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Research article

Non-null slant ruled surfaces

Mehmet Önder*

Delibekirli Village, Tepe Street, No. 63, 31440 Kırıkhan, Hatay, Turkey

* **Correspondence:** Email: mehmetonder197999@gmail.com.

Abstract: In this study, we define some new types of non-null ruled surfaces called slant ruled surfaces in the Minkowski 3-space E_1^3. We introduce some characterizations for a non-null ruled surface to be a slant ruled surface in E_1^3. Moreover, we obtain some corollaries which give the relationships between a non-null slant ruled surface and its striction line.

Keywords: non-null ruled surface; Frenet frame; slant ruled surface
Mathematics Subject Classification: 53A25, 53C50, 14J26

1. Introduction

In the study of curve theory, the curves whose curvatures satisfy some special conditions have an important role. The well-known of such curves is general helix defined by the classical definition that the tangent lines of the curve make a constant angle with a fixed straight line [5]. In 1802, M.A. Lancret stated a result on the helices which was first proved by B. de Saint Venant in 1845 [25]. Venant showed that a curve is a general helix if and only if the ratio of the curvatures κ and τ of the curve is constant, i.e., κ/τ is constant at all points of the curve. Helices have been studied not only in Euclidean spaces but also in Lorentzian spaces by some mathematicians and different characterizations of these curves have been obtained according to the properties of the spaces [7, 8, 13, 18].

Recently, Izumiya and Takeuchi have introduced a new curve called slant helix which is defined by the property that the normal lines of the curve make a constant angle with a fixed direction in the Euclidean 3-space E^3 [9]. Later, the spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix have been studied by Kula and Yaylı and they have obtained that the spherical images of a slant helix are spherical helices [14]. The position vector of a slant helix in E^3 has been studied by Ali [4]. Then the corresponding characterizations for the position vector of a timelike slant helix in Minkowski 3-space E_1^3 have been given by Ali and Turgut [3]. Moreover, Ali and Lopez have also given some new characterizations of slant helices in E_1^3 [2].

Analogue to the curves, ruled surfaces have orthonormal frames along their striction curves. So, the notions "helix" or "slant helix" can be considered for ruled surfaces. Before, Abdel-Baky considered the notion of "slant" for ruled surfaces by means of dual vector analysis [1]. Later, by considering the orthonormal frame along the striction curve of a ruled surface, Önder has defined slant ruled surfaces in the real Euclidean 3-space [24]. Moreover, Kaya and Önder have studied the position vectors and some differential equation characterizations for slant ruled surfaces in E^3 [10, 11, 23]. They have also studied this subject for null scrolls and defined slant null scrolls in E_1^3 [22].

In this paper, we define non-null slant ruled surfaces by considering the Frenet vectors of timelike and spacelike ruled surfaces in E_1^3. We give the conditions for a non-null ruled surface to be a slant ruled surface.

2. Preliminaries

Let E_1^3 be a Minkowski 3-space with natural Lorentz metric $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since this metric is not positive definite, for an arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in E_1^3 we have (i) $\langle \vec{v}, \vec{v} \rangle > 0$ and $\vec{v} = 0$, (ii) $\langle \vec{v}, \vec{v} \rangle < 0$ (iii) $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Then we have three types of vectors: spacelike, timelike or null (lightlike) if (i), (ii) or (iii) holds, respectively [16]. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha}'(s)$ satisfy (i), (ii) or (iii), respectively. For the vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, the vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2,\ x_1 y_3 - x_3 y_1,\ x_2 y_1 - x_1 y_2).$$

Analogue to the curves, a surface can be timelike or spacelike in E_1^3. The Lorentzian character of a surface in E_1^3 is determined by the induced metric on the surface. The surface is called timelike (spacelike), if the induced metric on the surface is a Lorentz metric (positive definite Riemannian metric) [6].

Let now I be an open interval in the real line IR. Let $\vec{k} = \vec{k}(u)$ be a curve in E_1^3 defined on I and $\vec{q} = \vec{q}(u)$ be a unit direction vector of an oriented line in E_1^3. Then we have following parametrization for a ruled surface N,

$$\vec{r}(u, v) = \vec{k}(u) + v\,\vec{q}(u). \tag{2.1}$$

The straight lines of surface are called rulings and the curve $\vec{k} = \vec{k}(u)$ is called base curve or generating curve. In particular, if the direction of $\vec{q}$ is constant, then ruled surface is said to be cylindrical, and non-cylindrical otherwise.

