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Research article

Existence of multiple non-trivial solutions for a nonlocal problem

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Abstract: In this paper, we are concerned with the following general nonlocal problem

$$\begin{cases} -\mathcal{L}_K u = \lambda_1 u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where λ_1 denotes the first eigenvalue of the nonlocal integro-differential operator $-\mathcal{L}_K$, $\Omega \subset \mathbb{R}^N$ is open, bounded domain with smooth boundary. Under several structural assumptions on f, we verify that the problem possesses at least two non-trivial solutions and locate the region in different parts of the Hilbert space by variational method. As a particular case, we derive an existence theorem for the following equation driven by the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Keywords: integro-differential operators; multiple solutions; variational method
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1. Introduction

In this paper, we study the following general nonlocal equation

$$\begin{cases} -\mathcal{L}_K u = \lambda_1 u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is open, bounded domain with smooth boundary, λ_1 is the first eigenvalue of the nonlocal integro-differential operator $\mathcal{L}_K$, which is defined by (see [1])

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} [u(x+y) + u(x-y) - 2u(x)]K(y)dy, \ \ x \in \mathbb{R}^N, \tag{1.2}$$

where $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ satisfies the following conditions:

(K_1) $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{|x|^2, 1\}$
(K_2) there exists $\delta > 0$ such that $K(x) \geq \delta |x|^{-(N+2s)}$, $\ \forall x \in \mathbb{R}^N$.

We remark that the Dirichlet datum is given in $\mathbb{R}^N \backslash \Omega$ and not simply on $\partial\Omega$, consistently with the non-local character of the operator $\mathcal{L}_K$.

A typical example for the function $K(x)$ is $K(x) = |x|^{-(N+2s)}$, then the operator $\mathcal{L}_K$ reduce to the so-called fractional Laplacian operator $(-\Delta)^s$ and (1.1) reduce to

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

which plays an increasingly significant role in both pure mathematical research and concrete applications, such as the thin obstacle problem [2, 3], minimal surfaces [4, 5], phase transitions [6], anomalous diffusion [7–9] and mathematical finance [10]. See [11–15] and references therein for an elementary introduction to the literature.

Different than the classical Laplacian operator $-\Delta$, the fractional Laplacian operator $(-\Delta)^s$ is known to be nonlocal and this difference may cause some difficulties to implement by variational methods. Over the past decades, problems similar to (1.1) have roused enough interest, many scholars have shown their concern in elliptic equation for bounded domains and unbounded domains, see [16–22] and the references therein.

On the other hand, there are few results concerned with the general nonlocal problem with the operator $\mathcal{L}_K$. In [1, 23], the authors study problem (1.1) by both the mountain pass theorem and the linking theorem. In [24], the authors obtained a Brezis-Nirenberg result for the operator $\mathcal{L}_K$. Infinitely many positive solutions and sign-changing solutions have been studied in [25, 26]. For other results about the operator $\mathcal{L}_K$ we refer to [12, 27] and their references therein.

Problem (1.1) has a variational character and the natural space where finding weak solutions for it is the functional space X, defined as follows (for more details we refer to [1] and [23], where this space was introduced and some properties of this space were proved).

Let

$$X := \{u : \mathbb{R}^N \to \mathbb{R} |\ u \text{ is Lebesgue measurable, } u|_\Omega \in L^2(\Omega) \text{ and } \int_Q (u(x) - u(y))^2 K(x-y)dxdy < \infty\},$$

where $Q = \mathbb{R}^{2N} \backslash (C\Omega \times C\Omega)$, $C\Omega := \mathbb{R}^N \backslash \Omega$. The space X is endowed with the following norm

$$\|u\|_X = |u|_2 + [u]_X,$$

and

$$[u]_X = (\int_Q |u(x) - u(y)|^2 K(x-y)dxdy)^{\frac{1}{2}}.$$

We can easily check that $\|\cdot\|_X$ is a norm on X. Define

$$E := \{u \in X : \ u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

From [1, 23], we have the following Poincaré type inequality: there exists a constant $C > 0$ such that for all $u \in E$,

