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Research article

Differential geometry of collective models

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Abstract: The classical astrophysical theory of Riemann ellipsoids and the quantum nuclear theory of Bohr and Mottelson share a common mathematical foundation in terms of the differential geometry of a principal bundle $\mathcal{P}$ and its associated vector bundle E, respectively. The bundle $\mathcal{P} = GL_{+}(3, \mathbb{R})$ is the connected component of the general linear group, the structure group G=SO(3) is the vorticity group, and the base manifold is the space of positive-definite real 3×3 symmetric matrices, identified geometrically with the space of inertia ellipsoids. The bundle is a Riemannian manifold whose metric is inherited from three-dimensional Euclidean space. A nonholonomic constraint force, like irrotational flow, determines a connection on the bundle.
Wave functions of the Bohr-Mottelson model are sections of the associated vector bundle E = $\mathcal{P} \times_{\rho}$V, where ρ denotes an irreducible representation of the vorticity group on the vector space V. Using the de Rham Laplacian $\triangle = \star d_{\nabla} \star d_{\nabla}$ for the kinetic energy introduces a "magnetic" term due to the connection between base manifold rotational and fiber vortex degrees of freedom. A class of Ehresmann connections creates a new model of nuclear rotation that predicts moments of inertia in agreement with experiment.

Keywords: bundle; connexion; vorticity; de Rham Laplacian; astrophysics; nuclear structure
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1. Introduction

Two well-established physical theories of collective motion, the Riemann ellipsoid astrophysical model of rotating stars and galaxies and the Bohr-Mottelson nuclear rotational model, have a heretofore unappreciated differential geometric structure.

In Part 1, this article formulates the classical theory of Riemann ellipsoids in terms of the differential geometry of the principal bundle $\mathcal{P}$, which is the connected component of the general linear group $GL_{+}(3, \mathbb{R})$. A gauge connection on the bundle $\mathcal{P}$ is equivalent physically to a nonholonomic constraint on the vortex velocity field. The bundle results may be considered a modern update of the discovery of

Riemann in 1860 [1] and its clarification by Lebovitz and Chandrasekhar in the 1960s [2, 3].

In the usual gauge method for isolated mechanical systems, the structure group is the rotation group, and the gauge conserved quantity is the angular momentum [4–9]. The gauge theory of Riemann ellipsoids is different because the vorticity group is the structure group and the Kelvin circulation is the gauge-conserved quantity. The angular momentum is a Noether conserved quantity too, but it is unrelated to the gauge symmetry.

Part 2 presents the quantum theory of Riemann ellipsoids which is a generalization of the Bohr-Mottelson nuclear model [10]. Wave functions are sections of an associated vector bundle. The quantum kinetic energy is proportional to the Laplace-de Rham operator and it depends on the exterior covariant derivative and the Riemannian structure. An apt choice of the Ehresmann connection, motivated by the classical Riemann theory, attains agreement between the bundle theory and experiment for moments of inertia of atomic nuclei.

2. Part 1: Classical Riemann ellipsoids

MacLaurin (1742) applied Newton's laws to describe rigidly-rotating stars. At small angular speeds, a rotating star has the shape of a spheroid, which is an ellipsoid with two equal axis lengths that rotates about its symmetry axis. Perhaps Maclaurin's most remarkable discovery was that a rigidly-rotating star attains a maximum angular speed at an eccentricity $e = 0.93$. While its angular momentum continues to increase as the star flattens out to a pancake shape, the limit of the angular speed is zero as $e \to 1$.

Jacobi (1834) proved a surprising theorem that, at a critical eccentricity $e = 0.81$, the star spontaneously changes its shape from a spheroid to a triaxial ellipsoid, resulting in a lower energy rotating star. Spheroids are, therefore, an insufficient class of shapes to describe rapidly-rotating stars.

In 1857, Dirichlet [11] posed the question: What happens if a star is rotating faster than Maclaurin's rigid body maximum? His Ockham's razor answer is that the kinematical motion group must be enlarged from SO(3), the rotation group, to GL(3,$\mathbb{R}$), the general linear group or, for incompressible fluids, to SL(3,$\mathbb{R}$), the special linear group.