The function defined by

$$\delta = \frac{\det(d\vec{k},\ \vec{q},\ d\vec{q})}{\langle d\vec{q},\ d\vec{q} \rangle}$$

is called distribution parameter (or drall) of ruled surface. Then, N is called developable surface if and only if $\delta = 0$ [15, 19, 21]. Then at all points of same ruling, the tangent planes are identical, i.e., tangent plane contacts the surface along a ruling. If $\det(d\vec{k},\ \vec{q},\ d\vec{q}) \neq 0$, then the tangent planes of N are distinct at all points of same ruling which is called nontorsal [19, 21].

Let consider the unit normal vector $\vec{m}$ of N defined by $\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$. So, at the points of a nontorsal ruling $u = u_1$ we have

$$\vec{a} = \lim_{v \to \infty} \vec{m}(u_1, v) = \frac{d\vec{q} \times \vec{q}}{\|d\vec{q}\|}.$$

which is called central tangent. The point at which the vectors $\vec{a}$ and $\vec{m}$ are orthogonal is called the striction point (or central point) C and the set of striction points of all rulings is called striction curve which has parametric representation

$$\vec{c}(u) = \vec{k}(u) - \frac{\langle d\vec{q},\ d\vec{k} \rangle}{\langle d\vec{q},\ d\vec{q} \rangle}\vec{q}(u). \tag{2.2}$$

It is clear that base curve is same with striction curve if and only if $\langle d\vec{q},\ d\vec{k} \rangle = 0$.

Since the vectors $\vec{a}$ and $\vec{q}$ are orthogonal, we can define an orthonormal frame on surface. For this purpose, let write $\vec{h} = \vec{a} \times \vec{q}$. The unit vector $\vec{h}$ is called central normal and the orthonormal frame $\{C;\ \vec{q}, \vec{h}, \vec{a}\}$ at central point C is called Frenet frame of N.

According to the Lorentzian casual characters of ruling and central normal, the Lorentzian character of surface N is classified as follows:

i) If the central normal vector $\vec{h}$ is spacelike and $\vec{q}$ is timelike, then the ruled surface N is said to be of type N_-.

ii) If the central normal vector $\vec{h}$ and the ruling $\vec{q}$ are both spacelike, then the ruled surface N is said to be of type N_+.

iii) If the central normal vector $\vec{h}$ is timelike, then the ruled surface N is said to be of type $N_\times$ [19,21].

The ruled surfaces of type N_- and N_+ are clearly timelike and ruled surface of type $N_\times$ is spacelike. By using these classifications and taking striction curve as base curve, parametrization of ruled surface N can be given as follows,

$$\vec{r}(s, v) = \vec{c}(s) + v\,\vec{q}(s), \tag{2.3}$$

where $\langle \vec{q}, \vec{q} \rangle = \varepsilon\,(= \pm 1)$, $\langle \vec{h}, \vec{h} \rangle = \pm 1$ and s is arc length of striction curve.

For the derivatives of vectors of Frenet frame $\{C;\ \vec{q}, \vec{h}, \vec{a}\}$ of ruled surface N with respect to arc length s of striction curve we have followings:

i) If the ruled surface N is a timelike ruled surface then we have

$$\begin{bmatrix} d\vec{q}/ds \\ d\vec{h}/ds \\ d\vec{a}/ds \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & \varepsilon k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{2.4}$$

and $\vec{q} \times \vec{h} = \varepsilon\vec{a}$, $\vec{h} \times \vec{a} = -\varepsilon\vec{q}$, $\vec{a} \times \vec{q} = -\vec{h}$ [19].

ii) If the ruled surface N is spacelike ruled surface then we have

$$\begin{bmatrix} d\vec{q}/ds \\ d\vec{h}/ds \\ d\vec{a}/ds \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{2.5}$$

and $\vec{q} \times \vec{h} = -\vec{a}$, $\vec{h} \times \vec{a} = -\vec{q}$, $\vec{a} \times \vec{q} = \vec{h}$ [21].