$$|u|_2 \leq C[u]_X.$$

Moreover, the norm

$$\|u\| := [u]_X = (\int_Q |u(x) - u(y)|^2 K(x-y)dxdy)^{1/2} = (\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy)^{1/2}$$

is an equivalent norm on E and $(E, \|\cdot\|)$ is a Hilbert space (see [1, Lemma 7]) with scalar product

$$(u, v) = \int_{\mathbb{R}^{2N}} [u(x) - u(y)][v(x) - v(y)]K(x-y)dxdy.$$

Note that $C_c^\infty(\Omega)$ is dense in E and the norm $\|\cdot\|$ involves the interaction between Ω and $\mathbb{R}^N \setminus \Omega$.

The weak formulation of problem (1.1) and (1.3) is given by

$$\int_{\mathbb{R}^{2N}} [u(x) - u(y)][v(x) - v(y)]K(x-y)dxdy = \lambda_1 \int_\Omega u(x)v(x)dx + \int_\Omega f(x, u)vdx, \quad \forall\, v \in X,$$

and

$$\int_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x-y|^{N+2s}}dxdy = \lambda_1 \int_\Omega u(x)v(x)dx + \int_\Omega f(x, u)vdx, \quad \forall\, v \in X,$$

which represents the Euler-Lagrange equation of the functional $I, I_0 : X \to \mathbb{R}$ defined as

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{\lambda_1}{2} \int_\Omega |u(x)|^2 dx - \int_\Omega F(x, u)dx,$$

and

$$I_0(u) := \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}dxdy - \frac{\lambda_1}{2} \int_\Omega |u(x)|^2 dx - \int_\Omega F(x, u)dx.$$

To give our result, first we recall the eigenvalues of the following problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.4}$$

Proposition 1.1. *[12, 23] Let $s \in (0,1)$, $N > 2s$, Ω be an open, bounded subset of $\mathbb{R}^N$, and let $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ be a function satisfying assumptions (K_1) and (K_2). Then*

(a) *problem* (1.4) *admits a eigenvalue λ_1 that is a positive and that can be characterized as follows:*

$$\lambda_1 = \min_{u \in E \backslash \{0\}} \frac{\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy}{\int_\Omega |u(x)|^2 dx};$$

(b) *there exists a nonnegative function* $\varphi_1 \in E$ *that is an eigenfunction corresponding to* λ_1*, that is,* $|\varphi_1|_2 = 1$ *and*

$$\lambda_1 = \int_{\mathbb{R}^{2N}} |\varphi_1(x) - \varphi_1(y)|^2 K(x-y)dxdy;$$

(c) λ_1 *is simple, that is, if* $u \in E$ *is a solution of* (1.4) *with* $\lambda = \lambda_1$*, then* $u = \zeta\varphi_1$ *for* $\zeta \in \mathbb{R}$*;*

(d) *the set of the eigenvalues of problem* (1.4) *consists of a sequence* $\{\lambda_k\}_{k\in\mathbb{N}}$ *with*

$$0 < \lambda_1 < \lambda_2 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty.$$

Remark 1.1. *[12] For the fractional Laplacian* $(-\Delta)^s$*, the first eigenfunction* φ_1 *is strictly positive in* Ω*.*

Motivated by the results mentioned above, the main aim of this paper is to establish the existence of multiple non-trivial solutions for (1.1) and (1.3) . To the best of authors' knowledge, the existence of multiple solutions for the problem (1.1) and (1.3) has not been well studied. The proof of our results borrow the ideas from [28] in which existence results of positive solutions are obtained for a class of Dirichlet problem. We assume that $f(x,t)$ is a Carathéodory function on $\Omega\times\mathbb{R}$ and satisfies

(f_1) $|f(x,t)| \leq \gamma|t|,\ \ x \in \Omega,\ t \in \mathbb{R}$, where $\gamma < (\lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}})\lambda_1^{\frac{1}{2}}$ and λ_1, λ_2 are the first two eigenvalues of problem (1.4).

