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Research article

Generalized k-fractional conformable integrals and related inequalities

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Abstract: In the paper, the authors introduce the generalized k-fractional conformable integrals, which are the k-analogues of the recently introduced fractional conformable integrals and can be reduced to other fractional integrals under specific values of the parameters involved. Hereafter, the authors prove the existence of k-fractional conformable integrals. Finally, the authors generalize some integral inequalities to ones for generalized k-fractional conformable integrals.

Keywords: left k-fractional conformable integral; right k-fractional conformable integral; fractional integral inequality
Mathematics Subject Classification: Primary 26A33; Secondary 26D10, 35A23, 47A63

1. Introduction

Conformable derivatives are nonlocal fractional derivatives. They can be called fractional since we can derive up to arbitrary order. However, since in the community of fractional calculus nonlocal fractional derivatives only are used to be called fractional, we prefer to replace conformable fractional by conformable (as a type of local fractional). Conformable derivatives and other types of local fractional derivatives or modified conformable derivatives in [7] can gain their importance by the ability of using them to generate more generalized nonlocal fractional derivatives with singular kernels (see [4, 23, 27]). Fractional calculus is the study of derivatives and integrals of non-integer order and is the generalized form of classical derivatives and integrals. It is as dated as classical calculus, but it acquires more importance in recent two decades, this is due to its applications in various fields such as physics, biology, fluid dynamics, control theory, image processing, signal processing, and computer networking. See [5, 11–18, 25, 26, 31, 32, 34, 58, 63–66]. In recent years, the research has been proceeded to generalize the existing inequalities through innovative ideas and approaches of fractional calculus. One of the most popular approaches among researchers is the use of fractional integral operators. Due to their potentials to be expended for the existence of nontrivial and positive solutions of several classes of fractional differential equations, the integral inequalities involving fractional integrals are considerably important.

A large bulk of existing literature consists of generalizations of numerous inequalities via fractional integral operators and their applications [9, 37, 42, 59, 62]. Mubeen and Iqbal [38] contributed the ongoing research by presenting the improved version of generalized Grüss type integral inequalities for k-Riemann-Liouville fractional integrals. Agarwal et al. [8] obtained certain Hermite-Hadamard type inequalities for generalized k-fractional integrals. Set et al. [52] presented an integral identity and generalized Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral. Mubeen et al. [39] established integral inequalities of Ostrowski type for k-fractional Riemann-Liouville integrals. Sarikaya and Budak [50] utilized local fractional integrals to derive a generalized inequality. Khan et al. [35] produced some important generalized inequalities for a finite class of positive decreasing functions for fractional conformable integrals. Jleli et al. [28] determined a Hartman-Winter type inequality involving fractional derivative with respect to another function. In the papers [6, 24, 56, 57, 61] and closely related references therein, there are more information on this topic.

The main object of this paper is to develop a new notion "generalized k-fractional conformable integral" which is the generalized form of fractional operators reported in [27]. Hereafter, we also generalize some integral inequalities given in [35] for a finite class of positive and decreasing functions to ones involving our newly introduced k-fractional conformable integrals. For details of those inequalities, their applications, and their stability, we refer readers to [2, 3, 33, 36, 54, 55].

2. Notations

The notion of left and right fractional conformable derivatives for a differentiable function f, introduced by Abdeljawad [1], can be expressed as

$$\mathfrak{T}^{\alpha}_{a^+} f(t) = (t-a)^{1-\alpha} f'(t) \quad \text{and} \quad \mathfrak{T}^{\alpha}_{b^-} f(t) = (b-t)^{1-\alpha} f'(t).$$

Correspondingly, left and right fractional conformable integrals for $0 < \alpha < 1$ can be represented by

$$\mathcal{H}^{\alpha}_{a^+} f(t) = \int_a^t \frac{f(x)}{(x-a)^{1-\alpha}}\,\mathrm{d}\,x \quad \text{and} \quad \mathcal{H}^{\alpha}_{b^-} f(t) = \int_t^b \frac{f(x)}{(b-x)^{1-\alpha}}\,\mathrm{d}\,x.$$

Let $\Gamma(z)$ for $\Re(z) > 0$ denote the classical gamma function [43, 45]. The left and right fractional conformable integral (LFCI and RFCI) operators of order $\beta \in \mathbb{C}$ for $\Re(\beta) > 0$ can be defined [27] respectively by

$${}^{\beta}\mathcal{H}^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}\,\mathrm{d}\,t$$

and

$${}^{\beta}\mathcal{H}^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left[\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}}\,\mathrm{d}\,t.$$

