



Research article

Progressive contractions, product contractions, quadratic integro-differential equations

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Abstract: Fixed point theory has been used very successfully to obtain properties of solutions of integral and integro-differential equations of the form

$$x(t) = g(t) + \int_0^t A(t-s)v(t, s, x(s))ds$$

because under general conditions the integral term may map bounded sets of continuous functions into equi-continuous sets. But quadratic integral equations have a coefficient of the integral terms of the form

$$f(t, x(t)) \int_0^t A(t-s)v(t, s, x(s))ds$$

which destroys the compactness of the map. Investigators have resorted to deep solutions often involving measures of non-compactness and Darbo's fixed point theorem.

In an effort to obtain some elementary approaches, in this paper we develop an apparently new technique by showing that by using progressive contractions we can show conditions under which the product of two contractions is a contraction.

We focus on integro-differential equations and use direct fixed point mappings which convert Lipschitz conditions into progressive contraction conditions.

Keywords: progressive contractions; quadratic equation; integro-differential equation; existence and uniqueness

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1. Introduction

This is a study of fixed point theory for integral equations including fractional quadratic [1] equations of the form

$$x(t) = g(t, x(t)) + f(t, x(t)) \int_0^t (t-s)^{q-1} v(t, s, x(s)) ds, \quad 0 < q < 1, \quad (1.1)$$

which are known to model a number of important problems in applied mathematics. A good discussion of these equations along with extensive references is found in Darwish [2] and later in Darwish and Henderson [1].

Those problems are challenging because of the product term

$$f(t, x(t)) \int_0^t (t-s)^{q-1} v(t, s, x(s)) ds$$

and the fact that $f(t, x(t))$ spoils the property that the integral would map bounded sets of continuous functions on an interval $[0, E]$ into an equicontinuous set.

Along with the integral equation (1.1) there can be found much literature on the parallel integro-differential equation

$$x'(t) = g(t, x(t)) + f(t, x(t)) \int_0^t (t-s)^{q-1} v(t, s, x(s)) ds, \quad x(0) = a. \quad (1.2)$$

In the search for existence and uniqueness properties of (1.1) we find assumptions that g is a contraction, while f and v are Lipschitz in x , among other severe restrictions on the magnitude of that product term. We would expect as much for (1.2), but here is where the surprise and contrast is found.

Using a technique which we call “direct fixed point mappings” introduced some decades ago, we convert (2.1) into a form in which g, f, v have all become contractions in the derivative of x on short intervals. Then we use a technique which we call “progressive contractions” to make the product term a contraction with sufficiently small contraction constant that on repeatedly short intervals the entire equation is a contraction and we obtain a unique solution (of the derivative) on $[0, E)$. Denoting the derivative of x by ϕ we then have the unique solution given as

$$x(t) = a + \int_0^t \phi(s) ds$$

on $[0, E]$. This is then parlayed into a unique solution on $[0, \infty)$. We had introduced this method for integro-differential equations without the coefficient function f in [3].

2. Inversion of (1.2) and assumptions

The equation with which we deal is a bit more general than (1.2) and is

$$x'(t) = g(t, x(t)) + f(t, x(t)) \int_0^t A(t, s) v(t, s, x(s)) ds, \quad x(0) = a \in \mathfrak{R}. \quad (2.1)$$

We will use it to define a fixed point mapping to prove that there is a unique solution on $[0, \infty)$. To invert this into a mapping equation we define a new function ϕ by the relation

$$x(t) = a + \int_0^t \phi(s) ds \quad (2.2)$$

so that $x'(t) = \phi(t)$ and (2.1) becomes

$$\begin{aligned} \phi(t) = & g\left(t, a + \int_0^t \phi(s) ds\right) \\ & + f\left(t, a + \int_0^t \phi(s) ds\right) \int_0^t A(t, s) v\left(t, s, a + \int_0^s \phi(u) du\right) ds. \end{aligned} \quad (2.3)$$

The natural mapping defined by (2.3) is called a *direct fixed point mapping*. Our problem now is to find a unique fixed point of this mapping on $[0, \infty)$.

Notice that if we find ϕ then (2.2) will define a solution of (2.1). Moreover, if there are two solutions of (2.1) with continuous derivatives, then those derivatives satisfy (2.3) contradicting uniqueness.

Since we are looking for solutions of (2.1) which are continuous on intervals $[0, E]$, it is natural to require that $g, f \in C([0, E] \times \mathfrak{R})$ and the integral $\int_0^t A(t, s) v(t, s, x(s)) ds$ be a continuous function on $[0, E]$ when $x \in C([0, E])$. However, the latter requirement allows A and v to have singularities as long as condition (2.7) is satisfied.

We assume that for each fixed $E > 0$ there exist positive constants α_E, K_E, r_E, M_E and a function $L_E : \{0 < s < t \leq E\} \rightarrow \mathfrak{R}$ so that if $|x|, |y| \leq r_E$, then

$$|g(t, x) - g(t, y)| \leq \alpha_E |x - y|, \quad (2.4)$$

$$|f(t, x) - f(t, y)| \leq K_E |x - y|, \quad (2.5)$$

$$|v(t, s, x) - v(t, s, y)| \leq L_E(t, s) |x - y|, \quad (2.6)$$

and

$$\int_0^t |A(t, s)| L_E(t, s) ds \leq M_E, \quad t \in [0, E]. \quad (2.7)$$

We will see that because of (2.2), all of these Lipschitz type conditions (2.4)–(2.6) will become contractions when we divide the interval $[0, E]$ into sufficiently short intervals by

$$0 = T_0 < T_1 < \dots < T_n = E, \quad T_i - T_{i-1} = T, \quad (i = 1, \dots, n)$$

for T small enough and $n \in \mathbb{N}$ with $nT = E$ defined by (4.1), below.