(f_2) $\tau_{\pm}(x) := \limsup\limits_{t\to\pm\infty} \frac{2F(x,t)}{t^2} \leq 0 (\not\equiv 0)$, $x \in \Omega$, where $F(x,u) = \int_0^u f(x,s)ds$.

(f_3) there are constants $r_1 > 0$ and $r_2 < 0$ such that $\int_\Omega F(x, r_j\varphi_1)dx > 0$ $(j = 1,2)$, where φ_1 is the eigenfunction of problem (1.4) corresponding to λ_1.

Our main result can be stated as follow.

Theorem 1.1. *Let* $s \in (0,1)$*,* $N > 2s$ *and* Ω *be an open bounded set of* $\mathbb{R}^N$ *with Lipschitz boundary, assume that* $K(x)$ *satisfies* $(K_1) - (K_2)$ *and the assumption* (f_1)*-*(f_3) *hold with*

$$\int_\Omega (\tau_+(x)(\varphi_1^+)^2 + \tau_-(x)(\varphi_1^-)^2)dx \leq 0, \tag{1.5}$$

or

$$\int_\Omega (\tau_+(x)(\varphi_1^-)^2 + \tau_-(x)(\varphi_1^+)^2)dx \leq 0, \tag{1.6}$$

then the problem (1.1) *possesses at least two non-trivial solution* u_1 *and* u_2*, one satisfying* $(u_1, \varphi_1) > 0$ *and the other satisfying* $(u_2, \varphi_1) < 0$*.*

In the non-local framework, the simplest example we can deal with is given by the fractional Laplacian $(-\Delta)^s$, according to the following result:

Theorem 1.2. *Let* $s \in (0,1)$*,* $N > 2s$ *and* Ω *be an open bounded set of* $\mathbb{R}^N$ *with Lipschitz boundary, assume that the assumption* (f_1)*-*(f_3) *hold, then the problem* (1.5) *possesses at least two non-trivial solution* u_1 *and* u_2*, one satisfying* $(u_1, \varphi_1) > 0$ *and the other satisfying* $(u_2, \varphi_1) < 0$*.*

The paper is organized as follows. In the forthcoming Section 2, we collect some necessary preliminary observations and devote ourselves to the proof of Theorems. Final, we give some conclusions in Section 3. Through the paper, we make use of following notations: $C, C_0, C_1, \cdots$ for positive constants (possibly different from line to line).

2. Preliminary results and Proof of Theorems

2.1. Preliminary results

Our results are based upon the following critical point theorems in Hilbert space.

Proposition 2.1. *([28]) Let $G \in C^1(E)$ for some Hilbert space E and is bounded on bounded sets. Assume that there exist constants $\theta < 1$, $d \in \mathbb{R}$, $u_0 \in E$ and a unit vector $\varphi_0 \in E$ such that*

$$(u_0, \varphi_0) \leq d \tag{2.1}$$

and

$$(G'(u), \varphi_0) \leq \theta \|G'(u)\| \ \text{ where } (u, \varphi_0) = d \text{ and } G(u) \geq G(u_0). \tag{2.2}$$

Assume also that there is a $\beta \geq 1$ satisfying

$$\limsup_{R\to\infty} R^{-\beta} \sup\{G(u) : \|u\| = R, (u, \varphi_0) \leq d\} \leq 0. \tag{2.3}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$(u_k, \varphi_0) \leq d, \ \ G(u_k) \to c, \ \ G(u_0) \leq c \leq \infty, \ \ \|G'(u_k)\| = o(\|u_k\|^{\beta-1} + 1). \tag{2.4}$$

In addition, if (2.1) − (2.3) are replaced by

$$(u_0, \varphi_0) \geq d, \tag{2.5}$$

$$(G'(u), \varphi_0) + \theta \|G'(u)\| \geq 0 \ \text{ where } (u, \varphi_0) = d \text{ and } G(u) \geq G(u_0) \tag{2.6}$$

and

$$\limsup_{R\to\infty} R^{-\beta} \sup\{G(u) : \|u\| = R, (u, \varphi_0) \geq d\} \leq 0, \tag{2.7}$$

respectively. Then there is a sequence $\{u_k\} \subset E$ satisfying

$$(u_k, \varphi_0) \geq d, \ \ G(u_k) \to c, \ \ G(u_0) \leq c \leq \infty, \ \ \|G'(u_k)\| = o(\|u_k\|^{\beta-1} + 1). \tag{2.8}$$

