{\sqrt{t}}\Big)\Big(1 + \log\frac{A}{u}\Big).$$

2. Proof of the main theorem

In this section we clue on the proof of our main result. To this end, we need two important lemmas.

Lemma 2.1. *Let the smooth positive function $f : M \times [0, T] \to \mathbb{R}$ satisfies (1.4) and denote $w = \frac{|\nabla f|^2}{(1-f)^2}$ then by assumptions of Theorem (1.1), we have*

$$\begin{aligned}\mathcal{L}(w) \geq\ & \frac{m^2(4(2m-1)(1-m) - n(1-m)^2)}{((1-m)f+m)^3}(1-f)^2 w^2 + \frac{m^2(3m(1-f)+f-2)}{((1-m)f+m)^2}w^2\\ & - \frac{2(1-m)|m+f|^k}{(1-m)f+m}w + \frac{m^2((1+m)f+2-m)}{((1-m)f+m)^2(1-f)}\nabla w \nabla f,\end{aligned}$$

where

$$\mathcal{L} = \frac{m^2}{(1-m)f+m}\Delta - \frac{\partial}{\partial t}.$$

Proof. We have,

$$\begin{aligned} w_t =& \frac{2h_{ij}f_if_j}{(1-f)^2} + \frac{2f_j\nabla_j(f_t)}{(1-f)^2} + \frac{2|\nabla f|^2 f_t}{(1-f)^3}\\ =& \frac{2h_{ij}f_if_j}{(1-f)^2} + \frac{2m^2 f_j\nabla_j(\Delta f)}{((1-m)f+m)(1-f)^2} - \frac{2m^2(1-m)\Delta f|\nabla f|^2}{((1-m)f+m)^2(1-f)^2}\\ &+ \frac{4m^2(2m-1)f_if_jf_{ij}}{((1-m)f+m)^2(1-f)^2} - \frac{4m^2(2m-1)(1-m)|\nabla f|^4}{((1-m)f+m)^3(1-f)^2}\\ &+ \frac{2m^2|\nabla f|^2\Delta f}{((1-m)f+m)(1-f)^3} + \frac{2m^2(2m-1)|\nabla f|^4}{((1-m)f+m)^2(1-f)^3}. \end{aligned}$$

Also, notice that

$$\begin{aligned} \nabla_i w &= \frac{2f_jf_{ij}}{(1-f)^2} + \frac{2|\nabla f|^2 f_i}{(1-f)^3}\\ &= \Delta w = \frac{2f_{ij}^2}{(1-f)^2} + \frac{2f_j\Delta(f_j)}{(1-f)^2} + \frac{8f_if_jf_{ij}}{(1-f)^3} + \frac{2\Delta f|\nabla f|^2}{(1-f)^3} + \frac{6|\nabla f|^4}{(1-f)^4}. \end{aligned}$$

The above computations follow that

$$\mathcal{L}(w) = \frac{2m^2 f_{ij}^2}{((1-m)f+m)(1-f)^2} + \frac{8m^2 f_if_jf_{ij}}{((1-m)f+m)(1-f)^3}$$

$$
\begin{aligned}
&+\frac{6m^2|\nabla f|^4}{((1-m)f+m)(1-f)^4}+\frac{2m^2(1-m)\Delta f|\nabla f|^2}{((1-m)f+m)^2(1-f)^2}+\frac{4m^2(1-m)(2m-1)|\nabla f|^2}{((1-m)f+m)^3(1-f)^2}\\
&+\frac{2m^2 f_j\Delta(f_j)}{((1-m)f+m)(1-f)^2}-\frac{2m^2 f_j\nabla_j(\Delta f)}{((1-m)f+m)(1-f)^2}-\frac{2R_{ij}f_if_j}{(1-f)^2}\\
&-\frac{4m^2(2m-1)f_if_jf_{ij}}{((1-m)f+m)^2(1-f)^2}-\frac{2m^2(2m-1)|\nabla f|^4}{((1-m)f+m)^2(1-f)^3}.
\end{aligned}
$$

