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Research article

On the denseness of certain reciprocal power sums

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Abstract: By $(\mathbb{Z}^+)^\infty$ we denote the set of all the infinite sequences $\mathcal{S}=\{s_i\}_{i=1}^\infty$ of positive integers (note that all the s_i are not necessarily distinct and not necessarily monotonic). Let $f(x)$ be a polynomial of nonnegative integer coefficients. For any integer $n\ge 1$, one lets $\mathcal{S}_n:=\{s_1,...,s_n\}$ and $H_f(\mathcal{S}_n):=\sum_{k=1}^n\frac{1}{f(k)^{s_k}}$. In this paper, we use a result of Kakeya to show that if $\frac{1}{f(k)}\le\sum_{i=1}^\infty\frac{1}{f(k+i)}$ holds for all positive integers k, then the union set $\bigcup\limits_{\mathcal{S}\in(\mathbb{Z}^+)^\infty}\{H_f(\mathcal{S}_n)|n\in\mathbb{Z}^+\}$ is dense in the interval $(0,\alpha_f)$ with $\alpha_f:=\sum_{k=1}^\infty\frac{1}{f(k)}$. It is well known that $\alpha_{x^2+1}=\frac{1}{2}(\pi\frac{e^{2\pi}+1}{e^{2\pi}-1}-1)\approx 1.076674$. Our dense result infers that for any sufficiently small $\varepsilon>0$, there are positive integers n_1 and n_2 and infinite sequences $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ of positive integers such that $1-\varepsilon<H_{x^2+1}(\mathcal{S}^{(1)}_{n_1})<1$ and $1<H_{x^2+1}(\mathcal{S}^{(2)}_{n_2})<1+\varepsilon$. Finally, we conjecture that for any polynomial $f(x)$ of integer coefficients satisfying that $f(m)\ne 0$ for any positive integer m and for any infinite sequence $\mathcal{S}=\{s_i\}_{i=1}^\infty$ of positive integers (not necessarily increasing and not necessarily distinct), there is a positive integer N such that for any integer n with $n\ge N$, $H_f(\mathcal{S}_n)$ is not an integer. Particularly, we guess that for any positive integer n, $H_{x^2+1}(\mathcal{S}_n)$ is never equal to 1.

Keywords: denseness; infinite series; reciprocal power sum; convergence
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1. Introduction

Let $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Q}$ be the set of integers, the set of positive integers and the set of rational numbers, respectively. Let $n\in\mathbb{Z}^+$. In 1915, Theisinger [9] showed that the n-th harmonic sum $1+\frac{1}{2}+...+\frac{1}{n}$ is never an integer if $n>1$. In 1923, Nagell [8] extended Theisinger's result by showing that if a and b are positive integers and $n\ge 2$, then the reciprocal sum $\sum_{i=0}^{n-1}\frac{1}{a+bi}$ is never an integer. Erdős and Niven [2] generalized Nagell's result by considering the integrality of the elementary symmetric functions of $\frac{1}{a},\frac{1}{a+b},...,\frac{1}{a+(n-1)b}$. In the recent years, Erdős and Niven's result [2] was extended to the general polynomial sequence, see [1], [4], [7], [10] and [11]. Another interesting and related topic is presented in [12].

By $(\mathbb{Z}^+)^\infty$ we denote the set of all the infinite sequence $\{s_i\}_{i=1}^\infty$ of positive integers (note that all the s_i are not necessarily distinct and not necessarily monotonic). For any given $\mathcal{S} = \{s_i\}_{i=1}^\infty \in (\mathbb{Z}^+)^\infty$, we let $\mathcal{S}_n := \{s_1, ..., s_n\}$. Associated to the infinite sequence $\mathcal{S}$ of positive integers and a polynomial $f(x)$ of nonnegative integer coefficients, one can form an infinite sequence $\{H_f(\mathcal{S}_n)\}_{n=1}^\infty$ of positive rational fractions with $H_f(\mathcal{S}_n)$ being defined as follows:

$$H_f(\mathcal{S}_n) := \sum_{k=1}^{n} \frac{1}{f(k)^{s_k}}.$$

Feng, Hong, Jiang and Yin [3] showed that when $f(x)$ is linear, the reciprocal power sum $H_f(\mathcal{S}_n)$ is never an integer if $n \geq 2$. Associated to any given infinite sequence $\mathcal{S}$ of positive integers, we let