Díaz and Pariguan [19] generalized the classical Pochhammer symbol $(\lambda)_n$, the classical gamma function $\Gamma(z)$, and the classical beta function $\mathrm{B}(u,v)$ respectively as

$$(\lambda)_{n,k} = \begin{cases} 1, & n = 0; \\ \lambda(\lambda+k)\cdots(\lambda+(n-1)k), & n \in \mathbb{N}, \end{cases} \qquad \Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{x/k-1}}{(x)_{n,k}},$$

and

$$B_k(u,v) = \frac{1}{k}\int_0^1 t^{u/k-1}(1-t)^{v/k-1}\,\mathrm{d}\,t.$$

See also [40, 41, 44, 46, 47]. It is not difficult to see that the k-gamma function $\Gamma_k(x)$ and the k-beta function $B_k(u,v)$ satisfy

$$\Gamma_k(x) = \int_0^\infty u^{x-1}e^{-u^k/k}\,\mathrm{d}\,u, \quad \Gamma(x) = \lim_{k\to 1}\Gamma_k(x), \quad \Gamma_k(x) = k^{x/k-1}\Gamma\Big(\frac{x}{k}\Big), \quad \Gamma_k(x+k) = x\Gamma_k(x),$$

and

$$B_k(u,v) = \frac{1}{k}B\Big(\frac{u}{k},\frac{v}{k}\Big), \quad B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}.$$

3. Generalized k-fractional conformable integrals

In this section, we introduce the generalized left and right k-fractional conformable integrals which generalize the Riemann-Liouville fractional integrals [49, p. 44], Hadamard fractional integrals [10], Katugampola fractional integrals [29], and generalized fractional integrals [51].

Definition 3.1. Let f be a continuous function on a finite real interval $[a,b]$. Then the generalized left and right k-fractional conformable integrals (k-FCI) of order $\beta \in \mathbb{C}$ for $\Re(\beta) > 0$ are respectively defined as

$${}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right]^{\beta/k-1}\frac{f(t)}{(t-a)^{1-\alpha}}\,\mathrm{d}\,t$$

and

$${}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_x^b \left[\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right]^{\beta/k-1}\frac{f(t)}{(b-t)^{1-\alpha}}\,\mathrm{d}\,t,$$

where $k > 0$ and $\alpha \in \mathbb{R}\backslash\{0\}$.

Some features of those concepts defined in Definition 3.1, such as the semi-group property, the derivative of functions, the Laplace transforms of functions using this derivative and the solution of IVP, can be found in [1].

Theorem 3.1. *Let $f \in L_1[a,b]$, $\alpha \in \mathbb{R}\backslash\{0\}$, and $k > 0$. Then both ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}f(x)$ and ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}f(x)$ exist for all $x \in [a,b]$ and $\Re(\beta) > 0$.*

Proof. Let $\Delta' = [a,b]\times[a,b]$ and $P' : \Delta' \to \mathbb{R}$ such that

$$P'(x,t) = [(x-a)^\alpha - (t-a)^\alpha]^{\beta/k-1}(t-a)^{\alpha-1}.$$

It is clear that $P' = P'_+ + P'_-$, where

$$P'_+(x,t) = \begin{cases} [(x-a)^\alpha - (t-a)^\alpha]^{\beta/k-1}(t-a)^{\alpha-1}, & a \le t \le x \le b; \\ 0, & a \le x \le t \le b \end{cases}$$

and

$$P'_-(x,t) = \begin{cases} [(t-a)^\alpha - (x-a)^\alpha]^{\beta/k-1}(x-a)^{\alpha-1}, & a \le t \le x \le b; \\ 0, & a \le x \le t \le b. \end{cases}$$

Since P' is measurable on Δ', we can write

$$\int_a^b P'(x,t)\,\mathrm{d}\,t = \int_a^x P'(x,t)\,\mathrm{d}\,t = \int_a^x [(x-a)^\alpha - (t-a)^\alpha]^{\beta/k-1}(t-a)^{\alpha-1}\,\mathrm{d}\,t = \frac{\alpha k}{\beta}(x-a)^{\alpha\beta/k}.$$