A Riemann ellipsoid (1860) is a self-gravitating, constant density fluid with an ellipsoidal boundary and a velocity field that is a linear function of the Cartesian position coordinates in its inertial centre of mass frame [1]. Riemann's classical theory was applied to the description of astrophysical systems [12–16] and gaseous plasmas [17].

With minor modifications, Riemann ellipsoid theory may be applied to fluids whose density is not uniform and to discrete systems of particles. Rapidly rotating atomic nuclei may be modeled as Riemann ellipsoids when the gravitational self-energy is replaced by the sum of the repulsive Coulomb self-energy among the protons and an attractive surface energy that approximates the strong interactions among the nucleons [18, 23]. In these more general settings, Riemann's main theorem does not apply. The theorem states that a solution to the second order virials of the Navier-Stokes equation for a Riemann ellipsoid is a solution to the Navier-Stokes equation itself. Because real stars and ellipsoidal galaxies are poorly modeled by a uniform density, Riemann's insightful theorem is of mostly mathematical and historical interest now.

The essential features of Riemann ellipsoid theory are as follows:

1. $\mathrm{GL}_+(3,\mathbb{R})$ Collective Motion. Consider a system of A point particles located at the vector Cartesian

coordinates $\vec{x}_\alpha$ in an inertial frame, $\alpha = 1, 2, \ldots, A$. In classical physics, a "point particle" should not to be taken literally. It is a system of particles centered at a point, say with the linear dimension of a human hair, which is large enough that quantum physics is not required. The Dirichlet-Riemann ansatz is that the motion of the entire system is constrained to an orbit of the general linear group in the Euclidean space $\mathbb{R}^{3A}$,

$$\xi \cdot (\vec{x}_1, \ldots, \vec{x}_A) = (\xi\, \vec{x}_1, \ldots, \xi\, \vec{x}_A), \tag{2.1}$$

for $\xi \in \mathrm{GL}_+(3,\mathbb{R})$. Because all the particles move in tandem, the motion is *collective*. For particles in general position, the isotropy subgroup is the identity and the orbit is identified with the Lie group itself. Hence, the Riemann ellipsoid configuration space is diffeomorphic to the connected component of the general linear group

$$\mathcal{P} = GL_+(3, \mathbb{R}) = \{\xi \in M_3(\mathbb{R}) \mid \det \xi > 0\}. \tag{2.2}$$

2. Ellipsoidal Shape. The space of all ellipsoids in $\mathbb{R}^3$ is identified with the manifold Q of all positive-definite real symmetric 3×3 matrices q. The relationship between $\mathcal{P}$ and Q is given by the surjective mapping π:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\pi} & Q \\ \xi & \longmapsto & q = \xi\xi^t \end{array} \tag{2.3}$$

   The strict Riemann hypothesis of a uniform fluid with a sharp ellipsoidal boundary is relaxed. An *inertia* ellipsoid is associated to every orbit point in $\mathbb{R}^{3A}$.
3. Structure (Gauge) Group G. Let G denote the special orthogonal group SO(3) that acts on the general linear group $\mathcal{P}$ by right multiplication, $\xi \longmapsto R_g\xi = \xi g^{-1}$ for $g \in G$. Its induced action on Q leaves the ellipsoid unchanged, $\pi(R_g\xi) = \xi g^{-1} g \xi^t = q$, since $g^{-1} = g^t$, or,

$$\pi \circ R_g = \pi, \text{ for all } g \in G. \tag{2.4}$$

   The projection π is right invariant with respect to the group G. Hence, the configuration space $\mathcal{P}$ is a principal fiber bundle over the base manifold Q with structure group G [20, 21].
4. $\mathcal{P}$ is a Riemannian manifold. For Newtonian physics applications, the metric on the configuration space must be inherited from the Euclidean metric on $\mathbb{R}^3$. This fundamental metric determines the kinetic energy and Lagrangian dynamics in the inertial frame. Right (or left) invariant vector fields on the general linear group are a convenient set of tangent vectors to an orbit. Suppose X is a 3×3 real matrix, regarded as a $\mathfrak{gl}(3, \mathbb{R})$ Lie algebra element, and denote its corresponding right invariant vector field by

$$(\mathcal{R}_X)_\xi = -\sum_{ij} (X \cdot \xi)_{ij} \frac{\partial}{\partial \xi_{ij}}. \tag{2.5}$$