In the equations (2.4) and (2.5), $k_1 = \frac{ds_1}{ds}$, $k_2 = \frac{ds_3}{ds}$ and s_1, s_3 are arc lengths of spherical curves generated by unit vectors $\vec{q}$ and $\vec{a}$, respectively.

Theorem 2.1. (**[17]**) *Let the striction curve $\vec{c} = \vec{c}(s)$ of ruled surface N be a unit speed curve and has the same Lorentzian casual character with the ruling and let also $\vec{c}(s)$ be the base curve of the surface. Then N is developable if and only if the unit tangent of the striction curve is the same with the ruling along the curve.*

3. Non-null q-slant ruled surfaces

In this section, we introduce the definition and characterizations of non-null q-slant ruled surfaces in E_1^3. First, we give the following definition.

Definition 3.1. Let N be a non-null ruled surface in E_1^3 given by the parametrization

$$\vec{r}(s,v)=\vec{c}(s)+v\,\vec{q}(s),\quad \left\|\vec{q}(s)\right\|=1, \tag{3.1}$$

where $\vec{c}(s)$ is striction curve of N and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame and non-zero invariants of N be $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and k_1, k_2, respectively. Then, N is called a q-slant ruled surface if the ruling $\vec{q}(s)$ makes a constant angle with a fixed non-null unit direction $\vec{u}$ in the space, i.e.,

$$\langle\vec{q},\vec{u}\rangle=c_q=constant. \tag{3.2}$$

Then we give following characterizations for q-slant ruled surfaces in E_1^3. Whenever we talk about N we will mean that the surface has properties as assumed in Definition 3.1.

Theorem 3.1. *The ruled surface N is a q-slant ruled surface if and only if the function k_1/k_2 is constant and given by*

$$k_1/k_2=\begin{cases}-\varepsilon c_a/c_q, & N\ is\ timelike\\ -c_a/c_q, & N\ is\ spacelike\end{cases} \tag{3.3}$$

where $c_q=\langle\vec{q},\vec{u}\rangle$, $c_a=\langle\vec{a},\vec{u}\rangle$ are non-zero constants.

Proof. Let N be a q-slant ruled surface in E_1^3. Then denoting by $\vec{u}$ the unit vector of fixed direction, N satisfies

$$\langle\vec{q},\vec{u}\rangle=c_q=constant. \tag{3.4}$$

Differentiating (3.4) with respect to s gives $\left\langle\vec{h},\vec{u}\right\rangle=0$. Therefore, $\vec{u}$ lies on the plane spanned by the vectors $\vec{q}$ and $\vec{a}$, i.e.,

$$\vec{u}=c_q\vec{q}+c_a\vec{a}, \tag{3.5}$$

where c_q and c_a are real constants. By differentiating (3.5) with respect to s it follows

$$0=\begin{cases}(c_qk_1+\varepsilon c_ak_2)\vec{h}; & N\ is\ timelike,\\ (c_qk_1+c_ak_2)\vec{h}; & N\ is\ spacelike,\end{cases} \tag{3.6}$$

and then we have that the function

$$k_1/k_2=\begin{cases}-\varepsilon c_a/c_q, & N\ is\ timelike\\ -c_a/c_q, & N\ is\ spacelike\end{cases}$$

is constant.

Conversely, given a non-null ruled surface N, let the equation (3.3) is satisfied. We define

$$\vec{u}=c_q\vec{q}+c_a\vec{a}, \tag{3.7}$$

where $\langle \vec{q}, \vec{u} \rangle = c_q$, $\langle \vec{a}, \vec{u} \rangle = c_a$ are non-zero constants. Differentiating (3.7) and using (3.3) it follows $\vec{u}' = 0$, i.e., $\vec{u}$ is a constant vector. On the other hand $\langle \vec{q}, \vec{u} \rangle = c_q = constant$. Then N is a q-slant ruled surface in E_1^3.