The natural mapping defined by (2.3) will be a contraction P_1 on $[0, T_1]$ and there will be a unique fixed point ξ_1 on that interval.

We will then take the complete metric space of continuous functions ϕ on $[0, T_2]$ satisfying $\phi(t) = \xi_1(t)$ on $[0, T_1]$. The natural mapping P_2 defined by (2.3) on $[0, T_2]$ will again be a contraction with a unique fixed point ξ_2 agreeing with ξ_1 on $[0, T_1]$.

Continuing in this way we obtain a unique continuous solution ξ on $[0, E]$. Now, to obtain a solution on $[0, \infty)$ we let $E = 1, 2, \dots, n$ and obtain solutions ϕ_i on $[0, i]$ which we then extend to $[0, \infty)$ by letting

$\phi_i(t) = \phi(i)$ for $t \geq i$. This sequence of functions converges uniformly on compact sets to a continuous function ϕ on $[0, \infty)$ which solves (2.3) because each ϕ_i solves (2.3) on $[0, i]$.

Remark 2.1. If L_E is continuous on the set $J := \{0 \leq s \leq t \leq E\}$ and $A(t, s) = A(t - s)$ with $|A|$ integrable on $[0, E]$, then condition (2.7) is satisfied, since

$$\left| \int_0^t A(t-s)L_E(t,s) ds \right| \leq \sup_{(t,s) \in J} |L_E(t,s)| \int_0^t |A(s)| ds.$$

Note that integrability of $|A(t,s)L_E(t,s)|$ on $[0, t]$ for $t > 0$ does not imply (2.7), as may easily be verified when $A(t,s) = (t-s)^{q-1}$, $L(t,s) = s^{p-1}$ with $0 < p, q$ and $p+q < 1$.

3. Creating building blocks for contractions

In this section we will see that each of the three terms on the right-hand-side of (2.3) is a contraction if the interval is sufficiently short, that the product is a contraction if the functions f and v are bounded; moreover if the interval is sufficiently short and if the functions f and v are bounded, then (2.3) defines a contraction.

To understand what must be done, we consider the mapping P of the Banach space $(\mathcal{B}, \|\cdot\|_E)$ of continuous functions $\phi : [0, E] \rightarrow \mathfrak{K}$ with the supremum norm as the natural mapping defined by (2.3). Thus, if $\phi, \psi \in \mathcal{B}$, then we have

$$\begin{aligned} |(P\phi)(t) - (P\psi)(t)| &\leq \alpha_E \int_0^t |\phi(s) - \psi(s)| ds \\ &+ \left| f\left(t, a + \int_0^t \phi(s) ds\right) \int_0^t A(t,s)v\left(t, s, a + \int_0^s \phi(u) du\right) ds \right. \\ &\left. - f\left(t, a + \int_0^t \psi(s) ds\right) \int_0^t A(t,s)v\left(t, s, a + \int_0^s \psi(u) du\right) ds \right|. \end{aligned} \quad (3.1)$$

When we examine the last two lines we realize that we have a product of two terms each of which could be a contraction if the interval is sufficiently short. We had introduced this work by dividing $[0, E]$ into n equal segments of length T . The idea is to use the integrals to make the terms contractions by taking T small. We need to prove an unusual type of theorem stating that the product of two contractions is a contraction if the mapping functions are bounded and the interval is sufficiently short. Google reveals that there is some literature on this, but it seems to be quite different than what is needed here.

Now in the lemma and its proof below, $\|\cdot\|$ will always denote the supremum on $[0, T]$ and recall that (2.7) holds.

Lemma 3.1. Let $E > 0$, $G_E \subset (\mathcal{B}, \|\cdot\|_E)$ and $m, h : G_E \rightarrow \mathcal{B}_E$ be such that $\|m(G_E)\|, \|h(G_E)\| \leq Q$, for some $Q > 0$. Assume that for the given $E > 0$ there exist $\alpha > 0, \beta > 0$ such that, if $0 < T \leq E$ and $\phi, \psi \in G_E$, then

$$|h(\phi)(t) - h(\psi)(t)| \leq \alpha T \|\phi - \psi\|, \quad t \in [0, T]$$

and

$$|m(\phi)(t) - m(\psi)(t)| \leq \beta T \|\phi - \psi\|, \quad t \in [0, T].$$

Under these conditions if $H(\phi) = h(\phi)m(\phi)$, then

$$\|H\phi - H\psi\| \leq TQ(\alpha + \beta)\|\phi - \psi\|. \quad (3.2)$$

Proof. We have, for $0 < T \leq E$ and $t \in [0, T]$,

$$\begin{aligned}
 & |H(\phi)(t) - H(\psi)(t)| \\
 &= |h(\phi)(t)m(\phi)(t) - h(\psi)(t)m(\psi)(t)| \\
 &= |h(\phi)(t)m(\phi)(t) - h(\phi)(t)m(\psi)(t) + h(\phi)(t)m(\psi)(t) - h(\psi)(t)m(\psi)(t)| \\
 &\leq |h(\phi)(t)[m(\phi)(t) - m(\psi)(t)]| + |m(\psi)(t)[h(\phi)(t) - h(\psi)(t)]| \\
 &\leq |h(\phi)(t)|\beta T\|\phi - \psi\| + |m(\psi)(t)|\alpha T\|\phi - \psi\| \\
 &= [\beta T|h(\phi)(t)| + \alpha T|m(\psi)(t)|]\|\phi - \psi\| \\
 &\leq (\alpha + \beta)TQ\|\phi - \psi\|
 \end{aligned}$$