Remark 2.1. *Note that the condition (2.2) allows us to restrict our attention to the region $(u, \varphi_1) \leq d$, and (2.3) can help us dispense with the requirement that we find two subsets A, B of E such that A links B and $\sup_A G \leq \inf_B G$. Conditions (2.6) and (2.7) are similar. Moreover, this results can allow us to consider problems in which the maximum principle and sub-super solutions do not apply.*

Lemma 2.1. *Suppose (f_1) and in addition that*

$$\int_\Omega (\tau_+(x)(u^+)^2 + \tau_-(x)(u^-)^2)dx \leq 0, \ \ (u, \varphi_1) \geq 0 \tag{2.9}$$

and

$$\int_\Omega (\tau_+(x)(\varphi_1^+)^2 + \tau_-(x)(\varphi_1^-)^2)dx < 0, \tag{2.10}$$

where $\tau_\pm$ is defined as (f_2). Assume also that

$$\int_\Omega F(x, r\varphi_1)dx > 0 \tag{2.11}$$

for some $r > 0$. Then there is at least one non-trivial solution of problem (1.1) *satisfying* $(u, \varphi_1) > 0$.

In addition, if we replace (2.9), (2.10) *and* (2.11) *by*

$$\int_\Omega (\tau_+(x)(u^+)^2 + \tau_-(x)(u^-)^2)dx \le 0, \ \ (u, \varphi_1) \le 0, \tag{2.12}$$

$$\int_\Omega (\tau_+(x)(\varphi_1^-)^2 + \tau_-(x)(\varphi_1^+)^2)dx < 0 \tag{2.13}$$

and

$$\int_\Omega F(x, -r\varphi_1)dx > 0, \tag{2.14}$$

then there is another solution satisfying $(u, \varphi_1) < 0$.

Proof. We only prove the first conclusion, the other is similar. Let

$$J(u) = -\|u\|^2 + \lambda_1|u|_2^2 + 2\int_\Omega F(x, u)dx,$$

and take $u_0 = r\varphi_1, d = 0, \beta = 2$ in Proposition 2.1. We first verify (2.7), suppose on the contrary that there is a sequence $\{u_k\} \subset E$ such that $(u_k, \varphi_1) \ge 0, \rho_k = \|u_k\| \to \infty$ and

$$\frac{J(u_k)}{\rho_k^2} \ge c > 0, \ k = 1, 2, \cdots . \tag{2.15}$$

Let $\tilde{u}_k = \frac{u_k}{\rho_k}$. Then we have that $(\tilde{u}_k, \varphi_1) \ge 0, \ \|\tilde{u}_k\| = 1$, and there exists a subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in E and a.e. in Ω. Therefore

$$\limsup_{k\to\infty} \frac{J(u_k)}{\rho_k^2} \le 2\int_\Omega \limsup_{k\to\infty} \frac{F(x, u_k)}{u_k^2}\tilde{u}_k^2 dx \le \int_\Omega (\tau_+(\tilde{u}^+)^2 + \tau_-(\tilde{u}^-)^2)dx \le 0,$$

which is a contradiction with (2.15). Next we claim that (2.6) holds. Note that from (f_1), for any $u \perp \varphi_1$ we have

$$|(J'(u)), \varphi_1| = 2|(f(x, u), \varphi_1)| \le 2|f(x, u)|_2|\varphi_1|_2 \le 2\gamma\frac{\|u\|}{(\lambda_1\lambda_2)^{\frac{1}{2}}},$$

where we take $\lambda_1|\varphi_1|_2^2 = \|\varphi_1\|^2 = 1$ and $\lambda_2|u|_2^2 \le \|u\|^2$ for $u \perp \varphi_1$. Therefore, for any $u \perp \varphi_1$ we have