Theorem 3.2. *Non-null ruled surface N is a q-slant ruled surface if and only if* $\det(\vec{q}', \vec{q}'', \vec{q}''') = 0$.
Proof. From the Frenet formulae in (2.4) and (2.5) we have

$$\det(\vec{q}', \vec{q}'', \vec{q}''') = \begin{cases} -\varepsilon k_1^3 k_2^2 \left(\frac{k_1}{k_2}\right)' ; & N \text{ is timelike} \\ k_1^3 k_2^2 \left(\frac{k_1}{k_2}\right)' ; & N \text{ is spacelike} \end{cases} \tag{3.8}$$

Let now N be a q-slant ruled surface in E_1^3. By Theorem 3.1 we have k_1/k_2 is constant. Then from (3.8) it follows that $\det(\vec{q}', \vec{q}'', \vec{q}''') = 0$.

Conversely, if $\det(\vec{q}', \vec{q}'', \vec{q}''') = 0$, since the curvatures are non-zero, from (3.8) it is obtained that k_1/k_2 is constant and Theorem 3.1 gives that N is a q-slant ruled surface in E_1^3.

Theorem 3.3. *Non-null ruled surface N is a q-slant ruled surface if and only if* $\det(\vec{a}', \vec{a}'', \vec{a}''') = 0$.
Proof. From the Frenet formulae in (2.4) and (2.5) it follows

$$\det(\vec{a}', \vec{a}'', \vec{a}''') = \begin{cases} -k_2^5 \left(\frac{k_1}{k_2}\right)' ; & N \text{ is timelike,} \\ k_2^5 \left(\frac{k_1}{k_2}\right)' ; & N \text{ is spacelike.} \end{cases} \tag{3.9}$$

Let now N be a q-slant ruled surface in E_1^3. By Theorem 3.1, we have k_1/k_2 is constant. Then from (3.9) it follows that $\det(\vec{a}', \vec{a}'', \vec{a}''') = 0$.

Conversely, if $\det(\vec{a}', \vec{a}'', \vec{a}''') = 0$, since the curvature k_2 is non-zero, from (3.9) it is obtained that k_1/k_2 is constant and Theorem 3.1 gives that N is a q-slant ruled surface in E_1^3.

Theorem 3.4. *Non-null ruled surface N is a q-slant ruled surface if and only if*

$$\vec{q}''' + m\vec{q}' = 3k_1'\vec{h}', \tag{3.10}$$

holds where

$$m = \begin{cases} -\frac{k_1''}{k_1} + \varepsilon(k_1^2 - k_2^2); & N \text{ is timelike,} \\ -\left(\frac{k_1''}{k_1} + k_1^2 + k_2^2\right); & N \text{ is spacelike.} \end{cases}$$

Proof. Assume that N is a timelike q-slant ruled surface. From (2.4) we get followings

$$\vec{q}'' = -\varepsilon k_1^2 \vec{q} + k_1' \vec{h} + k_1 k_2 \vec{a}, \tag{3.11}$$

$$\vec{q}''' = (-3\varepsilon k_1 k_1')\vec{q} + (k_1'' + \varepsilon k_1 k_2^2)\vec{h} + (2k_1' k_2 + k_1 k_2')\vec{a} - (\varepsilon k_1^2)\vec{q}'. \tag{3.12}$$

Since N is a q-slant ruled surface, k_1/k_2 is constant and by differentiation we have

$$k_1 k_2' = k_2 k_1', \tag{3.13}$$

and from (2.4)

$$\vec{h} = \frac{1}{k_1}\vec{q}'. \tag{3.14}$$

Substituting (3.13) and (3.14) in (3.12) gives

$$\vec{q}''' = \left(\frac{k_1''}{k_1} + \varepsilon(k_2^2 - k_1^2)\right)\vec{q}' + 3k_1'(-\varepsilon k_1\vec{q} + k_2\vec{a}). \tag{3.15}$$

Using the second equation of (2.4), (3.10) is obtained from (3.15).