from which (3.2) follows. \square

We will now show that every term in (3.1) satisfies a Lipschitz condition with constant containing T as a factor. When we are working on $[0, T]$, we always denote by $\|\cdot\|$ the supremum on $[0, T]$.

First, if $\phi, \psi : [0, T] \rightarrow \mathfrak{X}$, then

$$\begin{aligned}
 & \left| g\left(t, a + \int_0^t \phi(s)ds\right) - g\left(t, a + \int_0^t \psi(s)ds\right) \right| \\
 & \leq \alpha_E \int_0^t |\phi(s) - \psi(s)|ds \\
 & \leq T\alpha_E\|\phi - \psi\|.
 \end{aligned} \tag{3.3}$$

Next,

$$\left| f\left(t, a + \int_0^t \phi(s)ds\right) - f\left(t, a + \int_0^t \psi(s)ds\right) \right| \leq TK_E\|\phi - \psi\|. \tag{3.4}$$

Then,

$$\begin{aligned}
 & \int_0^t |A(t, s)| \left| v\left(t, s, a + \int_0^s \phi(u)du\right) - v\left(t, s, a + \int_0^s \psi(u)du\right) \right| ds \\
 & \leq \int_0^t |A(t, s)|L_E(t, s) \left[\int_0^s |\phi(u) - \psi(u)| du \right] ds \\
 & \leq T\|\phi - \psi\| \int_0^t |A(t, s)|L_E(t, s) ds \leq TM_E\|\phi - \psi\|.
 \end{aligned} \tag{3.5}$$

Now we want to show that the product of the functions generating (3.4) and (3.5) is a contraction, again with the coefficient T as a factor of the contraction constant. Going back to Lemma 3.1, we see that we will need to ask that f and v map G_E into G_E .

The Lipschitz condition controls growth rate so that a solution will not experience finite escape time and this is so important because there are no sign conditions. It also ensures uniqueness of solutions. The single property that seems unfortunate is the existence of the bound Q but it can be removed in the same way that so many fixed point theorems begin by requiring that the natural mapping maps a closed bounded convex set into itself. While that is a property which we cannot avoid, we note again that no sign conditions are demanded. In particular, the functions g , f , and v could all be combinations of $\sin x(t)$ and $\cos x(t)$.

4. A unique solution

Our main result is the following theorem.

Theorem 4.1. Let $E > 0$ be arbitrary and assume that there exist a closed convex and bounded set $G_E \subset (\mathcal{B}_E, \|\cdot\|_E)$, $\|G_E\| \leq r_E$ such that $P(G_E) \subset G_E$, and a $\phi_0 \in G_E$ with $\int_0^t A(t, s)v(t, s, \phi_0(s))ds$ bounded on $[0, E]$. If (2.4)–(2.7) hold, then the direct mapping (2.3) (and, consequently, equation (2.1)) has a unique solution on the interval $[0, E]$. This is then parlayed into a unique solution on $[0, \infty)$.

Proof. Let $E > 0$ be given. For our initial step we consider the space G_1 of the restrictions of continuous functions $\phi \in G_E$ on the set $[0, T]$ (with $T > 0$ defined below) along with the corresponding sup-norm, and note that this is a complete metric space. Furthermore, by the continuity of f , we have for any $\phi \in G_E$ and $t \in [0, E]$,

$$\left| f\left(t, a + \int_0^t \phi(s)ds\right) \right| \leq \sup \{|f(t, u)| : t \in [0, E], |u| \leq |a| + r_E E\},$$

which implies that $f\left(t, a + \int_0^t \phi(s)ds\right)$ is (uniformly) bounded for any $\phi \in G_E$. Also, by (2.6) we have

$$\begin{aligned} & \left| \int_0^t A(t, s)v\left(t, s, a + \int_0^s \phi(u)du\right) ds \right| \\ & \leq \left| \int_0^t A(t, s)v\left(t, s, a + \int_0^s \phi_0(u)du\right) ds \right| \\ & \quad + \int_0^t |A(t, s)| L(t, s) \int_0^s |\phi(u) - \phi_0(u)| duds \\ & \leq \left| \int_0^t A(t, s)v\left(t, s, a + \int_0^s \phi_0(u)du\right) ds \right| + 2Er_E \int_0^t |A(t, s)| L(t, s) ds, \end{aligned}$$

so, in view of (2.7) we see that $\int_0^t A(t, s)v\left(t, s, a + \int_0^s \phi(u)du\right) ds$ is (uniformly) bounded for all $\phi \in G_E$. In terms of Lemma 3.1, if Q is a bound of $\left| f\left(t, a + \int_0^t \phi(s)ds\right) \right|$ and $\int_0^t A(t, s)v\left(t, s, a + \int_0^s \phi(u)du\right) ds$ for all continuous $\phi \in G_E$, then we have