$$|(J'(u), \varphi_1)| \le \theta\|J'(u)\|,$$

where $\theta = \frac{\gamma}{(\lambda_1\lambda_2)^{\frac{1}{2}}}\frac{1}{1-\frac{\lambda_1+\gamma}{\lambda_2}} < 1$. So (2.6) holds and it follows from Proposition 2.1 that there exists a sequence satisfying (2.8). Assume $\rho_k = \|u_k\| \to \infty$ and let $\tilde{u}_k = \frac{u_k}{\rho_k}$. Then we have a subsequence converging weakly to a function $\tilde{u}$ in E and a.e. in Ω. Since

$$\frac{J(u_k)}{\rho_k^2} = 2\int_\Omega \frac{F(x, u_k)}{\rho_k^2}dx + \lambda_1|\tilde{u}_k|_2^2 - 1,$$

we can know that

$$0 \leq \int_\Omega (\tau_+(\tilde{u}^+)^2 + \tau_-(\tilde{u}^-)^2)dx + \lambda_1|\tilde{u}|_2^2 - 1 \leq \lambda_1|\tilde{u}|_2^2 - \|u\|^2 \leq 0.$$

Therefore,

$$\int_\Omega (\tau_+(\tilde{u}^+)^2 + \tau_-(\tilde{u}^-)^2)dx = 0 \tag{2.16}$$

and

$$\|\tilde{u}\|^2 = \lambda_1|\tilde{u}|_2^2 = 1. \tag{2.17}$$

Because λ_1 is a simple eigenvalue of $-L_k$ and $(\tilde{u}, \varphi_1) \geq 0$, it follows from (2.17) that $\tilde{u} = \varphi_1$. So (2.16) reduces to

$$\int_\Omega (\tau_+(\varphi_1^+)^2 + \tau_-(\varphi_1^-)^2)dx = 0,$$

which is a contradiction with (2.10). In consequence, ρ_k are bounded and standard methods show that there is a subsequence of $\{u_k\}$ converging in E to a solution satisfying

$$(u, \varphi_1) \geq 0, \quad J(u) = c, \quad J(u_0) \leq c, \quad J'(u) = 0.$$

Finally, we verify that $(u, \varphi_1) > 0$. If $(u, \varphi_1) = 0$, then

$$(\lambda_2 - \lambda_1)|u|_2^2 \leq \|u\|^2 - \lambda_1|u|_2^2 = (f(x, u), u) \leq \gamma|u|_2^2.$$

But $\gamma < \lambda_2 - \lambda_1$, which is a contradiction. We only replace φ_1 by $-\varphi_1$ in the above proof, the second conclusion is proved. Therefore, the proof of Lemma 2.1 is complete. □

2.2. *Proof of Theorem 1.1*

It follows from (f_2) that

$$\tau_\pm(x) \leq 0, \quad x \in \Omega,$$

then we have (2.9) and (2.12) hold. Therefore, the hypotheses of Lemma 2.1 are satisfied. Then, it follows from Lemma 2.1 that problem (1.1) has at least two non-trivial solution, one satisfying $(u, \varphi_1) > 0$ and the other one satisfying $(u, \varphi_1) < 0$. Therefore, the proof of Theorem 1.1 is complete.

2.3. *Proof of Theorem 1.2*

Note that $\varphi_1 > 0$ a.e. in Ω, then we can replace (2.10) with

$$\tau_+(x) \leq 0, \quad \tau_+(x) \not\equiv 0,$$

and (2.13) with

$$\tau_-(x) \leq 0, \quad \tau_-(x) \not\equiv 0.$$

Then Theorem 1.2 follows from Theorem 1.1.

3. Conclusion

In this paper, we generalize critical point theory to the nonlocal problem in which general methods such as maximum principle and sub-super solutions can not be applied. We can distinguish between different solutions via showing that they are located in different parts of the Hilbert space. Several recent results of the literatures are extended and improved.

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Conflict of interest

The authors declare no conflict of interest.

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