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Research article

Multiple closed form solutions to some fractional order nonlinear evolution equations in physics and plasma physics

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Abstract: Nonlinear evolution equations (NLEEs) of fractional order play important role to explain the inner mechanisms of complex phenomena in various fields of the real world. In this article, nonlinear evolution equations of fractional order; namely, the (3+1)-dimensional space-time fractional modified KdV-Zakharov-Kuznetsov equation, the time fractional biological population model and the space-time fractional modified regularized long-wave equation are revealed for seeking closed form analytic solutions. The offered equations are first transformed into ordinary differential equations of integer order with the help of a suitable composite transformation and the conformable fractional derivative. Then the rational (G'/G)-expansion method, which is reliable, efficient and computationally attractive, is employed to construct the traveling wave solutions successfully. The obtained solutions are appeared to be exact, much more new and general than the existing results in the literature.

Keywords: rational (G'/G)-expansion method; conformable fractional derivative; composite transformation; fractional order nonlinear evolution equation; exact solution
Mathematics Subject Classification: 34A08, 35R11

1. Introduction

Noticeable natural complex phenomena of the real world are described and formulated in the course of the differential equations. Fractional calculus has drawn the great interest of many researchers for their importance to depict the inner mechanisms of various complex physical phenomena of real world in broad sense. The fractional order nonlinear partial differential equations are more effective to explain widely the mechanisms of the nature of world than the classical differential equations of integer order. That is why; many researchers have recently paid deep attention to seek for the exact solutions to the nonlinear evolution equations (NLEEs) of fractional order. The study for revealing fractional order NLEEs is mainly due to their important appearance in different fields such as biology, physics, engineering, signal processing, systems identification, control theory, the finance, fractal dynamics and many other areas of science [1–5]. Several productive methods have been put forward to construct closed form analytic solutions to NLEEs of fractional order; namely the (G'/G)-expansion method and it's various modifications [6–10], the sub-equation method [11,12], the exp-function method [13,14], the first integral method [15,16], the functional variable method [17], the modified trial equation method [18,19], the simplest equation method [20], the Lie group analysis method [21], the characteristics method [22], the auxiliary equation method [23,24], the finite difference method [25], the finite element method [26], the differential transform method [27], the homotopy perturbation method [28], the Adomian decomposition method [29,30], the variational iteration method [31], the Tzou and Stehfest's algorithm [45], the spectral Gelarkin method [46], modified logistic model [47] and others [48–51].

The fractional order NLEEs can depict the physical phenomena more accurately than that of the integer order NLEEs [1–3,5]. Consequently, the aim of this study is to construct new and further general closed form analytic wave solutions to the fractional order nonlinear evolution equations mentioned above in the sense of fractional derivative. Sousa and Oliveira have recently introduced new fractional derivatives [52,53]. There are also some definitions of fractional derivative in fractional calculus. Some of them are given below:

(i) The derivative of non-integer order defined by Caputo [42] is

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt$$

(ii) Riemann-Liouville fractional derivative is given as [42]

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt$$

This definition is modified by Jumarie as [43]

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t) - f(0)\} dt$$

(iii) Ji-Huan He introduced the fractional derivative [44]

$$D_t^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (s-t)^{n-\alpha-1} \{f_0(s) - f(s)\} ds,$$

where $f_0(x)$ is a known function.
(iv) The conformable fractional derivative of order α is defined as follows [32]:

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, t > 0,\ \alpha \in (0,\ 1].$$

If the above limit exists, then f is called α-differentiable. Let $\alpha \in (0,\ 1]$ and $f,\ g$ be α-differentiable at a point $t > 0$, then T_{α} satisfies the following properties:

(i) $T_\alpha(a\,f + b\,g) = aT_\alpha(f) + bT_\alpha(g)$, for all $a,\ b \in R$
(ii) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in R$
(iii) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$
(iv) $T_\alpha(f\,g) = f\,T_\alpha(g) + g\,T_\alpha(f)$
(v) $T_\alpha(f/g) = \{g\,T_\alpha(f) - f\,T_\alpha(g)\}/g^2$
(vi) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

In this article, the rational (G'/G)-expansion method [33] is used for searching the exact analytic traveling wave solutions to the suggested equations in the sense of conformable fractional derivative [32].