$$h(\phi)(t) = f\left(t, a + \int_0^t \phi(s)ds\right), \quad \alpha = K_E,$$

$$m(\phi)(t) = \int_0^t A(t, s)v\left(t, s, a + \int_0^s \phi(u)du\right) ds, \quad \beta = M_E,$$

and

$$H(\phi) = h(\phi)m(\phi).$$

In view of (2.4), for the mapping P_1 defined by (2.3) on the space G_1 , we take $\phi, \psi \in G_E$ and $t \in [0, T]$ to obtain

$$\begin{aligned} |P_1(\phi)(t) - P_1(\psi)(t)| & \leq \left| g\left(t, a + \int_0^t \phi(s)ds\right) - g\left(t, a + \int_0^t \psi(s)ds\right) \right| \\ & \quad + |H(\phi)(t) - H(\psi)(t)| \end{aligned}$$

$$\leq T\alpha_E \|\phi - \psi\| + \|H\phi - H\psi\|.$$

Thus, by (3.2) we have

$$\|P_1(\phi) - P_1(\psi)\| \leq T\alpha_E \|\phi - \psi\| + TQ(K_E + M_E) \|\phi - \psi\|.$$

The contraction constant for P_1 of (3.1) is

$$T\alpha_E + TQ(K_E + M_E) = T[\alpha_E + Q(K_E + M_E)] < 1, \quad (4.1)$$

for a sufficiently small positive number T . This establishes the magnitude of T and a positive integer n which is chosen so that $nT = E$ for some properly chosen positive integer n . With such a choice, the mapping P_1 is a contraction, so by the contraction principle, we have that P_1 has a unique solution ξ_1 of (2.3), which, in turn, yields a unique solution of equation (2.1) on $[0, T]$.

If $T = E$, then there is nothing else to prove, so we may assume that $T < E$ and let $(G_2, \|\cdot\|)$ be the complete metric space (with the supremum metric) of continuous functions $\phi : [0, 2T] \rightarrow \mathfrak{R}$ which are restrictions on $[0, 2T]$ of the elements of G_E and all satisfy $\phi(t) = \xi_1(t)$ on $[0, T]$. Let P_2 be the mapping defined by (2.3) on the space G_2 .

Next, we want to compute (3.3), (3.4), (3.5) on $[0, 2T]$ assuming that $\phi(t) = \psi(t) = \xi_1(t)$ for $0 \leq t \leq T$. We have for $t \in [T, 2T]$,

$$\begin{aligned} \left| g\left(t, a + \int_0^t \phi(s) ds\right) - g\left(t, a + \int_0^t \psi(s) ds\right) \right| &\leq \alpha_E \int_0^t |\phi(s) - \psi(s)| ds \\ &\leq \alpha_E \int_T^{2T} |\phi(s) - \psi(s)| ds \\ &\leq \alpha_E T \|\phi - \psi\|_{[0, 2T]}, \end{aligned}$$

which is the same constant as in (3.3).

In the same way with (3.4), we have

$$\begin{aligned} \left| f\left(t, a + \int_0^t \phi(s) ds\right) - f\left(t, a + \int_0^t \psi(s) ds\right) \right| &\leq K_E \int_0^t |\phi(s) - \psi(s)| ds \\ &\leq K_E \int_T^{2T} |\phi(s) - \psi(s)| ds \\ &\leq TK_E \|\phi - \psi\|_{[0, 2T]}, \end{aligned}$$

exactly as in (3.4). Finally for (3.5) (and keeping (2.7) in mind) we have

$$\begin{aligned} \int_0^t |A(t, s)| \left| v\left(t, s, a + \int_0^s \phi(u) du\right) - v\left(t, s, a + \int_0^s \psi(u) du\right) \right| &\leq \int_0^t |A(t, s)| L_E(t, s) \int_0^s |\phi(u) - \psi(u)| du \\ &\leq \int_T^t |A(t, s)| L_E(t, s) \left[\int_T^{2T} |\phi(u) - \psi(u)| du \right] ds \end{aligned}$$

$$\begin{aligned} &\leq T\|\phi - \psi\| \int_T^t |A(t, s)|L_E(t, s) |ds \\ &= M_E T\|\phi - \psi\| \end{aligned}$$

which is the same contraction constant as in (3.5).

Because of the way T was chosen, we again have that P_2 is a contraction on G_2 so it has a unique fixed point ξ_2 agreeing with ξ_1 on $[0, T]$, which, in turn, is a unique solution of (2.1) defined on $[0, T_2]$. In $n - 2$ more steps we have a unique solution of equation (2.1) on $[0, E]$.

To obtain a solution on $[0, \infty)$, obtain solutions for $E = 1, 2, \dots$ named ϕ_1, ϕ_2, \dots and continue each with a horizontal line from $\phi_i(i)$ to ∞ . This set of functions converges uniformly on compact sets to a continuous function which is a global solution of (2.1). It should be noted that as E increases then T becomes smaller and Q becomes larger. But Q is always finite and $[0, E]$ can always be divided into pieces of the appropriate length, T . \square

Remark 4.2. The continuity of $A(t, s)v(t, s, x)$ on $\{0 \leq s \leq t \leq E\} \times [-r_E, r_E,]$ yields a uniform bound of $\int_0^t A(t, s)v(t, s, \phi(s))ds$ on $[0, E]$ for all $\phi \in G_E$. Note that if the constant function $0 \in G_E$, then boundedness of $\int_0^t A(t, s)v(t, s, a + \int_0^s \phi(u)du)ds$ is equivalent to boundedness of $\int_0^t A(t, s)v(t, s, a)ds$.