By the Bochner formula, we have

$$
f_i(f_{jji}-f_{ijj})=R_{ij}f_if_j
$$

and

$$
\frac{6m^2 f_if_jf_{ij}}{((1-m)f+m)(1-f)^3}=\frac{3m^2\nabla w\nabla f}{((1-m)f+m)(1-f)}-\frac{6m^2|\nabla f|^4}{((1-m)f+m)(1-f)^4},
$$

so, we deduce that

$$
\begin{aligned}
\mathcal{L}(w)=&\frac{2m^2 f_{ij}^2}{((1-m)f+m)(1-f)^2}+\frac{2m^2 f_if_jf_{ij}}{((1-m)f+m)(1-f)^3}\\
&+\frac{2m^2(1-m)\Delta f|\nabla f|^2}{((1-m)f+m)^2(1-f)^2}+\frac{4m^2(1-m)(2m-1)|\nabla f|^4}{((1-m)f+m)^3(1-f)^2}\\
&+\frac{2m^2R_{ij}f_if_j}{((1-m)f+m)(1-f)^2}+\frac{2h_{ij}f_if_j}{(1-f)^2}+\frac{2m^2(2m-1)|\nabla f|^4}{((1-m)f+m)^2(1-f)^3}\\
&+\frac{3m^2\nabla w\nabla f}{((1-m)f+m)(1-f)}-\frac{2m^2(2m-1)}{((1-m)f+m)^2}\nabla w\nabla f.
\end{aligned}
$$

We have the following estimate for the first three terms of the above formula,

$$
\begin{aligned}
&\frac{2m^2 f_{ij}^2}{((1-m)f+m)(1-f)^2}+\frac{2m^2 f_if_jf_{ij}}{((1-m)f+m)(1-f)^3}+\frac{2m^2(1-m)\Delta f|\nabla f|^2}{((1-m)f+m)^2(1-f)^2}\\
\geq&\frac{m^2}{(1-m)f+m}\sum_{i,j=1}^{n}\Big(\frac{f_{ij}}{1-f}+\frac{f_if_j}{(1-f)^2}\Big)^2-\frac{m^2f_i^2f_j^2}{((1-m)f+m)(1-f)^4}\\
&+\frac{m^2}{((1-m)f+m)(1-f)^2}\Big(\frac{f_{ij}}{\sqrt{n}}+\frac{\sqrt{n}(1-m)|\nabla f|^2}{(1-m)f+m}\Big)^2-\frac{nm^2(1-m)^2|\nabla|^2}{((1-m)f+m)^3(1-f)^2}\\
\geq&-\frac{m^2f_i^2f_j^2}{((1-m)f+m)(1-f)^4}-\frac{nm^2(1-m)^2|\nabla f|^4}{((1-m)f+m)^3(1-f)^2}.
\end{aligned}
$$

On the other hand,

$$
\frac{2m^2R_{ij}f_if_j}{((1-m)f+m)(1-f)^2}+\frac{2h_{ij}f_if_j}{(1-f)^2}\geq-\frac{2(1-m)k|m+f||\nabla f|^2}{((1-m)f+m)(1-f)^2},
$$

where we have used inequalities $-kg_{ij}\leq R_{ij}+h_{ij}\leq kg_{ij}$, $f\leq 0$ and $f_{ij}^2\geq\frac{f_{ii}^2}{n}$. Therefore,

$$
\mathcal{L}(w)\geq-\frac{m^2f_i^2f_j^2}{((1-m)f+m)(1-f)^4}-\frac{nm^2(1-m)^2|\nabla f|^4}{((1-m)f+m)^3(1-f)^2}
$$

$$
\begin{aligned}
&-\frac{2(1-m)k|m+f||\nabla f|^2}{((1-m)f+m)(1-f)^2}+\frac{4m^2(1-m)(2m-1)|\nabla f|^4}{((1-m)f+m)^3(1-f)^2}\\
&+\frac{2m^2(2m-1)|\nabla f|^4}{((1-m)f+m)^2(1-f)^3}+\frac{m^2((1+m)f+2-m)}{((1-m)f+m)^2(1-f)}\nabla w\nabla f\\
=&\frac{m^2(4(2m-1)(1-m)-n(1-m)^3)}{((1-m)f+m)^3}(1-f)^2w^2+\frac{m^2(3m(1-f)+f-2)}{((1-m)f+m)^2}w^2\\
&-\frac{2(1-m)|m+f|k}{(1-m)f+m}w+\frac{m^2((1+m)f+2-m)}{((1-m)f+m)^2(1-f)}\nabla w\nabla f.
\end{aligned}
$$