   For X, Y in $M_3(\mathbb{R})$, the inherited metric at the point $\xi \in \mathcal{P}$ is

$$\mathbf{g}_\xi((\mathcal{R}_X)_\xi, (\mathcal{R}_Y)_\xi) = \mathrm{tr}(X\, q\, Y^t), \tag{2.6}$$

   where $q = \pi(\xi) = \xi\xi^t$

2.1. Kinematics and Gauss coordinates

Left multiplication by the orthogonal group SO(3) on $\mathcal{P}$ is a rotation of the system with respect to the inertial frame. For a rotation group element $r \in SO(3)$, an ellipsoid with inertia tensor $q = \pi(\xi)$ is transformed into a rotated ellipsoid with inertia tensor $\pi(L_r\xi) = r\xi\xi^t r^t = rqr^t$.

The physics and geometry of left and right group actions by the orthogonal group are rendered explicit by making a double coset (or Gauss) decomposition of the general linear group, SO(3)$\backslash\mathcal{P}/$G. Each double coset contains a diagonal matrix.

The kinematical description of Riemann ellipsoids in the gauge formalism is attained by certain local trivializations of the bundle $\mathcal{P}$, which separate the degrees of freedom into rotational, vibrational, and vortex components. Every group element $\xi \in \mathcal{P}$ can be expressed as a product of three matrices, $\xi = R^tA\,S$, where R, S are real orthogonal matrices and A is a diagonal matrix with real positive entries a_i in descending order. The projection $q = \pi(\xi) = R^tA^2R$ in the ellipsoidal space of a bundle point ξ shows that the entries of the square of A are the eigenvalues of q and R is an orthogonal matrix that diagonalizes q. Physically R rotates the body into the principal axis frame, and the entries of A are the half-lengths of the inertia ellipsoid's principal axes. Because eigenvalues are unique, the diagonal matrix A is determined uniquely by q. The eigenspaces are also uniquely defined by q. If the eigenvalues are distinct, the eigenspaces are one dimensional and each row of R is unique up to a sign. Thus, when restricted to suitable open neighborhoods, the matrices R and A provide a local coordinate chart for the ellipsoidal space Q. Once R and A are determined by the local chart for the base manifold Q, the orthogonal matrix S in the structure group is given uniquely. A decomposition $\xi = R^tA\,S$, or $\xi = (q; S)$ for $q = R^tA^2R$ and $S = A^{-1}R\xi$, in an open neighborhood of $\mathcal{P}$ is a local trivialization of the bundle $\mathcal{P}$. The bundle $\mathcal{P}$ is only locally diffeomorphic to the Cartesian product of the base manifold Q and the structure group G.

With respect to left multiplication by elements r in the rotation group, the bundle point $\xi = R^tA\,S$ is transformed to $L_r\xi = (R\,r^t)^tA\,S$, or a rotation r is equivalent to right multiplication of the elements R of the subgroup $SO(3)$. With respect to right multiplication by elements g in the structure group G, the bundle point $\xi = R^tA\,S$ is transformed to $R_g\xi = R^tA(S\,g^{-1})$, or a gauge transformation g is equivalent to right multiplication of the elements S of the subgroup G.