Remark 4.3. In the special case that the kernel A is of fractional type as in (1.1), (1.2), i.e., $A(t, s) = (t - s)^{q-1}$ with $0 < q < 1$, then a sufficient condition yielding (2.7) is that there exist a continuous function $\ell : (0, \infty) \rightarrow \mathfrak{R}^+$ and a $p > 0$ with

$$L(t, s) \leq \ell(t) s^{p-1}, \quad 0 < s < t$$

and such that

$$\lim_{t \rightarrow 0^+} \ell(t) t^{q+p-1} \in \mathfrak{R},$$

since in such a case, we have

$$\begin{aligned} \int_0^t |A(t, s)|L(t, s) ds &\leq \ell(t) \int_0^t (t - s)^{q-1} s^{p-1} ds \\ &= \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)} \ell(t) t^{q+p-1}, \end{aligned}$$

and hence the function $\int_0^t |A(t, s)|L(t, s) ds$ is continuous on $[0, E]$.

Remark 4.4. The set G_E need not be a ball. For example, if $\phi : [0, E] \rightarrow \mathfrak{R}^+$ is continuous, then the set

$$G_\phi := \{x \in C([0, E], \mathbb{R}) : |x(t)| \leq \phi(t), t \in [0, E]\}$$

is a convex, closed and bounded subset of $(\mathcal{B}, \|\cdot\|_E)$. Indeed, if $(x_n) \in G_\phi$ with $x_n \xrightarrow{\|\cdot\|} x \in \mathcal{B}$ and $\varepsilon > 0$ is given, then for sufficiently large n we have for any $t \in [0, E]$,

$$|x(t)| \leq |x(t) - x_n(t)| + |x_n(t)| \leq \|x - x_n\| + \phi(t) \leq \varepsilon + \phi(t),$$

from which we have that

$$|x(t)| \leq \phi(t), \quad t \in [0, E],$$

so $x \in G_\phi$ and G_ϕ is closed. It is bounded by the bound of ϕ and it is convex since for any $x, y \in G_\phi$ and $\lambda \in [0, 1]$,

$$|\lambda x(t) + (1 - \lambda)y(t)| \leq \lambda|x(t)| + (1 - \lambda)|y(t)| \leq \phi(t), \quad t \in [0, E].$$

5. Two classical PDEs, higher order

In this section we leave the product problem to concentrate more directly on higher order problems and examples that will not be over shadowed by the difficulties posed by products. Here we will go through the details of a second order problem in such a way that the reader will readily see how to proceed with third order and higher. Moreover, we will discuss two examples from the classical literature showing how this elementary method of direct fixed point mappings and progressive contractions simplifies the work in establishing unique solutions on $[0, \infty)$.

We will note that the previous work simplifies and enhances some early work on partial integro-differential equations of Hyperbolic and Parabolic type. Two main examples are found in the fundamental treatise of Volterra [4, pp. 154-5]. One of the interesting aspects of this study is that as the order increases, the complexity does not. This is clearly seen in the second example. Volterra had obtained a first order integro-differential equation and, with some difficulty, transformed it into an integral equation in order to generate an existence and uniqueness theorem. In fact, that made things so much more complicated in that the second equation did not define a contraction, but the original one was a very nice progressive contraction yielding a unique global solution in a totally elementary way.

In both this section and the next we are dealing with equations without the product term which required the bound Q in the proof of Theorem 4.1. Thus, Theorem 4.1 will hold here without the requirement in the paragraph following (3.1) in which we ask for G and Q . Neither are needed and Lemma 3.1 is not used.

We begin with the Hyperbolic equation

$$\frac{\partial^2 u(z, t)}{\partial t^2} = \frac{\partial^2 u(z, t)}{\partial z^2} + \int_0^t \frac{\partial^2 u(z, \tau)}{\partial z^2} \psi(t, \tau) d\tau. \quad (5.1)$$

Using Fourier's method of separation of variables, we try for a solution of the form

$$u(z, t) = \sin m(z + \alpha)x(t) \quad (5.2)$$

in which m and α are constants.

The necessary and sufficient condition for this is that the function $x(t)$ should satisfy the ordinary integro-differential equation

$$\frac{d^2 x(t)}{dt^2} + m^2 \left(x(t) + \int_0^t \psi(t, \tau)x(\tau) d\tau \right) = 0. \quad (5.3)$$

In order to maximize our result we refer back to our work with general conditions on A

$$\frac{d^2 x(t)}{dt^2} = -m^2 \left(x(t) + \int_0^t A(t, \tau)x(\tau) d\tau \right). \quad (5.4)$$

This is an interesting problem for several reasons. First, it is not contrived, but is derived from a real problem of interest. Next, it is a simple linear problem so it is easy to follow the steps.

Regardless of the magnitude of the constant M , (5.4) will define a progressive contraction with a unique fixed point on any interval $[0, E]$.