$$H_f(\mathcal{S}) := \{H_f(\mathcal{S}_n) | n \in \mathbb{Z}^+\}$$

and

$$\alpha_f(\mathcal{S}) := \sum_{k=1}^{\infty} \frac{1}{f(k)^{s_k}}.$$

Put

$$\alpha_f := \sum_{k=1}^{\infty} \frac{1}{f(k)}. \tag{1}$$

Note that α_f may be $+\infty$. Then $\alpha_f(\mathcal{S}) \leq \alpha_f$ and $H_f(\mathcal{S}) \subseteq (\inf H_f(\mathcal{S}), \alpha_f(\mathcal{S}))$. It is clear that $H_f(\mathcal{S})$ is not dense (nowhere dense) in the interval $(\inf H_f(\mathcal{S}), \alpha_f(\mathcal{S}))$. However, if we put all the sets $H_f(\mathcal{S})$ together, then one arrives at the following interesting dense result that is the main result of this paper.

Theorem 1.1. *Let $f(x)$ be a polynomial of nonnegative integer coefficients and let U_f be the union set defined by*

$$U_f := \bigcup_{\mathcal{S} \in (\mathbb{Z}^+)^\infty} H_f(\mathcal{S}).$$

(i). *If $\deg f(x) = 1$, then U_f is dense in the interval $(\delta, +\infty)$ with $\delta := 1$ if $f(x) = x$, and $\delta := 0$ otherwise.*

(ii). *If $\deg f(x) \geq 2$ and*

$$\frac{1}{f(k)} \leq \sum_{i=1}^{\infty} \frac{1}{f(k+i)} \tag{2}$$

holds for all positive integers k, then U_f is dense in the interval $(0, \alpha_f)$ with α_f being given in (1).

It is well known that (see, for instance, [6])

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1} = \frac{1}{2}\Big(\pi \frac{e^{2\pi}+1}{e^{2\pi}-1} - 1\Big) := \alpha. \tag{3}$$

Furthermore, $\alpha \approx 1.076674$. Evidently, for any positive integer n, we have

$$0 < H_{x^2+1}(\mathcal{S}_n) \leq \sum_{k=1}^{n} \frac{1}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^2+1} < 2.$$

One can easily check that (2) is true when $f(x) = x^2 + 1$. So Theorem 1.1 infers that for any sufficiently small $\varepsilon > 0$, there are positive integers n_1 and n_2 and infinite sequences $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ of positive integers such that $1 - \varepsilon < H_{x^2+1}(\mathcal{S}_{n_1}^{(1)}) < 1$ and $1 < H_{x^2+1}(\mathcal{S}_{n_2}^{(2)}) < 1 + \varepsilon$. But it is unclear whether $H_{x^2+1}(\mathcal{S}_n)$ can take 1 as its value. We guess that the answer to this question is negative.

This paper is organized as follows. First, in Section 2, we recall the results due to Kakeya [5], and then show some preliminary lemmas which are needed in the proof of Theorem 1.1. Then in Section 3, we supply the proof of Theorem 1.1. The final section is devoted to some remarks. Actually, two conjectures are proposed there.

2. Auxiliary lemmas

In this section, we present several auxiliary lemmas that are needed in the proof of Theorem 1.1. Now let us state a result obtained by Kakeya in 1914.

Lemma 2.1. [5] *Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent infinite series of real numbers and let the set, denoted by SPS, of all the partial sums of the series $\sum_{k=1}^{\infty} a_k$ be defined by*

$$SPS := \Big\{ \sum_{i=1}^{m} a_{k_i} \Big| m \in \mathbb{Z}^+ \cup \{\infty\}, 1 \le k_1 < ... < k_m \Big\}.$$

Let $u := \inf SPS$ and $v := \sup SPS$ (note that u may be $-\infty$ and v may be $+\infty$). Then the set U consists of all the values in the interval (u, v) if and only if

$$|a_k| \le \sum_{i=1}^{\infty} |a_{k+i}|$$

holds for all $k \in \mathbb{Z}^+$.

Using Lemma 2.1, we can prove the following two useful results that play key roles in the proof of Theorem 1.1.