2.2. Velocity

Consider a curve $t \longmapsto \xi(t)$ in the bundle $\mathcal{P}$. Such a curve may be identified with the collective motion of a many-body system for which the trajectory of each particle α is constrained by $\mathbf{x}_\alpha(t) = \xi(t)\mathbf{y}_\alpha$, where $\mathbf{y}_\alpha$ is independent of time. The reference particle distribution $\mathbf{y}_\alpha$ is chosen so that its dimensionless inertia tensor is the identity matrix. With this choice the instantaneous inertia tensor of the many-body system simplifies to $q(t) = \xi\xi^t$. The point $\mathbf{y}_\alpha$ in $\mathbb{R}^{3A}$ is a $\mathrm{GL}_+(3,\mathbb{R})$ orbit representative.

The velocity vector for each particle is $\mathbf{v}_\alpha = \dot{\xi}\,\mathbf{y}_\alpha = u\,\mathbf{x}_\alpha$ for $u = \dot{\xi}\xi^{-1}$ and $\dot{\xi} = d\xi/dt$. Note that $\mathbf{v}_\alpha$ is a linear function of its position vector $\mathbf{x}_\alpha$. The velocity vector may be expressed as the value of a right invariant vector field on the group $\mathcal{P}$ at the point ξ,

$$V(t) = \sum_{ij}(\dot{\xi}\xi^{-1}\cdot\xi)_{ij}\frac{\partial}{\partial\xi_{ij}} = -(\mathcal{R}_u)_\xi. \tag{2.7}$$

With respect to a local trivialization, the curve is $t \longmapsto R(t)^tA(t)\,S(t)$. At each time t, define the antisymmetric matrix $\Omega(t) = \dot{R}R^t$ in the Lie algebra $\mathfrak{so}(3)$ of the rotation group and the antisymmetric matrix $\Lambda(t) = \dot{S}S^t$ in the Lie algebra $\mathfrak{g}$ of the structure group. A basis for the space of 3×3 antisymmetric matrices is given by e_i for $i = 1, 2, 3$, where $(e_i)_{jk} \equiv \varepsilon_{ijk}$. The matrix Ω determines the angular velocity vector ω, and Λ – the vortex velocity vector λ:

$$\Omega = \sum_i \omega_i e_i, \qquad \Lambda = \sum_i \lambda_i e_i. \tag{2.8}$$

For such a local trivialization, the velocity of the curve in the bundle can be shown to be a sum of rotational, vibrational, and vortex terms,

$$\begin{aligned} V(t) &= -(\mathcal{R}_\Omega)_R + \sum_i \dot{a}_i \left(\frac{\partial}{\partial a_i}\right)_A - (\mathcal{R}_\Lambda)_S \, . \\ &= \sum_i \left(-\omega_i \left(\mathcal{R}_{e_i}\right)_R + \dot{a}_i \left(\frac{\partial}{\partial a_i}\right)_A - \lambda_i \left(\mathcal{R}_{e_i}\right)_S\right). \end{aligned} \tag{2.9}$$

The velocity vector may be expressed alternatively as a sum of right invariant vector fields on the bundle $\mathcal{P}$ by using the identities,

$$\begin{aligned} (\mathcal{R}_{R^t\Omega R})_\xi &= -(\mathcal{R}_\Omega)_R = (\Omega R)_{ij} \left(\frac{\partial}{\partial R_{ij}}\right)_R \\ (\mathcal{R}_{R^t A^{-1}\dot{A}R})_\xi &= -\dot{a}_i \left(\frac{\partial}{\partial a_i}\right)_A \\ (\mathcal{R}_{R^t A\Lambda A^{-1}R})_\xi &= (\mathcal{R}_\Lambda)_S = -(\Lambda S)_{ij} \left(\frac{\partial}{\partial S_{ij}}\right)_S , \end{aligned} \tag{2.10}$$

when $\xi = R^t AS$. Here $(\mathcal{R}_\Omega)_R$ denotes a right invariant vector field on $SO(3)$ and $(\mathcal{R}_\Lambda)_S$ denotes a right invariant vector field on G.