Therefore, we obtain

$$\int_a^b \Big[\int_a^b P'(x,t)|f(x)|\,\mathrm{d}\,t\Big]\mathrm{d}\,x = \int_a^b |f(x)|\Big[\int_a^b P'(x,t)\,\mathrm{d}\,t\Big]\mathrm{d}\,x = \frac{\alpha k}{\beta}\int_a^b (x-a)^{\alpha\beta/k}|f(x)|\,\mathrm{d}\,x$$
$$\le \frac{\alpha k}{\beta}(b-a)^{\alpha\beta/k}\int_a^b |f(x)|\,\mathrm{d}\,x = \frac{\alpha k}{\beta}(b-a)^{\alpha\beta/k}\|f(x)\|_{L_1[a,b]} < \infty.$$

So, by Tonelli's theorem for iterated integrals [21, p. 147], the function $Q' : \Delta' \to \mathbb{R}$ such that $Q'(x,t) = P'(x,t)f(x)$ is integrable over Δ'. Hence, by Fubini's theorem, it follows that $\int_a^b P'(x,t)f(x)\,\mathrm{d}\,x$ is integrable over $[a,b]$ as a function of $t \in [a,b]$. This implies that ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}f(x)$ exists.

The existence of the right k-fractional conformable integral ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}f(x)$ can be proved in a similar manner. The proof of Theorem 3.1 is complete. □

4. Inequalities for generalized k-fractional conformable integrals

Fractional integral inequalities have been analyzed for many useful purposes. One of the most useful applications of such inequalities is the existence of nontrivial solutions of fractional differential equations. Many applications find in the literature for the existence of nontrivial solution eigenvalue problems by inequalities, see [42, 62]. Generalizing pre-existing inequalities by applying fractional integral operators is becoming a popular trend in the research field, see, for example, [22, 46, 48].

In this section, we present some k-analogues of inequalities in [53, 59, 60] for generalized k-fractional conformable integrals.

Theorem 4.1. *Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \le i \le n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a < x \le b$, $\eta > 0$, $\xi \ge \gamma_p > 0$ for $1 \le p \le n$. Then the left k-FCI operator ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}$ satisfies the inequality*

$$\frac{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x))}{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}(\prod_{i=1}^{n} g_i^{\gamma_i}(x))} \ge \frac{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x))}{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x))}. \tag{4.1}$$

Proof. Under given conditions, we have

$$[h^{\eta}(\rho) - h^{\eta}(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \ge 0.$$

Let us define a function

$$ {}_{k}^{\beta}\mathcal{J}_{a^+}^{\alpha}(x,\rho,\tau) = \frac{1}{k\Gamma_k(\beta)}\Big[\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\Big]^{\beta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)}{(\tau-a)^{1-\alpha}}[h^{\eta}(\rho)-h^{\eta}(\tau)][g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)]. \quad (4.2) $$

Accordingly, the function ${}_{k}^{\beta}\mathcal{J}_{a^+}^{\alpha}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating on both sides of the above equation (4.2) with respect to τ from a to x shows

$$ \begin{aligned} 0 \le \int_a^x {}_{k}^{\beta}\mathfrak{J}_{a^+}^{\alpha}(x,\rho,\tau)\,\mathrm{d}\tau &= \frac{1}{k\Gamma_k(\beta)}\int_a^x \Big[\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\Big]^{\beta/k-1} \\ &\times \prod_{i=1}^{n} g_i^{\gamma_i}(\tau)[h^{\eta}(\rho)-h^{\eta}(\tau)][g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)]\frac{\mathrm{d}\tau}{(\tau-a)^{1-\alpha}} \\ &= h^{\eta}(\rho)\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(\prod_{i\ne p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\Big)\Big] + g_p^{\xi-\gamma_p}(\rho)\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Big)\Big] \\ &-h^{\eta}(\rho)g_p^{\xi-\gamma_p}(\rho)\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Big)\Big] - \Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(h^{\eta}(x)\prod_{i\ne p}^{n} g_i^{\gamma_i}(x)\Big)\Big]. \end{aligned} \quad (4.3) $$