2. Description of the rational (G'/G)-expansion method

Consider the following nonlinear evolution equation of fractional order in the independent variables $t,\ x_1,\ x_2,\ ...,\ x_n$:

$$F(u_1,\ ...,\ u_k,\ \frac{\partial u_1}{\partial t},\ ...,\ \frac{\partial u_k}{\partial t},\ \frac{\partial u_1}{\partial x_1},\ ...,\ \frac{\partial u_k}{\partial x_1},\ ...,\ \frac{\partial u_1}{\partial x_n},\ ...,\ \frac{\partial u_k}{\partial x_n},$$

$$D_t^\alpha u_1,\ ...,D_t^\alpha u_k,\ D_{x_1}^\beta u_1,\ ...,\ D_{x_1}^\beta u_k,\ ...,\ D_{x_n}^\beta u_1,\ ...,\ D_{x_n}^\beta u_k,\ ...) = 0, \tag{2.1}$$

where $u_i = u_i(t,\ x_1,\ x_2,\ ...,\ x_n),\ i = 1,\ ...,\ k$ are unknown functions, F is a polynomial in u_i and it's various partial derivatives including the derivatives of fractional order.

Now, the main steps of the rational (G'/G)-expansion method are presented as follows:
Step 1: Making use of the traveling wave variable [34]

$$\xi = \xi(t,\ x_1,\ x_2,\ ...,\ x_n), u_i = u_i(t,\ x_1,\ x_2,\ ...,\ x_n) = U_i(\xi), \tag{2.2}$$

where t is the temporal variable and $x_i's$ are the spatial variables and ξ is called wave variable, Eq. (2.1) is turned into the following ordinary differential equation of integer order with respect to the variable ξ:

$$Q\,(U_1,\ ...,\ U_k,\ U_1',\ ...,\ U_k',\ U_{11}'',\ ...,\ U_{kk}'',\ ...) = 0. \tag{2.3}$$

Step 2: If possible take anti-derivative of Eq. (2.3) one or more times and integral constant can be set to zero as soliton solutions are hunted.

Step 3: Suppose the solution of Eq. (2.3) can be expressed as,

$$u\,(\xi) = \frac{a_0 + a_1(G'/G) + a_2(G'/G)^2 + \cdots + a_n(G'/G)^n}{b_0 + b_1(G'/G) + b_2(G'/G)^2 + \cdots + b_n(G'/G)^n}, \tag{2.4}$$

where $a_i,\ b_i\ (i = 0,\ 1,\ 2,\ ...,n)$ are constants with at least one of a_n and b_n is non-zero, while $G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G''(\xi) + \lambda\,G'(\xi) + \mu\,G(\xi) = 0, \tag{2.5}$$

where λ and μ are real parameters.

Eq. (2.5) provides the solutions,

$$\left(\frac{G'}{G}\right)=\begin{cases}-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi+C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi+C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}, & \lambda^2-4\mu>0\\ -\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi+C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi+C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}, & \lambda^2-4\mu<0\\ -\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\xi}, & \lambda^2-4\mu=0\end{cases} \quad (2.6)$$

where C_1 and C_2 are arbitrary constants.

Step 4: The positive integer n in Eq. (2.4) is fixed by taking homogeneous balance between the highest order derivative terms and the nonlinear terms in Eq. (2.3).

Step 5: Substituting Eq. (2.4) together with Eq. (2.5) into Eq. (2.3), we obtain a polynomial in (G'/G). By equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_i,\ b_i\ (i=0,\ 1,\ 2,\ ...,n),\lambda$ and μ. Solve this set of equations by the symbolic computation software, such as Maple for the parameters $a_i,\ b_i\ (i=0,\ 1,\ 2,\ ...,n),\lambda$ and μ.

Step 6: Using the values of $a_i,\ b_i\ (i=0,\ 1,\ 2,\ ...,n),\lambda$ and μ obtained in step 5 together with Eq. (2.6) into Eq. (2.4) provide the closed form traveling wave solutions of the nonlinear fractional partial differential Eq. (2.1).

3. Formulation of the solutions

In this section, the rational (G'/G)-expansion method is employed to derive the exact analytic solitary wave solutions to the (3+1)-dimensional space-time fractional mKdV-ZK equation, the time fractional biological population model and the space-time fractional modified regularized long-wave equation.