□

In order to get the desired result, we take a cut-off function Ψ by Li-Yau [6] on $B_{\frac{\rho}{2},T}$. Define a smooth function $\Psi : M\times[0,T]\to\mathbb{R}$ by $\Psi(x,t)=\widetilde{\Psi}(dis(x,x_0,t),t)$, supported in $B_{\frac{\rho}{2},T}$. The construction of Ψ depends on its properties as came in the following lemma.

Lemma 2.2. *[4] For a given $\tau\in(0,T]$, the smooth function $\widetilde{\Psi}$ satisfies the following properties:*

1. $0\le\widetilde{\Psi}\le 1$ *on* $[0,\rho]\times[0,T]$.
2. $\widetilde{\Psi}(r,t)=1$ *on* $[0,\frac{\rho}{2}]\times[\tau,T]$ *and* $\frac{\partial\widetilde{\Psi}}{\partial r}(r,t)=0$ *on* $[0,\frac{\rho}{2}]\times[0,T]$.
3. $\frac{|\partial_t\widetilde{\Psi}|}{\Psi^{\frac{1}{2}}}\le\frac{C}{\tau}$ *on* $[0,\infty)\times[,T]$, $C>0$ *and* $\widetilde{\Psi}(r,0)=0$ *where* $r\in[0,\infty)$.
4. $-\frac{C_a}{\rho}\le\frac{\partial_r\widetilde{\Psi}}{\widetilde{\Psi}^a}\le 0$ *and* $\frac{\partial_r^2\Psi}{\widetilde{\Psi}^a}\le\frac{C_a}{\rho^2}$ *for* $a\in(0,1)$.

Now we are prepare to prove our main theorem.

Proof of Theorem 1.1: Assume the same notation of f and w in the Lemma (2.1). Denote $\beta=-\frac{\nabla f}{1-f}$. Straightforward computations show that

$$
\begin{aligned}
\mathcal{L}(\Psi w)&=\mathcal{L}(w)\Psi+\frac{2m^2}{(1-m)f+m}\nabla\Psi\nabla w+\frac{m^2}{(1-m)f+m}\Delta\Psi.w-\Psi_t w\\
&\ge\frac{m^2(4(2m-1)(1-m)-n(1-m)^2)}{((1-m)f+m)^3}(1-f)^2\Psi w^2\\
&+\frac{m^2(3m(1-f)+f-2)}{((1-m)f+m)^2}\Psi w^2-\frac{2(1-m)|m+f|^k}{(1-m)f+m}\Psi w\\
&-\frac{m^2((1+m)f+2-m)}{((1-m)f+m)^2}\beta\nabla(\Psi w)+\frac{m^2((1+m)f+2-m)}{((1-m)f+m)^2}\beta\nabla\Psi.w\\
&+\frac{2m^2}{(1-m)f+m}\frac{\nabla\Psi}{\Psi}\nabla(\Psi w)-\frac{2m^2}{(1-m)f+m}\frac{|\nabla\Psi|^2}{\Psi}w\\
&+\frac{m^2}{(1-m)f+m}\Delta\Psi.w-\Psi_t w.
\end{aligned}
$$

Let (x_1,t_1) be a point, at which the function Ψw attains its maximum value and x_1 is not in the cut-locus of M by [6]. Then at the point (x_1,t_1) the following conditions are hold.

$$
\Delta(\Psi w)\le 0,\quad (\Psi w)_t\ge 0,\quad \nabla(\Psi w)=0.
$$

It follows that

$$
\begin{aligned}
&\frac{(1-m)(4(2m-1)-n(1-m))}{(1-m)f+m}(1-f)^2\Psi w^2+(3m(1-f)+f-2)\Psi w^2\\
&\le \frac{2(1-m)((1-m)f+m)|m+f|k}{m^2}\Psi w-((1+m)f+2-m)\beta\nabla\Psi.w\\
&+2((1-m)f+m)\frac{|\nabla\Psi|^2}{\Psi}w-((1-m)f+m)\Delta\Psi.w+\frac{((1-m)f+m)^2}{m^2}\Psi_t w.
\end{aligned}
\tag{2.1}
$$