Multiplying on both sides of the relation (4.3) by

$$ \frac{1}{k\Gamma_k(\beta)}\Big[\frac{(x-a)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\Big]^{\beta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{(\rho-a)^{1-\alpha}}. \quad (4.4) $$

and integrating on both sides with respect to ρ from a to x give

$$ \begin{aligned} 0 \le &\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(\prod_{i\ne p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\Big)\Big]\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Big)\Big] \\ -&\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(h^{\eta}(x)\prod_{i\ne p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\Big)\Big]\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Big)\Big]. \end{aligned} \quad (4.5) $$

Dividing on both sides of (4.5) by

$$ {}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Big)\Big[{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}\Big(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Big)\Big] $$

leads to (4.1). The proof of Theorem 4.1 is complete. □

Corollary 4.1. *Let* $\{g_i, 1 \le i \le n\}$ *be a sequence of continuous positive decreasing functions on the interval* $[a,b]$. *Let* $a < x \le b$, $\eta > 0$, $\xi \ge \gamma_p > 0$ *for* $1 \le p \le n$. *Then the left k-FCI operator* ${}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}$ *satisfies the inequality*

$$ \frac{{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}(\prod_{i\ne p}^{n} g_i^{\gamma_i}g_p^{\xi}(x))}{{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}(\prod_{i=1}^{n} g_i^{\gamma_i}(x))} \ge \frac{{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}((x-a)^{\eta}\prod_{i\ne p}^{n} g_i^{\gamma_i}g_p^{\xi}(x))}{{}_{k}^{\beta}\mathcal{H}_{a^+}^{\alpha}((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x))}. \quad (4.6) $$

Proof. This can be derived from taking $h(x) = x - a$ in Theorem 4.1. The proof of Corollary 4.1 is complete. □

Corollary 4.2. *Let $\{g_i, 1 \le i \le n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a < x \le b$, $\eta > 0$, $\xi \ge \gamma_p > 0$ for $1 \le p \le n$. Then the left k-FCI operator ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}$ satisfies the inequality*

$$\begin{aligned}
&\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&\ge \left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{4.7}$$

Proof. Multiplying on both sides of the relation

$$\begin{aligned}
0 \le \int_a^x {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a^+}(x,\rho,\tau)\,\mathrm{d}\tau &= \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^{\alpha}-(\tau-a)^{\alpha}}{\alpha}\right]^{\beta/k-1} \\
&\times \prod_{i=1}^{n} g_i^{\gamma_i}(\tau)[(\rho-a)^{\eta}-(\tau-a)^{\eta}][g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)]\frac{\mathrm{d}\tau}{(\tau-a)^{1-\alpha}} \\
&= (\rho-a)^{\eta}\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + g_p^{\xi-\gamma_p}(\rho)\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&-(\rho-a)^{\eta} g_p^{\xi-\gamma_p}(\rho)\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] - \left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)\right)\right]
\end{aligned}$$

by

$$\frac{1}{k\Gamma_k(\theta)}\left[\frac{(x-a)^{\alpha}-(\rho-a)^{\alpha}}{\alpha}\right]^{\theta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{(\rho-a)^{1-\alpha}} \tag{4.8}$$

and integrating on both sides with respect to ρ from a to x arrive at

$$\begin{aligned}
0 \le &\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&-\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&-\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{4.9}$$

Dividing on both sides of (4.9) by

$$\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + \left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

leads to (4.7). The proof of Corollary 4.2 is complete. □

Corollary 4.3. *Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the left k-FCI operator ${}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}$ satisfies the inequality*

$$\begin{aligned}
&\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\\
&\geq\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right].
\end{aligned}\tag{4.10}$$

Proof. Multiplying on both sides of the relation (4.3) by (4.8) and integrating on both sides with respect to ρ from a to x derive

$$\begin{aligned}
0\leq&\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\\
&-\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&-\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right].
\end{aligned}\tag{4.11}$$

Dividing on both sides of (4.11) by

$$\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]+\left[{}_k^{\theta}\mathcal{H}_{a^+}^{\alpha}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^{\beta}\mathcal{H}_{a^+}^{\alpha}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]$$

reveals (4.10). The proof of Corollary 4.3 is complete. □

Theorem 4.2. *Let $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the right k-FCI operator ${}_k^{\beta}\mathcal{H}_{b^-}^{\alpha}$ satisfies the inequality*

$$\frac{{}_k^{\beta}\mathcal{H}_{b^-}^{\alpha}(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x))}{{}_k^{\beta}\mathcal{H}_{b^-}^{\alpha}(\prod_{i=1}^{n}g_i^{\gamma_i}(x))}\geq\frac{{}_k^{\beta}\mathcal{H}_{b^-}^{\alpha}((b-x)^{\eta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x))}{{}_k^{\beta}\mathcal{H}_{b^-}^{\alpha}((b-x)^{\eta}\prod_{i=1}^{n}g_i^{\gamma_i}(x))}.\tag{4.12}$$