3.1. The (3+1)-dimensional space-time fractional mKdV-ZK equation

Consider the (3+1)-dimensional space-time fractional mKdV-ZK equation

$$D_t^{\alpha}u+\delta u^2D_x^{\alpha}u+D_x^{3\alpha}u+D_x^{\alpha}D_y^{2\alpha}u+D_x^{\alpha}D_z^{2\alpha}u=0, 0<\alpha\leq 1, \quad (3.1)$$

which is in the sense of conformable fractional derivative and δ is nonzero real constant. This equation is derived for plasma comprised of cool and hot electrons and a species of fluid ions [35].

The fractional complex transformation

$$u(x,\ y,\ z,\ t)=u(\xi),\xi=\frac{1}{\alpha}\{l\,x^{\alpha}+m\,y^{\alpha}+\ n\,z^{\alpha}-\omega\,t^{\alpha}\}, \quad (3.2)$$

where $l,\ m,\ n$ and ω are non-zero parameters, reduces Eq. (3.1) to the following ordinary differential equation with respect to the variable ξ:

$$-\omega u'+\delta l u^2u'+(l^3+l m^2+l n^2)u'''=0. \quad (3.3)$$

The anti-derivative of Eq. (3.3) with integral constant zero gives

$$-\omega u+\frac{\delta l}{3}u^{3}+(l^{3}+l m^{2}+l n^{2})u''=0. \tag{3.4}$$

Due to the homogeneous balance between u^3 and u'' the solution Eq. (2.4) takes the form

$$u(\xi)=\frac{a_0+a_1(G'/G)}{b_0+b_1(G'/G)}, \tag{3.5}$$

with at least one of a_1 and b_1 is non-zero.

The substitution of Eq. (3.5) with the help of Eq. (2.5) into Eq. (3.4) yields a polynomial in (G'/G). Setting like terms of this polynomial to zero makes available a set of algebraic equations for $a_0,\ a_1,\ b_0,\ b_1,\ \omega,\ \lambda$ and μ. Solving these equations by Maple gives the following set of solutions:

Set-1: $a_0=\pm\frac{1}{2\delta}(2b_1\mu-b_0\lambda)\sqrt{-6\delta(l^2+m^2+n^2)}$, $a_1=\pm\frac{1}{2\delta}(b_1\lambda-2b_0)\sqrt{-6\delta(l^2+m^2+n^2)}$,

$$\omega=\frac{l}{2}(4\mu-\lambda^2)(l^2+m^2+n^2),\delta\neq 0, \tag{3.6}$$

where $b_0,\ b_1,\ \lambda$ and μ are arbitrary constants.

Set-2: $a_0=\pm\frac{b_1}{4\delta}(\lambda^2-4\mu+\lambda\sqrt{4\mu-\lambda^2})\sqrt{-6\delta(l^2+m^2+n^2)}$, $b_0=\frac{b_1}{2}(\lambda+\sqrt{4\mu-\lambda^2})$,

$$a_1=\pm\frac{b_1}{2\delta}\sqrt{-6\delta(l^2+m^2+n^2)(4\mu-\lambda^2)},\omega=\frac{l}{2}(4\mu-\lambda^2)(l^2+m^2+n^2),\delta\neq 0, \tag{3.7}$$

where $b_1,\ \lambda$ and μ are arbitrary constants.

Set-3: $a_0=\pm\frac{b_1(4\mu-\lambda^2)\sqrt{-6\delta(l^2+m^2+n^2)}}{4\delta}$, $\omega=\frac{l}{2}(4\mu-\lambda^2)(l^2+m^2+n^2)$, $a_1=0$,

$$b_0=\frac{1}{2}b_1\lambda,\delta\neq 0, \tag{3.8}$$

where $b_1,\ \lambda$ and μ are arbitrary constants.

Set-4: $a_0=\pm\frac{b_0\lambda}{2\delta}\sqrt{-6\delta(l^2+m^2+n^2)}$, $a_1=\pm\frac{b_0}{\delta}\sqrt{-6\delta(l^2+m^2+n^2)}$, $b_1=0$,

$$\omega=\frac{l}{2}(4\mu-\lambda^2)(l^2+m^2+n^2),\delta\neq 0, \tag{3.9}$$

where $b_0,\ \lambda$ and μ are arbitrary constants.