2.3. Riemannian structure

The kinetic energy is proportional to the squared length of the velocity

$$K = (\mathcal{I}/8)\, \mathbf{g}_\xi(V(t), V(t)), \tag{2.11}$$

where $\mathcal{I}$ is a constant with the units of the moment of inertia. Expanding the velocity into the three types of motion, Eq. (2.9), the kinetic energy becomes

$$K = (\mathcal{I}/4) \left(-\mathrm{tr}(A^2\Omega^2) + \mathrm{tr}(\dot{A}^2) - \mathrm{tr}(A^2\Lambda^2) + 2\,\mathrm{tr}(\Omega A\Lambda A)\right). \tag{2.12}$$

The last term is due to Coriolis coupling between the rotational and vortex degrees of freedom. The derivatives of the kinetic energy with respect to the angular velocity and vortex velocity are the vectors of angular momentum and circulation, respectively,

$$L_k = \frac{\partial K}{\partial \omega_k} = (\mathcal{I}/2)\, [(a_i^2 + a_j^2)\omega_k - 2a_i a_j \lambda_k] \tag{2.13}$$

$$C_k = -\frac{\partial K}{\partial \lambda_k} = (\mathcal{I}/2)\, [2a_i a_j \omega_k - (a_i^2 + a_j^2)\lambda_k], \tag{2.14}$$

where i, j, k are cyclic.

The equations of motion are found using the Lagrangian formalism [22, 23]. Suppose that the potential energy $V(A)$ is a smooth function of the principal axes lengths. Then the potential is left and right invariant with respect to the rotation group and the structure group, respectively. But the metric, and hence the kinetic energy, is also left and right invariant with respect to the rotation and structure groups. Since the Lagrangian is the difference between the kinetic and potential energies, the two invariances, according to Noether's theorem, imply conservation laws. These are the angular momentum and Kelvin circulation. In the rotating body-fixed frame, the angular momentum and Kelvin circulation vectors precess

$$\frac{d\mathbf{L}}{dt} = -\omega \times \mathbf{L} \quad \text{and} \quad \frac{d\mathbf{C}}{dt} = -\lambda \times \mathbf{C}. \tag{2.15}$$

The two vector conservation laws in the inertial centre of mass frame are

$$\frac{d}{dt}(R^t\,\mathbf{L}) = 0 \quad \text{and} \quad \frac{d}{dt}(S^t\,\mathbf{C}) = 0. \tag{2.16}$$

The equations of motion form a Hamiltonian dynamical system [24] and a finite-dimensional Lax system [19].

2.4. Connections on the bundle $\mathcal{P}$

Constraint forces, in addition to forces described by the potential energy $V(A)$, are typically significant. The simplest case is the rigid body for which the vortex velocity vanishes, $\lambda = 0$. This is a holonomic constraint which reduces the configuration space to $Q \cong \mathcal{P}/G$. But constraints are not typically holonomic. For example, an irrotational fluid (like a water droplet) has zero circulation, $\mathbf{C} = 0$. Another example is the so-called "falling cat" [25, 26], for which the angular momentum vanishes, $\mathbf{L} = 0$. In these cases the vortex velocity is proportional to the angular velocity [3]

$$\frac{\lambda_k}{\omega_k} = \begin{cases} \dfrac{2a_i a_j}{a_i^2 + a_j^2}, & \text{irrotational flow} \\[2ex] \dfrac{a_i^2 + a_j^2}{2a_i a_j}, & \text{falling cat} \end{cases} \tag{2.17}$$

where i, j, k are cyclic. A nonholonomic constraint for a Riemann ellipsoid is a proportionality between the vortex and angular velocity components, $\lambda_k = A_k(a_1, a_2, a_3)\,\omega_k$ with a factor A_k that depends on the axis lengths. This proportionality is equivalent to a connection on the bundle $\mathcal{P}$, as it will be shown next.

For each point ξ in the bundle, denote the tangent space by $T_\xi\mathcal{P}$. By definition, the vertical space V_ξ is the subspace of $T_\xi\mathcal{P}$ consisting of the tangents to curves in the fiber G,

$$V_\xi = \left\{X \in T_\xi\mathcal{P} \mid \pi_* X = 0\right\}. \tag{2.18}$$

If $\Lambda \in \mathfrak{g}$ is a Lie algebra element, then the fundamental vector field, denoted by Λ^*, is the left invariant vector field on the fiber G. A basis for V_ξ is the set of fundamental vector fields $\{e_a^*, a = 1, 2, 3\}$, where $(e_a)_{bc} = \varepsilon_{abc}$.