In (5.4) we let

$$x''(t) = \xi(t), \quad 0 \leq t < \infty \quad (5.5)$$

so that

$$x'(t) = x'(0) + \int_0^t \xi(u) du, \quad (5.6)$$

and

$$x(t) = x(0) + \int_0^t \left[x'(0) + \int_0^s \xi(u) du \right] ds \quad (5.7)$$

which we write as

$$x(t) = a + \int_0^t \left[b + \int_0^s \xi(u) du \right] ds \quad (5.8)$$

and

$$x'(t) = b + \int_0^t \xi(u) du. \quad (5.9)$$

Then (5.4) becomes

$$\begin{aligned} \xi(t) = & -m^2 \left(a + \int_0^t \left[b + \int_0^s \xi(u) du \right] ds \right) \\ & - m^2 \int_0^t |A(t, s)| \left\{ a + \int_0^s \left[\int_0^u \xi(v) dv \right] du \right\} ds. \end{aligned} \quad (5.10)$$

We then form (5.10) with ξ and η and let P be the natural mapping defined by (5.10) and compute

$$\begin{aligned} |P\xi - P\eta| & \leq m^2 \int_0^t \left[\int_0^s \|\xi - \eta\| du \right] ds \\ & + m^2 \int_0^t |A(t, s)| \left\{ \int_0^s \left[\int_0^u \|\xi - \eta\| dv \right] du \right\} ds \\ & \leq m^2 t^2 \|\xi - \eta\| + m^2 t^2 \|\xi - \eta\| \int_0^t |A(t, s)| ds \\ & \leq m^2 t^2 \|\xi - \eta\| \left[1 + \int_0^t |A(t, s)| ds \right]. \end{aligned}$$

With a bound on $\int_0^t |A(t, s)| ds$ on an interval $[0, T]$ (which is automatically yielded when A is continuous) this is all we need to complete the proof of Theorem 4.1 for this problem and obtain a unique solution on $[0, \infty)$. Thus we obtain an elementary solution to a difficult classical problem.

The next example of Volterra [4, pp. 154-5] is even more interesting. Volterra begins with the Parabolic partial derivative integro-differential equation

$$\frac{\partial u(x, t)}{\partial x^2} - \frac{\partial^2 u(x, t)}{\partial x^2} - \int_0^t A(t, s) \frac{\partial^2 u(x, s)}{\partial x^2} ds = 0$$

and applies separation of variables

$$u = f(t)(a \sin kx + b \cos kx)$$

and obtains the equation for f as

$$\frac{df}{dt} = -k^2 f(t) - \int_0^t k^2 A(t, s) f(s) ds.$$

This is readily solved as in Theorem 4.1 for a global solution because all of the terms become contractions. However Volterra transforms it into the integral equation

$$f(t) = h - \int_0^t k^2 \left(1 + \int_s^t A(u, s) du \right) f(s) ds$$

which is very difficult to solve since no term is a contraction. Thus, the method of Theorem 4.1 reduces a classical problem to an elementary one.

Though both of the above results concern linear equations, it is not difficult to see that progressive contractions may well be applied to nonlinear second (and higher) order integro-differential equations. For example, we may consider the equation

$$x''(t) = g(t, x(t)) + f(t, x(t)) \int_0^t A(t, s) v(t, s, x(s)) ds, t \geq 0. \quad (5.11)$$

Using the notation in (5.5)–(5.9) equation (5.11) becomes

$$\begin{aligned} \xi(t) &= g\left(t, a + bt + \int_0^t (t-s)\xi(s)ds\right) \\ &+ f\left(t, a + bt + \int_0^t (t-s)\xi(s)ds\right) \\ &\times \int_0^t A(t, s) v\left(t, s, a + bs + \int_0^s (s-u)\xi(u)du\right) ds, \end{aligned}$$

and one can easily see that the result in Theorem 4.1 is also true for equation (5.11). Note that possible presence of derivatives (up to second order) of x in f or v can be treated in the same way as above as long as Lipschitz type conditions hold. Similar type results may also be obtained for higher order integro-differential equations.

6. Strategy

In this section again we are not dealing with a product involving the integral and so the bound on the functions in G mentioned in Theorem 4.1 is not required.

In a more modern note, Miller [5, pp. 10–11, 83–147] shows that integro-differential equations can always be transformed to integral equations, while integral equations can be transformed into integro-differential equations if the functions are smooth. Thus, Miller takes the course of dealing only with integral equations and gives the long treatment pp. 83–147 with those results.

In view of our results in Section 4, one can prove global existence and uniqueness of integro-differential equations under much weaker conditions than those required for integral equations.

On p. 10 Miller shows that the scalar equation

$$x'(t) = G(t, x(t)) + \int_0^t k(t, s)H(s, x(s))ds, x(0) = \xi \quad (6.1)$$

can be written as the integral equation

$$x(t) = \xi + \int_0^t G(s, x(s))ds + \int_0^t \int_s^t k(u, s)duH(s, x(s))ds. \quad (6.2)$$

Lipschitz conditions with respect to x on G and H translate into progressive contraction conditions for global existence and uniqueness for (6.1), so this is not productive strategy for existence and uniqueness.

Moreover, on p. 11 Miller shows that if f and g in

$$x(t) = f(t) + \int_0^t g(t, s, x(s))ds \quad (6.3)$$

are smooth then (6.3) can be written as (6.1). This is productive strategy if $g(t, s, x)$ and $(\partial g(t, s, x)/\partial t)$ are Lipschitz in x because Theorem 4.1 may yield a unique global solution in an elementary way.