Lemma 2.2. *Let $\sum_{k=1}^{\infty} a_k$ be a convergent infinite series of positive real numbers and*

$$V := \Big\{ \sum_{i=1}^{m} a_{k_i} \Big| m \in \mathbb{Z}^+, 1 \le k_1 < ... < k_m \Big\}.$$

If

$$a_k \le \sum_{i=1}^{\infty} a_{k+i} \tag{4}$$

holds for all $k \in \mathbb{Z}^+$, then the set V is dense in the interval $(0, v)$ with $v := \sum_{k=1}^{\infty} a_k$.

Proof. From the condition (4) and Lemma 2.1, we know that the set

$$SPS = \Big\{ \sum_{i=1}^{m} a_{k_i} \Big| m \in \mathbb{Z}^+ \cup \{\infty\}, 1 \le k_1 < ... < k_m \Big\}$$

consists of all the values in the interval $(0, v)$ since here $\inf SPS = 0$. Let r be any given real number in $(0, v)$ and ε be any sufficiently small positive number (one may let $\varepsilon < \min(r, v - r)$). Then $r \in SPS$

which implies that there is an integer $m \in \mathbb{Z}^+ \cup \{\infty\}$ and there are m integers $k_1, ..., k_m$ with $1 \le k_1 < ... < k_m$ such that $r = \sum_{i=1}^{m} a_{k_i}$.

If $m \in \mathbb{Z}^+$, then $r \in V$. So Lemma 2.2 is true in this case.

If $m = \infty$, then $r = \sum_{i=1}^{\infty} a_{k_i}$. That is, $\text{limit}_{n\to\infty} \sum_{i=1}^{n} a_{k_i} = r$. Thus there is a positive integer m' such that $|r - \sum_{i=1}^{m'} a_{k_i}| < \varepsilon$. Noticing that all a_{k_i} are positive, we deduce that $r - \varepsilon < \sum_{i=1}^{m'} a_{k_i} < r$ as desired.

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $\sum_{k=1}^{\infty} a_k$ be a divergent infinite series of positive real numbers with a_k decreasing as k increasing and $a_k \to 0$ as $k \to \infty$. Define*

$$V := \Big\{ \sum_{i=1}^{m} a_{k_i} \Big| m \in \mathbb{Z}^+, 1 \le k_1 < ... < k_m \Big\}.$$

Then the set V is dense in the interval $(0, +\infty)$.

Proof. Let r be any given real number in $(0, +\infty)$ and ε be any sufficiently small positive number (one may let $\varepsilon < r$). Let $a_0 := 0$ and $m_0 = 0$. Since the series $\sum_{k=0}^{\infty} a_k$ is divergent, there exists a unique integer $m_1 \ge 0$ such that

$$\sum_{k=m_0}^{m_1} a_k < r$$

and

$$\sum_{k=m_0}^{m_1} a_k + a_{m_1+1} \ge r.$$

On the one hand, since a_k decreases as k increases and $a_k \to 0$ as $k \to \infty$, there is an integer m_2 with $m_2 > m_1 + 1$ and

$$a_{m_2} < r - \sum_{k=m_0}^{m_1} a_k \le a_{m_1+1}.$$

Moreover, there exists an integer m_3 satisfying that $m_3 \ge m_2$ and

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k < r$$

and

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + a_{m_3+1} \ge r$$

since $\sum_{k=m_2}^{\infty} a_k$ also diverges.

Continuing in this way, we can form an increasing sequence $\{m_k\}_{k=0}^{\infty}$ such that

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k < r$$

but

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k + a_{m_{2t+1}+1} \ge r$$

for any nonnegative integer t. Obviously, one has

$$\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k \in V.$$

On the other hand, since $\lim_{k\to+\infty} a_k = 0$, it follows that there exists a nonnegative integer t_0 such that $a_{m_{2t_0+1}+1} < \varepsilon$. That is, we have

$$r - \varepsilon < r - a_{m_{2t_0+1}+1} \leq \sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t_0}}^{m_{2t_0+1}} a_k < r.$$

Hence V is dense in the interval $(0, +\infty)$.