Proof. Under given conditions, we have

$$[(b-\rho)^\eta-(b-\tau)^\eta][g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)]\geq 0.$$

Let us define a function

$$\begin{aligned}&{}_k^\beta\mathfrak{I}_{b^-}^\alpha(x,\rho,\tau)\\&=\frac{1}{k\Gamma_k(\beta)}\Big[\frac{(b-x)^\alpha-(b-\tau)^\alpha}{\alpha}\Big]^{\beta/k-1}\frac{\prod_{i=1}^n g_i^{\gamma_i}(\tau)}{(b-\tau)^{1-\alpha}}[(b-\rho)^\eta-(b-\tau)^\eta][g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)].\end{aligned}\tag{4.13}$$

Consequently, the function ${}_k^\beta\mathfrak{I}_{b^-}^\alpha(x,\rho,\tau)$ is positive for all $\tau\in(a,b]$. Integrating on both sides of the above equation (4.13) with respect to τ from x to b gives

$$\begin{aligned}0&\leq\int_x^b{}_k^\beta\mathfrak{I}_{b^-}^\alpha(x,\rho,\tau)\,\mathrm{d}\tau=\frac{1}{k\Gamma_k(\beta)}\int_x^b\Big[\frac{(b-x)^\alpha-(b-\tau)^\alpha}{\alpha}\Big]^{\beta/k-1}\\&\times\prod_{i=1}^n g_i^{\gamma_i}(\tau)[(b-\rho)^\eta-(b-\tau)^\eta][g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)]\frac{\mathrm{d}\tau}{(b-\tau)^{1-\alpha}}\\&=(b-\rho)^\eta\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^\xi(x)\Big)\Big]+g_p^{\xi-\gamma_p}(\rho)\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\Big)\Big]\\&-(b-\rho)^\eta g_p^{\xi-\gamma_p}(\rho)\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big(\prod_{i=1}^n g_i^{\gamma_i}(x)\Big)\Big]-\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big((b-x)^\eta\prod_{i\neq p}^n g_i^{\gamma_i}(x)\Big)\Big].\end{aligned}\tag{4.14}$$

Multiplying the relation (4.14) by

$$\frac{1}{k\Gamma_k(\beta)}\Big[\frac{(b-x)^\alpha-(b-\rho)^\alpha}{\alpha}\Big]^{\beta/k-1}\frac{\prod_{i=1}^n g_i^{\gamma_i}(\rho)}{(b-\rho)^{1-\alpha}}\tag{4.15}$$

and integrating on both sides with respect to ρ from x to b produce

$$\begin{aligned}0&\leq\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^\xi(x)\Big)\Big]\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\Big)\Big]\\&-\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big((b-x)^\eta\prod_{i\neq p}^n g_i^{\gamma_i}g_p^\xi(x)\Big)\Big]\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big(\prod_{i=1}^n g_i^{\gamma_i}(x)\Big)\Big].\end{aligned}\tag{4.16}$$

Dividing on both sides of (4.16) by

$${}_k^\beta\mathcal{H}_{b^-}^\alpha\Big((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\Big)\Big[{}_k^\beta\mathcal{H}_{b^-}^\alpha\Big(\prod_{i=1}^n g_i^{\gamma_i}(x)\Big)\Big]$$

yields (4.12). The proof of Theorem 4.2 is complete. □

Corollary 4.4. *Let $\{g_i, 1\leq i\leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a<x\leq b$, $\eta>0$, $\xi\geq\gamma_p>0$ for $1\leq p\leq n$. Then the right k-FCI operator* ${}_k^\beta\mathcal{H}_{b^-}^\alpha$

satisfies the inequality

$$\begin{aligned}
&\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\\
&\geq \left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{4.17}$$

Proof. Multiplying the relation (4.14) by

$$\frac{1}{k\Gamma_k(\theta)}\left[\frac{(b-x)^{\alpha}-(b-\rho)^{\alpha}}{\alpha}\right]^{\theta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{(b-\rho)^{1-\alpha}} \tag{4.18}$$

and integrating on both sides with respect to ρ from x to b procure

$$\begin{aligned}
0 \leq &\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\\
&-\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((x-a)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\
&-\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{4.19}$$