7. An example

Consider the equation (2.1) with $x(0) = a > 0$,

$$g(t, x) := c_1 \frac{t\sqrt{|x|+1}}{t^3+10}, \quad 0 \leq t, \quad x \in \mathfrak{R},$$

$$f(t, x) := c_2 \frac{\sqrt[3]{x^2+1}}{\sqrt{t^8+10}}, \quad 0 \leq t, \quad x \in \mathfrak{R},$$

$$v(t, s, x) := \frac{x^2+s}{\sqrt[6]{ts}} + |x|, \quad 0 < s < t, \quad x \in \mathfrak{R},$$

and

$$A(t, s) := \frac{(t-s)^{\frac{8}{15}-1}}{\sqrt[5]{t+s}}, \quad 0 < s < t,$$

i.e., the equation

$$\begin{aligned} x'(t) = & c_1 \frac{t\sqrt{|x(t)|+1}}{t^3+10} \\ & + c_2 \frac{\sqrt[3]{x^2(t)+1}}{\sqrt{t^8+10}} \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{t+s}} \left[\frac{x^2(s)+s}{\sqrt[6]{ts}} + |x(s)| \right] ds, \end{aligned} \quad (7.1)$$

where c_1, c_2 are sufficiently small nonnegative constants.

Note that the (non-convolution) kernel A is singular at $t = s$ and at $t = 0$, also the function $v(t, s, x)$ is singular at $t = 0$.

Setting

$$x'(t) = \phi(t)$$

equation (7.1) is transformed into the equation

$$\phi(t) = c_1 \frac{t \sqrt{|a + \int_0^t \phi(s) ds| + 1}}{t^3 + 10} + c_2 \frac{\sqrt[3]{\{a + \int_0^t \phi(s) ds\}^2 + 1}}{\sqrt{t^8 + 10}} \times \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{t+s}} \left[\frac{\{a + \int_0^t \phi(s) ds\}^2 + s}{\sqrt[6]{ts}} + |x(s)| \right] ds.$$

Let $E > 0$ be arbitrary. Firstly we'll show that (2.4)–(2.6) are fulfilled for $t \in [0, E]$ and $|x|, |y| \leq r_E = 1$. We have

$$\begin{aligned} |g(t, x) - g(t, y)| &= c_1 \frac{t |\sqrt{|x| + 1} - \sqrt{|y| + 1}|}{t^3 + 10} \\ &= \frac{c_1 t ||x| - |y||}{(t^3 + 10) |\sqrt{|x| + 1} + \sqrt{|y| + 1}|} \leq \frac{c_1 E}{20} |x - y|, \end{aligned}$$

so (2.4) is satisfied with $\alpha_E = c_1 \frac{E}{20}$.

Also,

$$|f(t, x) - f(t, y)| = c_2 \left| \frac{\sqrt[3]{x^2 + 1} - \sqrt[3]{y^2 + 1}}{\sqrt{t^8 + 10}} \right| \leq \frac{2c_2}{3\sqrt{10}} |x - y|,$$

thus (2.5) is satisfied with $K_E = \frac{2c_2}{3\sqrt{10}}$.

Yet, for $0 < s < t \leq E$, $|x|, |y| \leq r_E = 1$, we have

$$|v(t, s, x) - v(t, s, y)| = \left| \frac{x^2 - y^2}{\sqrt[6]{ts}} + |x| - |y| \right| \leq \left(\frac{2}{\sqrt[6]{ts}} + 1 \right) |x - y|$$

and (2.6) is satisfied with $L(t, s) := \frac{2}{\sqrt[6]{ts}} + 1$, $0 < s \leq t$.

Finally, for $0 < s < t \leq E$ we find

$$\begin{aligned} \int_0^t |A(t, s)| L(t, s) ds &= \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{t+s}} \left(\frac{2}{\sqrt[6]{ts}} + 1 \right) ds \\ &\leq \frac{1}{\sqrt[5]{t}} \int_0^t (t-s)^{-\frac{7}{15}} \left(\frac{2}{\sqrt[6]{s^2}} + 1 \right) ds \\ &= \frac{1}{\sqrt[5]{t}} \left[2 \int_0^t (t-s)^{\frac{8}{15}-1} s^{\frac{10}{15}-1} ds + \int_0^t (t-s)^{-\frac{7}{15}} ds \right] \\ &= t^{-1/5} \left[2 \frac{\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{10}{15}\right)}{\Gamma\left(\frac{18}{15}\right)} t^{1/5} + \frac{15}{8} t^{8/15} \right] \\ &\leq 2 \frac{\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{18}{15}\right)} + \frac{15}{8} E^{8/15} \end{aligned}$$

and condition (2.7) is satisfied for any $E > 0$.