Using Eqs. (3.6)–(3.9) in Eq. (3.5) possess the following respective results:

$$u_1(\xi)=\pm\sqrt{-6\delta(l^2+m^2+n^2)}\times\frac{(b_1\lambda-2b_0)+(2b_1\mu-b_0\lambda)(G'/G)}{2\delta\{b_0+b_1(G'/G)\}}, \tag{3.10}$$

$$u_2(\xi) = \frac{\pm(\lambda^2 - 4\mu + \lambda\sqrt{4\mu - \lambda^2})\sqrt{-6\delta(l^2 + m^2 + n^2)} \pm 2\sqrt{-6\delta(l^2 + m^2 + n^2)(4\mu - \lambda^2)}(G'/G)}{2\delta(\lambda + \sqrt{4\mu - \lambda^2}) + 2(G'/G)}, \tag{3.11}$$

$$u_3(\xi) = \frac{\pm(4\mu - \lambda^2)\sqrt{-6\delta(l^2 + m^2 + n^2)}}{2\delta\{\lambda + 2(G'/G)\}}, \tag{3.12}$$

$$u_4(\xi) = \pm\frac{\lambda}{2\delta}\sqrt{-6\delta(l^2 + m^2 + n^2)} \pm \frac{1}{\delta}\sqrt{-6\delta(l^2 + m^2 + n^2)}(G'/G), \tag{3.13}$$

where $\xi = \frac{1}{2\alpha}\{2(l\,x^\alpha + m\,y^\alpha + n\,z^\alpha) - l(4\mu - \lambda^2)(l^2 + m^2 + n^2)\,t^\alpha\}$.

Eq. (3.10) with the aid of Eq. (2.6) grants three types of traveling wave solutions of Eq. (2.1) as follows:

When $\lambda^2 - 4\mu > 0$, the expression for the hyperbolic function solution is

$$u_1^1(\xi) = \frac{\sqrt{-6\delta(l^2 + m^2 + n^2)}}{2\delta} \times \frac{\pm(b_1\lambda - 2b_0) \pm (2b_1\mu - b_0\lambda)\left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}\right)}{b_0 + b_1\left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}\right)}. \tag{3.14}$$

In particular case, for $C_1 \neq 0,\ C_2 = 0$ Eq. (3.14) becomes

$$u_1^1(\xi) = \pm\frac{\sqrt{-6\delta(l^2 + m^2 + n^2)}}{4\delta} \times \frac{2(b_1\lambda - 2b_0) - (2b_1\mu - b_0\lambda)\{\lambda - \sqrt{\lambda^2 - 4\mu}\tanh(\sqrt{\lambda^2 - 4\mu}/2)\xi\}}{2b_0 - b_1\{\lambda - \sqrt{\lambda^2 - 4\mu}\tanh(\sqrt{\lambda^2 - 4\mu}/2)\xi\}}, \tag{3.15}$$

where $\xi = \frac{1}{2\alpha}\{2(l\,x^\alpha + m\,y^\alpha + n\,z^\alpha) - l(4\mu - \lambda^2)(l^2 + m^2 + n^2)\,t^\alpha\}$.

When $\lambda^2 - 4\mu < 0$, the trigonometric function solution is gained as

$$u_1^2(\xi) = \frac{\sqrt{-6\delta(l^2 + m^2 + n^2)}}{2\delta} \times \frac{\pm(b_1\lambda - 2b_0) \pm (2b_1\mu - b_0\lambda)\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}\right)}{b_0 + b_1\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}\right)}. \tag{3.16}$$

For particular case, if we choose $C_1 \neq 0,\ C_2 = 0$ Eq. (3.16) turns into

$$u_1^2(\xi) = \pm\frac{\sqrt{-6\delta(l^2 + m^2 + n^2)}}{4\delta} \times \frac{2(b_1\lambda - 2b_0) - (2b_1\mu - b_0\lambda)\{\lambda + \sqrt{4\mu - \lambda^2}\tan(\sqrt{4\mu - \lambda^2}/2)\xi\}}{2b_0 - b_1\{\lambda + \sqrt{4\mu - \lambda^2}\tan(\sqrt{4\mu - \lambda^2}/2)\xi\}}, \tag{3.17}$$

where $\xi = \frac{1}{2\alpha}\{2(l\,x^\alpha + m\,y^\alpha + n\,z^\alpha) - l(4\mu - \lambda^2)(l^2 + m^2 + n^2)\,t^\alpha\}$.

The above acquired closed form solutions to the nonlinear space-time fractional mKdV-ZK equation are new and more general. If we choose $b_1 = 0$, the solutions (3.15) and (3.17) coincide with those constructed by the (G'/G)-expansion method [36]. Furthermore, Eqs. (3.11)-(3.13) under the same procedure as above also provide much more new and general solutions which are not recorded here to keep the readers away from the inconvenience.