Conversely, let us assume that (3.10) holds. Differentiating (3.14) gives

$$\vec{h}' = -\left(\frac{k_1'}{k_1^2}\right)\vec{q}' + \left(\frac{1}{k_1}\right)\vec{q}'', \tag{3.16}$$

and so,

$$\vec{h}'' = -\left(\frac{k_1'}{k_1^2}\right)'\vec{q}' - 2\left(\frac{k_1'}{k_1^2}\right)\vec{q}'' + \left(\frac{1}{k_1}\right)\vec{q}'''. \tag{3.17}$$

Substituting (3.10) in (3.17) it follows

$$\vec{h}'' = -2\left(\frac{k_1'}{k_1^2}\right)\vec{q}'' - \left[\left(\frac{k_1'}{k_1^2}\right)' + \frac{m}{k_1}\right]\vec{q}' + 3\left(\frac{k_1'}{k_1}\right)\vec{h}'. \tag{3.18}$$

Now, writing (3.11) in (3.18) and using (2.4) we have

$$\vec{h}'' = -\left[\left(\frac{k_1'}{k_1^2}\right)' + \frac{m}{k_1}\right]\vec{q}' - (\varepsilon k')\vec{q} - 2\left(\frac{k_1'}{k_1}\right)^2\vec{h} + \left(\frac{k_2k_1'}{k_1}\right)\vec{a}. \tag{3.19}$$

On the other hand, from (2.4) it is obtained

$$\vec{h}'' = -(\varepsilon k_1)\vec{q}' - (\varepsilon k_1')\vec{q} + (\varepsilon k_2^2)\vec{h} + k_2'\vec{a}. \tag{3.20}$$

Substituting (3.20) in (3.19) we have

$$\frac{k_2'}{k_2} = \frac{k_1'}{k_1}. \tag{3.21}$$

Integrating (3.21), we get that k_1/k_2 is constant and by Theorem 3.1, N is a q-slant ruled surface.

If N is a spacelike ruled surface, then by the similar way it is obtained that N is a q-slant ruled surface if and only if (3.10) holds for $m = -\left(\frac{k_1''}{k_1} + k_1^2 + k_2^2\right)$.

Theorem 3.5. *Let N be a developable non-null ruled surface in E_1^3. Then N is a q-slant ruled surface if and only if the striction line $\vec{c}(s)$ is a general helix in E_1^3.*

Proof. Since N is a developable non-null ruled surface in E_1^3, from Theorem 2.1 we have $\vec{c}'(s) = \vec{t}(s) = \vec{q}(s)$ where $\vec{t}(s)$ is the unit tangent of $\vec{c}(s)$. Then from Definition 3.1, it is clear that N is a q-slant ruled surface if and only if the striction line $\vec{c}(s)$ is a general helix in E_1^3.

4. Non-null h-slant ruled surfaces

In this section, we introduce the definition and characterizations of non-null h-slant ruled surfaces in E_1^3. First, we give the following definition.

Definition 4.1. Let N be a non-null ruled surface in E_1^3 given by the parametrization

$$\vec{r}(s,v)=\vec{c}(s)+v\,\vec{q}(s),\quad \left\|\vec{q}(s)\right\|=1, \tag{4.1}$$

where $\vec{c}(s)$ is striction curve of N and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame and non-zero invariants of N be $\left\{\vec{q},\vec{h},\vec{a}\right\}$ and k_1, k_2, respectively. Then, N is called a h-slant ruled surface if the central normal vector $\vec{h}$ makes a constant angle with a fixed non-zero unit direction $\vec{u}$ in the space, i.e.,

$$\left\langle \vec{h},\vec{u}\right\rangle = c_h = constant. \tag{4.2}$$

Then, under the assumptions given in Definition 4.1, we can give the following theorems characterizing non-null h-slant ruled surfaces.

Theorem 4.1. *N is a non-null h-slant ruled surface if and only if the function*

$$f=\begin{cases}\dfrac{k_1^2}{(\varepsilon(k_2^2-k_1^2))^{\frac{3}{2}}}\left(\dfrac{k_2}{k_1}\right)'; & N\ is\ timelike,\\ \dfrac{k_1^2}{(k_1^2+k_2^2)^{\frac{3}{2}}}\left(\dfrac{k_2}{k_1}\right)'; & N\ is\ spacelike.\end{cases} \tag{4.3}$$

is constant.