It remains to show that for the closed, convex, and bounded set $G_E := \{x \in BC[0, E] : \|x\| \leq 1\}$ we have $P(G_E) \subset G_E$. Indeed, in view of $a > 0$, for $\|\phi\| \leq 1$, we obtain

$$\left| g\left(t, a + \int_0^t \phi(s) ds\right) \right| = c_1 \frac{t \sqrt{|a + \int_0^t \phi(s) ds| + 1}}{t^3 + 10} \leq c_1 \frac{t \sqrt{a + t + 1}}{t^3 + 10},$$

while

$$\begin{aligned}
& \left| f\left(t, a + \int_0^t \phi(s) ds\right) \int_0^t A(t, s) v\left(t, s, a + \int_0^s \phi(u) du\right) ds \right| \\
& \leq c_2 \frac{\sqrt[3]{\left[a + \int_0^t \phi(s) ds\right]^2 + 1}}{\sqrt{t^8 + 10}} \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{t+s}} \times \\
& \quad \times \left[\frac{\left(a + \int_0^s \phi(u) du\right)^2 + s}{\sqrt[6]{ts}} + \left| a + \int_0^s \phi(u) du \right| \right] ds \\
& \leq c_2 \frac{\sqrt[3]{(a+t)^2 + 1}}{\sqrt{t^8 + 10}} \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{t+s}} \left[\frac{[a+s]^2 + s}{\sqrt[6]{ts}} + |a+s| \right] ds \\
& \leq c_2 \frac{\sqrt[3]{(a+t)^2 + 1} [(a+t)^2 + t]}{\sqrt{t^8 + 10}} \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{2s} \sqrt[6]{s^2}} ds \\
& \quad + c_2 \frac{\sqrt[3]{(a+t)^2 + 1} |a+t|}{\sqrt{t^8 + 10}} \int_0^t \frac{(t-s)^{-\frac{7}{15}}}{\sqrt[5]{s}} ds \\
& = c_2 \frac{\sqrt[3]{(a+t)^2 + 1} [t^2 + (2a+1)t + a^2]}{\sqrt[5]{2} \sqrt{t^8 + 10}} \int_0^t (t-s)^{\frac{8}{15}-1} s^{\frac{7}{15}-1} ds + \\
& \quad + c_2 \frac{\sqrt[3]{(a+t)^2 + 1} |a+t|}{\sqrt[5]{2} \sqrt{t^8 + 10}} \int_0^t (t-s)^{\frac{8}{15}-1} s^{\frac{4}{5}-1} ds \\
& \leq \frac{c_2 \sqrt[3]{(a+t)^2 + 1} [t^2 + (2a+1)t + a^2]}{\sqrt[5]{2} \sqrt{t^8 + 10}} \frac{\Gamma\left(\frac{8}{15}\right) \Gamma\left(\frac{7}{15}\right)}{\Gamma(1)} \\
& \quad + c_2 \frac{\sqrt[3]{(a+t)^2 + 1} |a+t| \Gamma\left(\frac{8}{15}\right) \Gamma\left(\frac{4}{5}\right)}{\sqrt[5]{2} \sqrt{t^8 + 10} \Gamma\left(\frac{20}{15}\right)} t^{\frac{1}{3}} \\
& = c_2 \frac{\Gamma\left(\frac{8}{15}\right) \Gamma\left(\frac{7}{15}\right)}{\sqrt[5]{2} \Gamma(1)} \frac{\sqrt[3]{(a+t)^2 + 1} [t^2 + (2a+1)t + a^2]}{\sqrt{t^8 + 10}} \\
& \quad + c_2 \frac{\Gamma\left(\frac{8}{15}\right) \Gamma\left(\frac{4}{5}\right)}{\sqrt[5]{2} \Gamma\left(\frac{20}{15}\right)} \frac{\sqrt[3]{(a+t)^2 + 1} |a+t|}{\sqrt{t^8 + 10}} t^{\frac{1}{3}}
\end{aligned}$$

i.e., for $|x|, |y| \leq 1$ we have for $\phi \in G_E, t \geq 0$

$$\begin{aligned}
& |P(\phi)(t)| \\
& \leq \left| g\left(t, a + \int_0^t \phi(s) ds\right) \right| \\
& \quad + \left| f\left(t, a + \int_0^t \phi(s) ds\right) \int_0^t A(t, s) v\left(t, s, a + \int_0^s \phi(u) du\right) ds \right| \\
& \leq c_1 \frac{t \sqrt{a+t+1}}{t^3 + 10} + c_2 \frac{\Gamma\left(\frac{8}{15}\right) \Gamma\left(\frac{7}{15}\right)}{\sqrt[5]{2} \Gamma(1)} \frac{\sqrt[3]{(a+t)^2 + 1} [t^2 + (2a+1)t + a^2]}{\sqrt{t^8 + 10}}
\end{aligned}$$

$$+c_2 \frac{\Gamma\left(\frac{8}{15}\right)\Gamma\left(\frac{4}{5}\right)}{\sqrt[5]{2}\Gamma\left(\frac{4}{3}\right)} \frac{\sqrt[3]{(a+t)^2+1}|a+t|}{\sqrt{t^8+10}} t^{1/3}.$$

Since the functions

$$\frac{t\sqrt{a+t+1}}{t^3+10}, \quad \frac{\sqrt[3]{(a+t)^2+1}[t^2+(2a+1)t+\alpha^2]}{\sqrt{t^8+10}}, \quad \frac{\sqrt[3]{(a+t)^2+1}|a+t|}{\sqrt{t^8+10}} t^{1/3}$$

are bounded on $[0, \infty)$, we may conclude that for sufficiently small values of the constants c_1, c_2 we have

$$|P(\phi)(t)| \leq 1, \quad t \geq 0,$$

thus $P(G_E) \subset G_E$ for any $E > 0$.

As E is arbitrary, by Theorem 4.1 we have that the integro-differential equation (7.1) has a unique solution x defined on the whole $[0, \infty)$. Note that for this solution x

$$|x'(t)| \leq 1, \quad \text{and,} \quad a-t \leq x(t) \leq a+t, \quad t \geq 0.$$

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