3.2. The time fractional biological population model

The nonlinear time fractional biological population model is

$$D_t^{\alpha} u - D_x^2 u - D_y^2 u - \rho(u^2 - \eta) = 0, 0 < \alpha \leq 1, \tag{3.18}$$

where ρ, η are constants, u represents the population density and $\rho(u^2 - \eta)$ represents the population supply due to births and deaths. A biological population model is a mathematical model which helps us to understand the dynamical procedure of population changes and provides valuable predictions. The universe that range from simple to dynamic is full of interactions. Most of the earth's processes affect human life. Procedures in population modeling have significantly enhanced our understanding of biology and the natural world. A population model that is applied to the study of population dynamics is a type of mathematical model which provides us a good understanding of how complicated interactions and procedures work.

Making use of the fractional compound transformation

$$u(x, y, t) = u(\xi), \quad \xi = k\,x + i\,k\,y - \frac{c\,t^{\alpha}}{\alpha}, i^2 = -1 \tag{3.19}$$

Eq. (3.18) is converted into the integer order ODE,

$$-\,c\,u' - \rho(u^2 - \eta) = 0. \tag{3.20}$$

Balancing the highest order derivative and the nonlinear term appearing in Eq. (3.20), the solution Eq. (2.4) reduces to the form

$$u(\xi) = \frac{a_0 + a_1 (G'/G)}{b_0 + b_1 (G'/G)}, \tag{3.21}$$

where at least one of a_1 and b_1 is nonzero.

Substituting Eq. (3.21) along with Eq. (2.5) into Eq. (3.20) yields a polynomial in (G'/G). Setting each coefficient of this polynomial to zero, offered a system of algebraic equations for a_0, a_1, b_0, b_1, c, λ and μ. Solving this set of equations with the aid of computer algebra, like Maple, provides the following results:

$$a_0 = \frac{b_1}{2c}(2\rho\eta \pm \sqrt{\eta}c\lambda), a_1 = \pm\sqrt{\eta}b_1, b_0 = \frac{b_1}{2c}(c\lambda \pm 2p\sqrt{\eta}), c \neq 0, \tag{3.22}$$

where b_1, c and λ are arbitrary constants.

Inserting the values appearing in Eq. (3.22) into Eq. (3.21) possesses

$$u(\xi) = \frac{(2\rho\eta \pm \sqrt{\eta}c\lambda) \pm 2c\sqrt{\eta}(G'/G)}{(c\lambda \pm 2\rho\sqrt{\eta}) + 2c(G'/G)}, \tag{3.23}$$

where $\xi = k\,x + i\,k\,y - \frac{c\,t^{\alpha}}{\alpha}$, $i^2 = -1$.

Eq. (3.23) along with Eq. (2.6) makes available the following three types of closed form traveling wave solutions:
When $\lambda^2 - 4\mu > 0$, the hyperbolic function solution is gained as

$$u_1(\xi) = \frac{(2\rho\eta \pm \sqrt{\eta}c\lambda) \pm 2c\sqrt{\eta}\left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}\right)}{(c\lambda \pm 2\rho\sqrt{\eta}) + 2c\left(-\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi + C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}\right)}. \tag{3.24}$$

We might choose the arbitrary constants as $C_1 = r_1\cosh\theta,\ C_2 = r_1\sinh\theta$ in Eq. (3.24) and simplify the solution to

$$u_1(\xi) = \frac{2\rho\eta \pm r_1 c\sqrt{\eta(\lambda^2-4\mu)}\tanh\{(\sqrt{\lambda^2-4\mu}/2)\xi + \theta\}}{\pm 2\rho\sqrt{\eta} + cr_1\sqrt{\lambda^2-4\mu}\tanh\{(\sqrt{\lambda^2-4\mu}/2)\xi + \theta\}}, \tag{3.25}$$

where $r_1 = \sqrt{C_1^2 - C_2^2}$, $\theta = \tanh^{-1}(C_2/C_1)$ and $\xi = k\,x + i\,k\,y - \frac{c\,t^{\alpha}}{\alpha}$, $i^2 = -1$.