A connexion [21] is a smooth assignment of a horizontal subspace H_ξ of the tangent space $T_\xi\mathcal{P}$ to each point $\xi \in \mathcal{P}$ such that

$$(1)\ T_\xi\mathcal{P} = H_\xi \oplus V_\xi \tag{2.19}$$

$$(2)\ H_{\xi\cdot g} = (R_g)_* H_\xi\,. \tag{2.20}$$

Because the kernel of π_* at $\xi \in \mathcal{P}$ is the vertical subspace V_ξ, its image is T_qQ, and the tangent space $T_\xi\mathcal{P}$ is a direct sum of vertical and horizontal subspaces, the linear transformation π_* is an isomorphism from the horizontal subspace onto the tangent space of the base manifold $\pi_* : H_\xi \longrightarrow T_qQ$, where $q = \pi(\xi)$. If $T \in T_qQ$ is a tangent vector to the base manifold, then its horizontal lift is the unique horizontal vector $\tilde{T} \in H_\xi$ such that $\pi_*\tilde{T} = T$. Given any basis of smooth vector fields in an open neighborhood of the base manifold, $\{\mathbf{f}_m, m = 1, \ldots, \dim Q\}$, their unique horizontal lifts are denoted by $\{\mathbf{F}_m, m = 1, \ldots, \dim Q\}$. The set $\{(\mathbf{F}_m)_\xi\}$ is a basis for the horizontal subspace H_ξ and

$$(\mathbf{F}_m)_\xi = (\mathbf{f}_m)_q - \sum_a A^a_m(\xi)(e^*_a)_S\,, \tag{2.21}$$

where, in a local trivialization, $\xi = (q; S)$, and the coefficients A^a_m are smooth real-valued functions on the bundle $\mathcal{P}$.

The second defining property of a connexion, Eq. (2.20), asserts that $(\mathbf{F}_m)_{\xi\cdot g} = (R_g)_*(\mathbf{F}_m)_\xi$. In particular, when $\xi = (q; I)$, where I is the structure group identity and $g = S^{-1} \in G$, the right translation of a horizontal basis vector at the structure group identity is

$$\begin{aligned}(\mathbf{F}_m)_{(q;S)} &= (R_{S^{-1}})_*(\mathbf{F}_m)_{(q;I)} \\ &= (\mathbf{f}_m)_q - \sum_a A^a_i(q; I)(Ad_{S^{-1}}e_a)^*_S \\ &= (\mathbf{f}_m)_q + \sum_a A^a_m(q)(\mathcal{R}_{e_a})_S\,.\end{aligned} \tag{2.22}$$

The functions $A^a_m(q) \equiv A^a_m(q; I)$ are the connection coefficients.

Consider now a basis $\{(\mathbf{f}_m)_q, m = 1, \ldots, 6\}$ for the tangent space at $q \in Q$ that consists of the three right invariant vector fields $(\mathcal{R}_{\mathbf{e}_i})_R$ on the rotation group $SO(3)$ and the three vibrational vector fields $(\partial/\partial a_i)_A$. A tangent vector to a curve in the base manifold is

$$T(t) = \sum_{i=1}^{3}\left(-\omega_i\,(\mathcal{R}_{\mathbf{e}_i})_R + \dot{a}_i\,(\partial/\partial a_i)_A\right). \tag{2.23}$$

The curve's lift to the bundle is required to have the tangent $V(t)$ of Eq. (2.9),

$$V(t) = -\sum_{i=1}^{3}\omega_i\left((\mathcal{R}_{\mathbf{e}_i})_R + \frac{\lambda_i}{\omega_i}\,(\mathcal{R}_{e_i})_S\right) + \sum_{i=1}^{3}\dot{a}_i\,(\partial/\partial a_i)_A. \tag{2.24}$$