Proof. Assume that N is a non-null h-slant ruled surface and let N be timelike. Let $\vec{u}$ be a fixed non-zero constant vector such that $\left\langle \vec{h},\vec{u}\right\rangle = c_h = constant$. Then for the vector $\vec{u}$ we have

$$\vec{u}=b_1(s)\vec{q}(s)+c_h\vec{h}(s)+b_2(s)\vec{a}(s), \tag{4.4}$$

where $b_1=b_1(s)$ and $b_2=b_2(s)$ are smooth functions of arc length parameter s. Since $\vec{u}$ is constant, differentiation of (4.4) gives

$$\begin{cases} b_1'-\varepsilon c_h k_1=0,\\ b_1k_1+\varepsilon b_2k_2=0,\\ b_2'+c_hk_2=0.\end{cases} \tag{4.5}$$

From the second equation of system (4.5) we have

$$b_1=-\varepsilon b_2\frac{k_2}{k_1}. \tag{4.6}$$

Moreover,

$$\langle\vec{u},\vec{u}\rangle=\varepsilon b_1^2+c_h^2-\varepsilon b_2^2=constant. \tag{4.7}$$

Substituting (4.6) in (4.7) gives

$$\varepsilon b_2^2\left(\left(\frac{k_2}{k_1}\right)^2-1\right)=n^2=constant. \tag{4.8}$$

Then from (4.8) it is obtained that

$$b_2=\pm\frac{n}{\sqrt{\varepsilon\left[\left(\frac{k_2}{k_1}\right)^2-1\right]}}. \tag{4.9}$$

Considering the third equation of system (4.5), from (4.9) we have

$$\frac{d}{ds}\left[\pm\frac{n}{\sqrt{\varepsilon\left[\left(\frac{k_2}{k_1}\right)^2-1\right]}}\right]=-c_hk_2.$$

This can be written as

$$\frac{k_1^2}{\left(\varepsilon(k_2^2-k_1^2)\right)^{\frac{3}{2}}}\left(\frac{k_2}{k_1}\right)'=\frac{c_h}{n}=constant,$$

which is desired.

Conversely, assume that N is timelike and the function in (4.3) is constant, i.e.,

$$\frac{k_1^2}{\left(\varepsilon(k_2^2-k_1^2)\right)^{\frac{3}{2}}}\left(\frac{k_2}{k_1}\right)'=constant=d.$$

We define

$$\vec{u}=\frac{k_2}{\sqrt{\varepsilon(k_2^2-k_1^2)}}\vec{q}-d\vec{h}-\frac{\varepsilon k_1}{\sqrt{\varepsilon(k_2^2-k_1^2)}}\vec{a},\quad \varepsilon(k_2^2-k_1^2)>0. \tag{4.10}$$

Differentiating (4.10) with respect to s and using (4.3) we have $\vec{u}'=0$, i.e., $\vec{u}$ is a constant vector. On the other hand $\left\langle\vec{h},\vec{u}\right\rangle=constant$. Thus, N is a non-null h-slant ruled surface.

If N is considered as a spacelike ruled surface, then making the similar calculations, it is obtained that N is a h-slant ruled surface if and only if the function $\frac{k_1^2}{(k_1^2+k_2^2)^{\frac{3}{2}}}\left(\frac{k_2}{k_1}\right)'$ is constant.

At the following theorem we give a special case for which the first curvature k_1 is equal to 1 and obtain second curvature for N to be a non-null h-slant ruled surface.

Theorem 4.2. *Let N be a non-null ruled surface with first curvature $k_1\equiv1$. Then the central normal vector $\vec{h}$ makes a constant angle θ with a fixed non-null direction $\vec{u}$, i.e., N is a h-slant ruled surface if and only if the second curvature is given as follows:*

i) *If N is a timelike ruled surface then*

$$k_2(s)=\pm\frac{s}{\sqrt{s^2+\varepsilon\mu^2(\theta)}},\quad (s^2+\varepsilon\mu^2(\theta)>0), \tag{4.11}$$

where

$$\mu(\theta)=\begin{cases}\coth\theta; & if\ \vec{u}\ is\ a\ timelike\ vector,\\ \tanh\theta; & if\ \vec{u}\ is\ a\ spacelike\ vector.\end{cases}$$

ii) *If N is a spacelike ruled surface then*

$$k_2(s)=\pm\frac{s}{\sqrt{\eta^2(\theta)-s^2}},\quad (\eta^2(\theta)-s^2>0), \tag{4.12}$$

where

$$\eta(\theta) = \begin{cases} \tanh\theta; & if\ \vec{u}\ is\ a\ timelike\ vector, \\ \coth\theta; & if\ \vec{u}\ is\ a\ spacelike\ vector. \end{cases}$$