For $\lambda^2 - 4\mu < 0$, the trigonometric function solution is found as follows:

$$u_2(\xi) = \frac{(2\rho\eta \pm \sqrt{\eta}c\lambda) \pm 2c\sqrt{\eta}\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}\right)}{(c\lambda \pm 2\rho\sqrt{\eta}) + 2c\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}\right)}. \tag{3.26}$$

If the arbitrary constants are assigned as $C_1 = r_2\cos\phi,\ C_2 = r_2\sin\phi$ in Eq. (3.26), then it becomes to the simplest form

$$u_2(\xi) = \frac{2\rho\eta \mp cr_2\sqrt{\eta(4\mu-\lambda^2)}\tan\{(\sqrt{4\mu-\lambda^2}/2)\xi - \phi\}}{\pm 2\rho\sqrt{\eta} - cr_2\sqrt{4\mu-\lambda^2}\tan\{(\sqrt{4\mu-\lambda^2}/2)\xi - \phi\}}, \tag{3.27}$$

where $r_2 = \sqrt{C_1^2 - C_2^2}$, $\phi = \tanh^{-1}(C_2/C_1)$ and $\xi = k\,x + i\,k\,y - \frac{c\,t^{\alpha}}{\alpha}$, $i^2 = -1$.

If $\lambda^2 - 4\mu = 0$, the rational function solution is

$$u_3(\xi) = \frac{(2\rho\eta \pm \sqrt{\eta}c\lambda) \pm 2c\sqrt{\eta}\left(-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}\right)}{(c\lambda \pm 2\rho\sqrt{\eta}) + 2c\left(-\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi}\right)}. \tag{3.28}$$

In particular, for $C_1 = 0$ Eq. (3.26) becomes

$$u_3(\xi) = \frac{2\rho\eta\xi \pm 2c\sqrt{\eta}}{2c \pm 2\rho\sqrt{\eta}\xi}, \tag{3.29}$$

where $\xi = k\,x + i\,k\,y - \frac{c\,t^{\alpha}}{\alpha}$, $i^2 = -1$.

The exact solutions obtained above to the biological population model are new and general. Abdel Salam and Gumma [37] constructed two traveling wave solutions in terms of hyperbolic function only. But we gained six closed form traveling wave solutions in terms of hyperbolic function, trigonometric function and rational function. So far we hunt; no one achieved these results ever.

3.3. The space-time fractional modified regularized long-wave equation

The following nonlinear space-time fractional modified regularized long-wave equation is considered to be examined for further exact traveling wave solutions:

$$D_t^{\alpha} u + \delta\, D_x^{\alpha} u + \tau\, u^2 D_x^{\alpha} u - \eta\, D_t^{\alpha} D_x^{2\alpha} = 0, 0 < \alpha \leq 1 \tag{3.30}$$

where δ, τ and η are all constants. This equation proposed by Benjamin et al. to describe approximately the unidirectional propagation of long waves in certain dispersive systems is supposed to be alternative to the modified KdV equation. Eq. (3.30) is formulated to demonstrate some physical phenomena like transverse waves in shallow water and magneto hydrodynamic waves in plasma and photon packets in nonlinear crystals [38–40].

The wave variable transformation

$$u(x,\, t) = u\,(\xi), \quad \xi = \frac{x^{\alpha}}{\alpha} - \frac{c\, t^{\alpha}}{\alpha}, \tag{3.31}$$

reduces Eq. (3.30) to the ODE

$$(\delta - c)u' + \tau\, u^2 u' + c\, \eta\, u''' = 0. \tag{3.32}$$

Integrating Eq. (3.32) and setting integral constant to zero gives

$$(\delta - c)u + \frac{\tau}{3}\, u^3 + c\, \eta\, u'' = 0. \tag{3.33}$$

Considering homogeneous balance for Eq. (3.33) the solution Eq. (2.4) is appeared as

$$u(\xi) = \frac{a_0 + a_1\,(G'/G)}{b_0 + b_1\,(G'/G)} \tag{3.34}$$

in which at least one of a_1 and b_1 is nonzero.

Put Eq. (3.34) with the help of Eq. (2.5) in Eq. (3.33); collect the coefficients of like powers of (G'/G) and equate them to zero we obtain a set of equations for a_0, a_1, b_0, b_1, c, λ and μ. Calculating these equations by Maple gives the results

Set-1:

$$a_0 = \pm\frac{\sqrt{3\delta\eta}(2b_1\mu - b_0\lambda)}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}}, a_1 = \mp\frac{\sqrt{3\delta\eta}(2b_0 - b_1\lambda)}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}}, c = -\frac{2\delta}{4\mu\eta - \lambda^2\eta - 2}, \tag{3.35}$$

where $b_0,\ b_1,\ \lambda$ and μ are arbitrary constants.