Dividing on both sides of (4.19) by

$$\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

demonstrates (4.17). The proof of Corollary 4.4 is complete. □

Theorem 4.3. *Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the right k-FCI operator ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}$ satisfies the inequality*

$$\frac{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x))}{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}(\prod_{i=1}^{n} g_i^{\gamma_i}(x))} \geq \frac{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x))}{{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x))}. \tag{4.20}$$

Proof. Under given conditions, we have

$$[h^{\eta}(\rho) - h^{\eta}(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \geq 0.$$

Let us define a function

$$ {}^{\beta}_{k}\mathcal{J}^{\alpha}_{b^-}(x,\rho,\tau) = \frac{1}{k\Gamma_k(\beta)}\left[\frac{(b-x)^{\alpha}-(b-\tau)^{\alpha}}{\alpha}\right]^{\beta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)}{(b-\tau)^{1-\alpha}}[h^{\eta}(\rho) - h^{\eta}(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)]. \quad (4.21)$$

Thus, the function ${}^{\beta}_{k}\mathcal{J}^{\alpha}_{b^-}(x,\rho,\tau)$ is positive for all $\tau \in (a,b]$. Integrating on both sides of the above equation (4.21) with respect to τ from x to b results in

$$\begin{aligned} 0 \leq \int_x^b {}^{\beta}_{k}\mathfrak{J}^{\alpha}_{b^-}(x,\rho,\tau)\,\mathrm{d}\tau &= \frac{1}{k\Gamma_k(\beta)}\int_x^b\left[\frac{(b-x)^{\alpha}-(b-\tau)^{\alpha}}{\alpha}\right]^{\beta/k-1} \\ &\times\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)[h^{\eta}(\rho) - h^{\eta}(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)]\frac{\mathrm{d}\tau}{(b-\tau)^{1-\alpha}} \\ &= h^{\eta}(\rho)\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + g_p^{\xi-\gamma_p}(\rho)\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\ &- h^{\eta}(\rho)g_p^{\xi-\gamma_p}(\rho)\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] - \left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)\right)\right]. \end{aligned} \quad (4.22)$$

Multiplying the relation (4.22) by (4.15) and integrating on both sides with respect to ρ from x to b yield

$$\begin{aligned} 0 \leq &\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\ &-\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]. \end{aligned} \quad (4.23)$$

Dividing on both sides of (4.23) by

$${}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

leads to (4.20). The proof of Theorem 4.3 is complete. □

Corollary 4.5. *Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a,b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for*

$1 \le p \le n$. Then the right k-FCI operator ${}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}$ satisfies the inequality

$$\begin{aligned}
&\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&\ge \left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{4.24}$$

Proof. Multiplying the relation (4.22) by (4.18) and integrating on both sides with respect to ρ from x to b give

$$\begin{aligned}
0 \le &\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&-\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&-\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{4.25}$$

Dividing on both sides of (4.25) by

$$\left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + \left[{}^{\theta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{\beta}_{k}\mathcal{H}^{\alpha}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

concludes (4.24). The proof of Corollary 4.5 is complete. □

5. Conclusions

In this paper, we have presented the left and right k-fractional conformable integrals and generalized some important integral inequalities to ones for our newly introduced k-FCI operators related to a finite sequence of positive and decreasing functions. Our work produces k-analogues of many pre-existing results in the literature. Further, many special cases for other integral operators can be derived from our generalizations. The results obtained can be employed to confirm the existence of nontrivial solutions of fractional differential equations of different classes. The k-FCI operators in this paper are different from those introduced by Katugampola [30] as their kernels depend on the boundary points a and b and need a different convolution theory under conformable Laplace. Our k-fractional conformable integrals in this paper generalize well-known fractional integral operators such as Caputo integral operators [49,

p. 44], Riemann-Liouville integral operators [49, p. 44], Hadamard integral operators [10], and their k-analogues.

Finally we state that possible future works can be in proving new inequalities in the frame of new generalized integrals. The integrals correspond to certain fractional derivatives with nonsingular kernels, for example. See the papers [4, 23].

Remark 5.1. This paper is a slightly revised version of the preprint [20].

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Conflict of interest

The authors declare no conflict of interest.

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