Proof. Let N be a timelike ruled surface with first curvature $k_1 \equiv 1$ and let N be a h-slant ruled surface. Then for a fixed constant timelike unit vector $\vec{u}$ we have

$$\left\langle \vec{h}, \vec{u} \right\rangle = \sinh\theta = constant. \tag{4.13}$$

Differentiating (4.13) with respect to s gives

$$\langle -\varepsilon\vec{q} + k_2\vec{a},\ \vec{u} \rangle = 0, \tag{4.14}$$

and from (4.14) we have

$$\langle \vec{q}, \vec{u} \rangle = \varepsilon k_2 \langle \vec{a}, \vec{u} \rangle. \tag{4.15}$$

If we put $\langle \vec{a}, \vec{u} \rangle = \varepsilon x$, we can write

$$\vec{u} = (\varepsilon k_2 x)\vec{q} + (\sinh\theta)\vec{h} - x\vec{a}. \tag{4.16}$$

Since $\vec{u}$ is unit, from (4.16) we have

$$x = \pm\frac{\cosh\theta}{\sqrt{\varepsilon(1-k_2^2)}}, \quad (\varepsilon(1-k_2^2) > 0). \tag{4.17}$$

Then, the vector $\vec{u}$ is given as follows

$$\vec{u} = \pm\frac{\varepsilon k_2\cosh\theta}{\sqrt{\varepsilon(1-k_2^2)}}\vec{q} + (\sinh\theta)\vec{h} \mp \frac{\cosh\theta}{\sqrt{\varepsilon(1-k_2^2)}}\vec{a}. \tag{4.18}$$

Differentiating (4.14) with respect to s, it follows

$$\left\langle -\varepsilon(1-k_2^2)\vec{h} + k_2'\vec{a},\ \vec{u} \right\rangle = 0. \tag{4.19}$$

Writing $\langle \vec{a}, \vec{u} \rangle = \varepsilon x$ and (4.13) in (4.19) we have

$$x = \frac{(1-k_2^2)\sinh\theta}{k_2'}. \tag{4.20}$$

From (4.17) and (4.20) we obtain the following differential equation,

$$\pm\coth\theta\frac{\varepsilon k_2'}{\left(\varepsilon(1-k_2^2)\right)^{3/2}} - 1 = 0. \tag{4.21}$$

By integration from (4.21) we get

$$\pm\coth\theta\frac{\varepsilon k_2}{\sqrt{\varepsilon(1-k_2^2)}} - s + c = 0, \tag{4.22}$$

where c is integration constant. By making parameter change $s \to -s + c$, (4.22) can be written as

$$\pm \coth\theta \frac{\varepsilon k_2}{\sqrt{\varepsilon(1-k_2^2)}} = -s \tag{4.23}$$

which gives us $k_2(s) = \pm \frac{s}{\sqrt{s^2+\varepsilon\coth^2\theta}}$.

Conversely, assume that $k_2(s) = \pm \frac{s}{\sqrt{s^2+\varepsilon\coth^2\theta}}$ holds and let us put

$$x = \pm \frac{\cosh\theta}{\sqrt{\varepsilon(1-k_2^2)}} = \pm \frac{\cosh\theta}{\sqrt{\varepsilon - \frac{\varepsilon s^2}{s^2+\varepsilon\coth^2\theta}}} = \pm \sinh\theta \sqrt{s^2+\varepsilon\coth^2\theta}, \tag{4.24}$$

where θ is constant. Then, $k_2 x = s\sinh\theta$. Let now consider the vector $\vec{u}$ defined by

$$\vec{u} = \sinh\theta\left(\varepsilon s\vec{q} + \vec{h} \mp \left(\sqrt{s^2+\varepsilon\coth^2\theta}\right)\vec{a}\right) \tag{4.25}$$

By differentiating (4.25) and using Frenet formulae we have $\vec{u}' = 0$, i.e., the direction of $\vec{u}$ is constant and $\left\langle \vec{h}, \vec{u} \right\rangle = \sinh\theta = constant$. Then N is a non-null h-slant ruled surface.