Set-2:

$$a_0 = \pm\frac{b_1\sqrt{3\delta\eta}(4\mu - \lambda^2)}{2\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}}, a_1 = 0, b_0 = \frac{1}{2}b_1\lambda, c = -\frac{2\delta}{4\mu\eta - \lambda^2\eta - 2}, \tag{3.36}$$

where $b_1,\ \lambda$ and μ are arbitrary constants.

Set-3:

$$a_0 = \pm\frac{b_0\lambda\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta-\lambda^2\eta-2)}}, a_1 = \pm\frac{2\sqrt{3\delta\eta}b_0}{\sqrt{\tau(4\mu\eta-\lambda^2\eta-2)}}, b_1 = 0, c = -\frac{2\delta}{4\mu\eta-\lambda^2\eta-2}, \tag{3.37}$$

where b_0, λ and μ are arbitrary constants.

Set-4:

$$a_0 = \pm\frac{b_1\delta\{(4\mu-\lambda^2)(\lambda\eta\pm\sqrt{6\eta-3\eta^2(4\mu-\lambda^2)})-2\lambda\}}{\sqrt{-\delta\tau}(4\mu\eta-\lambda^2\eta-2)}, a_1 = \pm\frac{b_1\delta}{\sqrt{-\delta\tau}},$$

$$b_0 = \frac{b_1}{6\eta}\{3\eta\lambda \pm \sqrt{6\eta-3\eta^2(4\mu-\lambda^2)}\}, c = -\frac{2\delta}{4\mu\eta-\lambda^2\eta-2}, \tag{3.38}$$

where b_1, λ and μ are arbitrary constants.

Utilizing the values appeared in Eqs. (3.35)–(3.38), the Eq. (3.34) provide the following expressions for desired solutions:

$$u_1(\xi) = \frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta-\lambda^2\eta-2)}} \times \frac{\pm(2b_1\mu-b_0\lambda) \mp (2b_0-b_1\lambda)(G'/G)}{b_0+b_1(G'/G)}, \tag{3.39}$$

$$u_2(\xi) = \pm\frac{\sqrt{3\delta\eta}(4\mu-\lambda^2)}{\sqrt{\tau(4\mu\eta-\lambda^2\eta-2)}} \times \frac{1}{\lambda+2(G'/G)}, \tag{3.40}$$

$$u_3(\xi) = \pm\frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta-\lambda^2\eta-2)}} \times \{\lambda+2(G'/G)\}, \tag{3.41}$$

$$\begin{aligned} u_4(\xi) = &\frac{6\eta\delta}{\sqrt{-\delta\tau}(4\mu\eta-\lambda^2\eta-2)} \\ &\times\frac{\pm\{(4\mu-\lambda^2)(\lambda\eta\pm\sqrt{6\eta-3\eta^2(4\mu-\lambda^2)})-2\lambda\}\pm(4\mu\eta-\lambda^2\eta-2)(G'/G)}{\{3\eta\lambda\pm\sqrt{6\eta-3\eta^2(4\mu-\lambda^2)}\}+6\eta(G'/G)}, \end{aligned} \tag{3.42}$$

where $\xi = \frac{x^\alpha}{\alpha} + \frac{2\delta t^\alpha}{\alpha(4\mu\eta-\lambda^2\eta-2)}$.

Each of Eqs. (3.39)–(3.42) together with Eq. (2.6) makes available exact traveling wave solutions to the space-time fractional modified regularized long-wave equation of three types, such as hyperbolic function solution, trigonometric function solution and rational function solution. For convenience of the readers, we record here the solutions only for Eq. (3.39) as follows:

When $\lambda^2-4\mu > 0$, the hyperbolic function solution is formed as follows:

$$\begin{aligned} u_1^1(\xi) = &\frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta-\lambda^2\eta-2)}} \\ &\times\frac{\pm(2b_1\mu-b_0\lambda)\mp(2b_0-b_1\lambda)\left(-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi+C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi+C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}\right)}{b_0+b_1\left(-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}/2)\xi+C_2\cosh(\sqrt{\lambda^2-4\mu}/2)\xi}{C_1\cosh(\sqrt{\lambda^2-4\mu}/2)\xi+C_2\sinh(\sqrt{\lambda^2-4\mu}/2)\xi}\right)}. \end{aligned} \tag{3.43}$$