This concludes the proof of Lemma 2.3. □

3. Proof of Theorem 1.1

In the section, we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$V_f := \Big\{ \sum_{i=1}^{m} \frac{1}{f(k_i)} \Big| m \in \mathbb{Z}^+, 1 \leq k_1 < ... < k_m \Big\}$$

and

$$\bar{V}_f := \Big\{ \sum_{i=1}^{m} \frac{1}{f(k_i)} \Big| m \in \mathbb{Z}^+, 2 \leq k_1 < ... < k_m \Big\}.$$

Pick any given real number r in $(\inf U_f, \sup U_f)$ and let ε be any sufficiently small positive number (one may let $\varepsilon < \min(r - \inf U_f, \sup U_f - r)$).

(i). Since $f(x)$ is a polynomial of nonnegative integer coefficients and degree one, it follows that $\sum_{k=1}^{\infty} \frac{1}{f(k)}$ (resp. $\sum_{k=2}^{\infty} \frac{1}{f(k)}$) is a divergent infinite series of positive real numbers with $\{\frac{1}{f(k)}\}_{k=1}^{\infty}$ (resp. $\{\frac{1}{f(k)}\}_{k=2}^{\infty}$) directly decreasing to 0 as k increases. By Lemma 2.3, we know that V_f (resp. $\bar{V}_f$) is dense in the interval $(0, +\infty)$. Clearly, we have $\sup U_f = \sup V_f = +\infty$.

If $f(1) = 1$, then $f(x) = x$ which implies that $f(2) > 1$, $\inf U_f = 1$ and $r \in (\inf U_f, \sup U_f) = (1, +\infty)$. Since $\bar{V}_f$ is dense in the interval $(0, +\infty)$, there is an element

$$\sum_{i=1}^{m} \frac{1}{f(k_i)} \in \Big(r - 1 - \varepsilon, r - 1 - \frac{\varepsilon}{2}\Big) \tag{5}$$

with $2 \leq k_1 < \cdots < k_m$. Now let $s_k = 1$ for $k \in \{k_1, \ldots, k_m\}$ and $s_k > \frac{\log \frac{2k_m}{\varepsilon}}{\log f(2)}$ for $k \in \{2, 3, \ldots, k_m\} \setminus \{k_1, \ldots, k_m\}$. Then

$$0 \leq \sum_{\substack{k=2 \\ k\notin\{k_1,\ldots,k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{f(2)^{\frac{\log \frac{2k_m}{\varepsilon}}{\log f(2)}}} = \frac{\varepsilon}{2}. \tag{6}$$

It follows from (5) and (6) that

$$\sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} = 1 + \sum_{\substack{k=2 \\ k\notin\{k_1,\dots,k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^{m} \frac{1}{f(k_i)^{s_{k_i}}} \in (r-\varepsilon, r).$$

That is, U_f is dense in the interval $(\inf U_f, \sup U_f) = (1, +\infty)$ in this case.

If $f(1) > 1$, then $\inf U_f = 0$ and $r \in (\inf U_f, \sup U_f) = (0, +\infty)$. Since V_f is dense in the interval $(0, +\infty)$, there is an element

$$\sum_{i=1}^{m} \frac{1}{f(k_i)} \in \left(r-\varepsilon, r-\frac{\varepsilon}{2}\right) \tag{7}$$

with $1 \le k_1 < \cdots < k_m$. Now, let $s_k = 1$ for $k \in \{k_1, \dots, k_m\}$ and $s_k > \frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}$ for $k \in \{1, 2, \dots, k_m\} \setminus \{k_1, \dots, k_m\}$. One has

$$0 \le \sum_{\substack{k=1 \\ k\notin\{k_1,\dots,k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{f(1)^{\frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}}} = \frac{\varepsilon}{2}, \tag{8}$$

and so by (7) and (8),

$$\sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} = \sum_{\substack{k=1 \\ k\notin\{k_1,\dots,k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^{m} \frac{1}{f(k_i)^{s_{k_i}}} \in (r-\varepsilon, r).$$

Namely, U_f is dense in the interval $(\inf U_f, \sup U_f) = (0, +\infty)$ in this case.

(ii). First of all, since $f(x)$ is a polynomial of nonnegative integer coefficients and $\deg f(x) \ge 2$, we know that $\sum_{k=1}^{\infty} \frac{1}{f(k)}$ is a convergent infinite series of positive real numbers. With the hypothesis $\frac{1}{f(k)} \le \sum_{i=1}^{\infty} \frac{1}{f(k+i)}$ for any positive integer k, Lemma 2.2 yields that V_f is dense in the interval $(0, \sup V_f)$.