Since $m\in(1-\dfrac{4}{8+n},1)$ the first term on the left side of the above inequality is positive,

$$
\frac{(1-m)(4(2m-1)-n(1-m))}{(1-m)f+m}\ge 0.
$$

Because, $0<f\le 1-\dfrac{1}{m}$, the second term satisfies

$$
3m(1-f)+f-2>(1-m)f+m.
$$

The above inequalities together with (2.1), yield that

$$
\begin{aligned}
\Psi w^2\le&\frac{2(1-m)|m+f|k}{m^2}\Psi w-\frac{(1+m)f+2-m}{(1-m)f+m}\beta\nabla\Psi.w\\
&+2\frac{|\Psi|^2}{\Psi}w-\Delta\Psi.w+\frac{(1-m)f+2-m}{m^2}\Psi_t w.
\end{aligned}
\tag{2.2}
$$

In foregoing, we estimate each term on the right hand side of (2.2). Using $\Psi^{\frac{1}{2}}\le 1$, for the fist term we have

$$
\frac{2(1-m)|m+f|k}{m^2}\Psi w\le\frac{1}{8}\Psi w^2+\frac{(1-m)^2(m+f)^2}{m^4}c_1k^2,
$$

where c_1 is a positive constant. And for the second term, we proceed as in the following

$$
\begin{aligned}
-\frac{(1+m)f+2-m}{(1-m)f+m}\beta\nabla\Psi.w&\le\frac{(1+m)f+2-m}{(1-m)f+m}\nabla\Psi.w^{\frac{3}{2}}\\
&\le\Psi^{\frac{3}{4}}w^{\frac{3}{2}}.\frac{(1+m)f+2-m}{(1-m)f+m}\frac{|\nabla\Psi|}{\Psi^{\frac{3}{4}}}\\
&\le\frac{1}{8}\Psi w^2+\frac{((1+m)f+2-m)^4}{((1-m)f+m)^4}\frac{c_2}{\rho^4}
\end{aligned}
$$

with chosen positive constant c_2. Straightforward computations show that the following inequalities hold

$$
\begin{aligned}
2\frac{|\nabla\Psi|^2}{\Psi}w&=\Psi^{\frac{1}{2}}w.2\frac{|\nabla\Psi|^2}{\Psi^{\frac{3}{2}}}\le\frac{1}{8}\Psi w^2+\frac{c_3}{\rho^4}\\
-\Delta\Psi.w&\le\frac{1}{8}\Psi w^2+\frac{c_4}{\rho^4}+c_4k^2,
\end{aligned}
$$

where c_3 and $c_4 = c_4(n)$ are positive constants. Finally, for the last term, denote $\gamma = \dfrac{(1-m)f+m}{m^2}$ and it follows that

$$\begin{aligned}\gamma\frac{\partial\Psi}{\partial t}w &\leq \gamma|\frac{\partial\bar{\Psi}}{\partial t}|w + \gamma|\frac{\partial\bar{\Psi}}{\partial r}||\frac{\partial}{\partial t}dist|w \\ &\leq \frac{1}{16}\Psi w^2 + \gamma^2\frac{c_5}{\tau^2} + \gamma\frac{C_{\frac{1}{2}}}{\rho}\Psi^{\frac{1}{2}}.\sup\int_0^{dist}|Ric\Big(\frac{d\zeta(s)}{ds},\frac{d\zeta(s)}{ds}\Big)|ds \\ &\leq \frac{1}{16}\Psi w^2 + \gamma^2\frac{c_5}{\tau^2} + \gamma C_{\frac{1}{2}}kw\Psi^{\frac{1}{2}} \\ &\leq \frac{1}{18}\Psi w^2 + \gamma^2\frac{c_5}{\tau^2} + c_6\gamma^2k^2 \qquad c_5, c_6 > 0.\end{aligned}$$