Assigning the arbitrary constants as $C_1 \neq 0,\ \ C_2 = 0$ to Eq. (3.43) and simplifying one may obtain

$$u_1^1(\xi) = \frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}} \times \frac{\pm 2(2b_1\mu - b_0\lambda) \pm (2b_0 - b_1\lambda)\{\lambda - \sqrt{\lambda^2 - 4\mu}\tanh(\sqrt{\lambda^2 - 4\mu}/2)\xi\}}{2b_0 - b_1\{\lambda - \sqrt{\lambda^2 - 4\mu}\tanh(\sqrt{\lambda^2 - 4\mu}/2)\xi\}}, \tag{3.44}$$

where $\xi = \frac{x^\alpha}{\alpha} + \frac{2\delta t^\alpha}{\alpha(4\mu\eta - \lambda^2\eta - 2)\alpha}$.

When $\lambda^2 - 4\mu < 0$, the trigonometric function solution is

$$\begin{aligned} u_1^2(\xi) = &\frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}} \\ &\times \frac{\pm(2b_1\mu - b_0\lambda) \mp (2b_0 - b_1\lambda)\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}\right)}{b_0 + b_1\left(-\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\cos(\sqrt{4\mu-\lambda^2}/2)\xi}{C_1\cos(\sqrt{4\mu-\lambda^2}/2)\xi + C_2\sin(\sqrt{4\mu-\lambda^2}/2)\xi}\right)}. \end{aligned} \tag{3.45}$$

Since C_1 and C_2 are arbitrary constants, if we choose $C_1 \neq 0,\ \ C_2 = 0$, Eq. (3.45) after simplification becomes

$$u_1^2(\xi) = \frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}} \times \frac{\pm 2(2b_1\mu - b_0\lambda) \pm (2b_0 - b_1\lambda)\{\lambda + \sqrt{\lambda^2 - 4\mu}\tan(\sqrt{\lambda^2 - 4\mu}/2)\xi\}}{2b_0 - b_1\{\lambda + \sqrt{\lambda^2 - 4\mu}\tan(\sqrt{\lambda^2 - 4\mu}/2)\xi\}}, \tag{3.46}$$

where $\xi = \frac{x^\alpha}{\alpha} + \frac{2\delta t^\alpha}{\alpha(4\mu\eta - \lambda^2\eta - 2)\alpha}$.

For $\lambda^2 - 4\mu = 0$, the rational function solution is

$$u_1^3(\xi) = \frac{\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}} \times \frac{\pm(2b_1\mu - b_0\lambda) \mp (2b_0 - b_1\lambda)\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right)}{b_0 + b_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}\right)}. \tag{3.47}$$

In particular, if $C_1 \neq 0,\ \ C_2 = 0$, Eq. (3.47) is simplified to the form

$$u_1^3(\xi) = \pm\frac{(2b_1\lambda - 4b_0)\sqrt{3\delta\eta}}{\sqrt{\tau(4\mu\eta - \lambda^2\eta - 2)}} \times \frac{1}{2b_1 + (2b_0 - b_1\lambda)\xi}, \tag{3.48}$$

where $\xi = \frac{x^\alpha}{\alpha} + \frac{2\delta t^\alpha}{\alpha(4\mu\eta - \lambda^2\eta - 2)}$.

The closed form traveling wave solutions to the nonlinear space-time fractional modified regularized long-wave equation were successfully constructed in this effort. The solutions obtained by Kaplan et al. [41] and also by Abdel Salam and Gumma [37] are only in terms of hyperbolic, where as we achieved those in terms of hyperbolic, trigonometric and rational. On comparison, our solutions are general and much more in number than those of [37,41].

4. Conclusion

In this article, our core aim was to explore further new and general closed form traveling wave solutions to the (3+1)-dimensional space-time fractional mKdV-ZK equation, the time fractional biological population model and the space-time fractional modified regularized long-wave equation. The desired solutions have been successfully achieved by the rational (G'/G)-expansion method in terms of hyperbolic, trigonometric and rational. To the best of our knowledge, the results obtained throughout this article are not recorded in the literature. The suggested method has shown high performance to construct traveling wave solutions in closed form which will be helpful to analyze important phenomena in the nature of real world.

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Conflict of interest

The authors declare that they have no competing interests.

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