If we assume that $\vec{u}$ is spacelike then we have $\left\langle \vec{h}, \vec{u} \right\rangle = \cosh\theta = constant$ and making the similar calculations we obtain $k_2(s) = \pm \frac{s}{\sqrt{s^2+\varepsilon\coth^2\theta}}$. Then we can write (4.11).

If the ruled surface N is a spacelike ruled surface then following the same procedure, it is easily obtained that N is a h-slant ruled surface if and only if the second curvature is given by $k_2(s) = \pm \frac{s}{\sqrt{\eta^2(\theta)-s^2}}$ where $\eta(\theta) = \tanh\theta$, if $\vec{u}$ is a timelike vector; and $\eta(\theta) = \coth\theta$, if $\vec{u}$ is a spacelike vector.

On the other hand, if the striction line $\vec{c}(s)$ is a geodesic on N, then principal normal vector $\vec{n}$ of $\vec{c}(s)$ and central normal vector $\vec{h}$ of N coincide. Then, we have the following corollary.

Corollary 4.1. *Let the striction line $\vec{c}(s)$ be a geodesic on N. Then N is a non-null h-slant ruled surface if and only if $\vec{c}(s)$ is a slant helix in E_1^3.*

If the non-null ruled surface N is developable, then by Theorem 2.1, the Frenet frame $\left\{\vec{t}, \vec{n}, \vec{b}\right\}$ of striction line $\vec{c}(s)$ coincides with the frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and we can give the following corollary.

Corollary 4.2. *Let N be a non-null developable surface in E_1^3. Then N is a h-slant ruled surface if and only if striction line is a slant helix in E_1^3.*

5. Non-null a-slant ruled surfaces

In this section we introduce the definition of a-slant ruled surfaces in E_1^3.

Definition 5.1. Let N be a non-null ruled surface in E_1^3 given by the parametrization

$$\vec{r}(s,v) = \vec{c}(s) + v\,\vec{q}(s), \quad \left\|\vec{q}(s)\right\| = 1,$$

where $\vec{c}(s)$ is striction curve of N and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame and non-zero invariants of N be $\{\vec{q}, \vec{h}, \vec{a}\}$ and k_1, k_2, respectively. Then, N is called a a-slant ruled surface if the central tangent vector $\vec{a}$ makes a constant angle with a fixed non-zero direction $\vec{u}$ in the space, i.e.,

$$\langle \vec{a}, \vec{u} \rangle = c_a = constant.$$

From (3.5) it is clear that a non-null ruled surface N is a-slant ruled surface if and only if it is a q-slant ruled surface. So, all the theorems given in Section 3 also characterize the a-slant ruled surfaces.

After these definitions and characterizations of non-null slant ruled surfaces we can give the followings:

Let N_1 and N_2 be two non-null ruled surfaces in E_1^3 with Frenet frames $\{\vec{q}_1, \vec{h}_1, \vec{a}_1\}$ and $\{\vec{q}_2, \vec{h}_2, \vec{a}_2\}$, respectively. If N_1 and N_2 have common central normals i.e., $\vec{h}_1 = \vec{h}_2$ at the corresponding points of their striction lines, then N_1 and N_2 are called Bertrand offsets [12]. Similarly, if $\vec{a}_1 = \vec{h}_2$ at the corresponding points of their striction lines, then the surface N_2 is called a Mannheim offset of N_1 and the ruled surfaces N_1 and N_2 are called Mannheim offsets [20]. Considering these definitions we come to the following corollaries:

Corollary 5.1. *Let N_1 be a non-null h-slant ruled surface. Then the Bertrand offsets of N_1 form a family of non-null h-slant ruled surfaces.*

Corollary 5.2. *Let N_1 and N_2 form a Mannheim offset. Then N_1 is a non-null q-slant (or a-slant) ruled surface if and only if N_2 is a non-null h-slant ruled surface.*

6. Conclusion

The notion of "slant" given for the curves is considered for the non-null ruled surfaces in the Minkowski 3-space and some new types of non-null ruled surfaces called "slant ruled surfaces" are defined and characterized. These characterizations are obtained according to the curvatures of the non-null surface which are defined on the striction line of the surface. Of course, new characterizations can be obtained for these surfaces and moreover, the subject can be studied in different spaces in which ruled surfaces are defined.

Conflict of interest

The author declares no conflict of interest.

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