We claim that $f(1) > 1$. Otherwise, $f(1) = 1$. Then $f(x) = x^m$ with $m \ge 2$. However,

$$\frac{1}{f(1)} = 1 > \frac{\pi^2}{6} - 1 = \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \ge \sum_{i=1}^{\infty} \frac{1}{f(1+i)},$$

which contradicts with our hypothesis. So we must have $f(1) > 1$. The claim is proved.

In the following, we let $f(1) > 1$. Then $\inf U_f = 0$, $\sup U_f = \sup V_f = \alpha_f$ and $r \in (\inf U_f, \sup U_f) = (0, \alpha_f)$. Since V_f is dense in the interval $(0, \sup V_f) = (0, \alpha_f)$, there is an element

$$\sum_{i=1}^{m} \frac{1}{f(k_i)} \in \left(r-\varepsilon, r-\frac{\varepsilon}{2}\right)$$

with $1 \le k_1 < \cdots < k_m$. Then letting $s_k = 1$ for $k \in \{k_1, \dots, k_m\}$ and $s_k > \frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}$ for $k \in \{1, 2, \dots, k_m\} \setminus \{k_1, \dots, k_m\}$ gives us that

$$0 \le \sum_{\substack{k=1 \\ k\notin\{k_1,\dots,k_m\}}}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{f(1)^{\frac{\log \frac{2k_m}{\varepsilon}}{\log f(1)}}} = \frac{\varepsilon}{2}.$$

It infers that

$$\sum_{k=1}^{k_m}\frac{1}{f(k)^{s_k}}=\sum_{\substack{k=1\\ k\notin\{k_1,...,k_m\}}}^{k_m}\frac{1}{f(k)^{s_k}}+\sum_{i=1}^{m}\frac{1}{f(k_i)^{s_{k_i}}}\in(r-\varepsilon,r).$$

In other words, U_f is dense in the interval $(0,\alpha_f)$. So part (ii) is proved.

The proof of Theorem 1.1 is complete. □

4. Final remarks

We let $f(x)$ be a polynomial of nonnegative integer coefficients and of degree at least two, and let U_f be the union set given in Theorem 1.1. Then part (ii) of Theorem 1.1 says that the condition (2) is a sufficient condition such that the union set U_f is dense in the interval $(0,\alpha_f)$. One may ask the following interesting question: What is the sufficient and necessary condition on $f(x)$ for the union set U_f to be dense in the interval $(0,\alpha_f)$? We propose the following conjecture to answer this problem.

Conjecture 4.1. *Let $f(x)$ be not a monomial and be a polynomial of nonnegative integer coefficients and of degree at least two. Then the set U_f is dense in the interval $(0,\alpha_f)$ if and only if the following inequality holds:*

$$\frac{1}{f(1)}-\frac{1}{f(1)^2}\leq\sum_{k=2}^{\infty}\frac{1}{f(k)}.$$

By Theorem 1.1, one knows that for any sufficiently small $\varepsilon>0$, there are positive integers n_1 and n_2 and infinite sequences $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ of positive integers such that $1-\varepsilon<H_{x^2+1}(\mathcal{S}_{n_1}^{(1)})<1$ and $1<H_{x^2+1}(\mathcal{S}_{n_2}^{(2)})<1+\varepsilon$. But it is not clear whether $H_{x^2+1}(\mathcal{S}_n)$ can take 1 as its value. We believe that the answer to this question is negative. As the conclusion of this paper, we suggest the following conjecture.

Conjecture 4.2. *Let $f(x)$ be a polynomial of integer coefficients satisfying that $f(m)\neq 0$ for any positive integer m and $\mathcal{S}=\{s_i\}_{i=1}^{\infty}$ be an infinite sequence of positive integers (not necessarily increasing and not necessarily distinct). Then there is a positive integer N such that for any integer n with $n\geq N$, $H_f(\mathcal{S}_n)$ is not an integer. In particular, for any positive integer n, $H_{x^2+1}(\mathcal{S}_n)$ is never equal to 1.*

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Conflict of interest

We declare that we have no conflict of interest.

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