Adding these inequalities into (2.2), we deduce

$$\begin{aligned}\Psi^2w^2 &\leq \Psi w^2 \\ &\leq C''\Big[\Big(\frac{(1-m)^2(m+f)^2}{m^4} + \frac{((1-m)f+m)^2}{m^4} + 1\Big)k^2 \\ &\quad + \Big(\frac{((1+m)f+2-m)^4}{((1-m)f+m)^4} + 1\Big)\frac{1}{\rho^4} + \frac{((1-m)f+m)^2}{m^4}\frac{1}{\tau^2}\Big]\end{aligned}$$

at (x_1, t_1) with $C'' = C''(n) = 2\max\{c_1, \ldots, c_6\}$ is a positive real. Applying the inequality $\sqrt{x^2+y^2} \leq x + y$ which holds for $x, y \geq 0$ and using $\Psi(x,\tau) \leq 1$, then for all $x \in M$ we have the following estimate

$$\begin{aligned}w(x,\tau) = (\Psi w)(x,\tau) &\leq (\Psi w)(x_1, t_1) \\ &\leq C'^2\Big[\Big(\frac{(1-m)^2|m+f|}{m^2} + \frac{(1-m)f+m}{m^2} + 1\Big)k \\ &\quad + \Big(\frac{((1+m)f+2-m)^2}{((1-m)f+m)^2} + 1\Big)\frac{1}{\rho^2} + \frac{(1-m)f+m}{m^2}\frac{1}{\tau}\Big],\end{aligned}$$

where $C' = \sqrt{C''}$. Since $\tau \in (0, T]$ was chosen arbitrary, we obtain

$$\begin{aligned}\frac{|\nabla f(x,t)|}{1-f(x,t)} &\leq C'\Big[\Big(\frac{\sqrt{(1-m)|m+f|}}{m} + \frac{\sqrt{(1-m)f+m}}{m} + 1\Big)k \\ &\quad + \Big(\frac{(1+m)f+2-m}{(1-m)f+m} + 1\Big)\frac{1}{\rho} + \frac{\sqrt{(1-m)f+m}}{m}\frac{1}{\sqrt{t}}\Big].\end{aligned}$$

Since $0 < f \leq 1 - \dfrac{1}{m}$, we know

$$\frac{(1+m)f+2-m}{(1-m)f+m} + 1 \leq 2.$$

Notice that

$$|m+f| = \frac{m}{1-m}|u^{1-m} - m| \leq \frac{m(1+m)}{1-m},$$

$$(1-m)f + m = mu^{1-m} \leq m.$$

Finally we obtain

$$\frac{|\nabla f|}{1-f} \leq C\Big[(\frac{1}{\sqrt{m}}+1)\sqrt{k}+\frac{1}{\rho}+\frac{1}{\sqrt{mt}}\Big]$$

with $C = C(n)$. Now by replacing f with u, the above inequality yields

$$m\frac{|\nabla u|}{u^m} \leq C\Big[(\frac{1}{\sqrt{m}}+1)\sqrt{k}+\frac{1}{\rho}+\frac{1}{\sqrt{mt}}\Big]\frac{1-mu^{1-m}}{1-m},$$

which finishes the proof of the theorem. □

Now we present the proof of Corollary 1.2 . Indeed, when $m \to 1$ and $0 < u \leq A$, we have

$$\lim_{m\to 1}\frac{1-mu^{1-m}}{1-m} = \lim_{m\to 1}(1-f) = 1+\log\frac{A}{u}$$

Then

$$\frac{|\nabla u|}{u} \leq C\Big(\sqrt{k}+\frac{1}{\rho}+\frac{1}{\sqrt{t}}\Big)\Big(1+\log\frac{A}{u}\Big)$$

which completes the desired result.

3. Conclusion

Fast diffusion equations are important types of partial differential equations. These equations play an important role in describing physical processes of diffusion in plasma, gas kinetics, thin liquid film dynamics and so on. Also, this equation also arises in many geometric phenomena. In this paper, we considered the fast diffusion equations

$$\frac{\partial u}{\partial t} = \Delta u^m, m < 1 \tag{3.1}$$

on complete noncompact Riemannian manifold M with evolving metric under the general geometric flow. Under some curvature constraints, we established a Hamiltons gradient estimates for this equation under general geometric flows. Depending on the physical problem and how the metric evolves, this estimate will have important interpretation in that physical phenomena.

Conflict of interest

The author declares that there is no conflicts of interest in this paper.

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