The lift is horizontal if and only if H_ξ is spanned by

$$\mathbf{F}_i = (\mathcal{R}_{e_i})_R + \left(\frac{\lambda_i}{\omega_i}\right)(\mathcal{R}_{e_i})_S \tag{2.25}$$

$$\mathbf{F}_{i+3} = \frac{\partial}{\partial a_i}, \tag{2.26}$$

for $i = 1, 2, 3$. The Riemann ellipsoid connection coefficients vanish for the vibrational vectors and simplify to a diagonal form for the rotational vectors

$$A_i^a(q) = \delta_i^a \left(\frac{\lambda_i}{\omega_i}\right). \tag{2.27}$$

In particular, the special rotational modes correspond to the following Christoffel symbols:

$$A_k^k = \begin{cases} 0, & \text{rigid;} \\ 2a_i a_j/(a_i^2 + a_j^2), & \text{irrotational;} \\ (a_i^2 + a_j^2)/(2a_i a_j), & \text{falling cat,} \end{cases} \tag{2.28}$$

where i, j, k are cyclic. The coefficients are just functions of the axis lengths due to rotational invariance of the horizontal subspace, $(L_r)_* H_\xi = H_{r\xi}$.

To conclude the analysis of Riemann ellipsoids, the next two subsections show that the physical connections for irrotational flow and the falling cat are mathematically natural, corresponding to the Riemannian connection and to the invariant connection.

2.5. Riemannian connection

The horizontal subspace H_ξ^{IF} for irrotational flow is defined as the orthogonal complement to the vertical subspace V_ξ. Denote the vector space of all 3×3 symmetric matrices by $\mathfrak{m}$. The orthogonal complement $V_\xi^\perp$ is given explicitly by

$$H_\xi^{\mathrm{IF}} = \left\{(\mathcal{R}_Y)_\xi \in T_\xi \mathcal{P} \mid Y \in \mathfrak{m}\right\}. \tag{2.29}$$

To prove this, suppose $(\mathcal{R}_\Lambda)_S$, $\Lambda \in \mathfrak{g}$, is a vertical vector and $(\mathcal{R}_Y)_\xi$, $Y \in \mathfrak{m}$, is a horizontal vector. These two vectors are orthogonal,

$$\begin{aligned} \mathbf{g}_\xi((\mathcal{R}_\Lambda)_S, (\mathcal{R}_Y)_\xi) &= \mathbf{g}_\xi((\mathcal{R}_{R^t A\Lambda A^{-1} R})_\xi, (\mathcal{R}_Y)_\xi) \\ &= \mathrm{tr}\left(R^t A\Lambda A^{-1} R(\xi\xi^t) Y^t\right) \\ &= \mathrm{tr}\,(R^t A\Lambda A R Y) \\ &= -\mathrm{tr}\,(Y^t R^t A\Lambda A R) \\ &= 0. \end{aligned} \tag{2.30}$$

Since the sums of the dimensions of the vertical space and the horizontal space add up to the dimension of the tangent space $T_\xi\mathcal{P}$, the tangent space is a direct sum of the horizontal and vertical subspaces. If $(\mathcal{R}_Y)_\xi$ is a horizontal vector and $g \in G$, then right invariance implies

$$(R_g)_*(\mathcal{R}_Y)_\xi = (\mathcal{R}_Y)_{\xi g^{-1}}, \tag{2.31}$$

or $(R_g)_* H_\xi^{\mathrm{IF}} = H_{\xi g^{-1}}^{\mathrm{IF}}$. Since the assignment of the horizontal subspace H_ξ^{IF} is also smooth, it defines a connexion on $\mathcal{P}$.

The vibrational vectors are horizontal since $Y = R^t A^{-1}\dot{A}R$ is a symmetric matrix. But the rotational vectors are not horizontal because

$$\mathbf{g}_\xi((\mathcal{R}_{e_i})_R, (\mathcal{R}_{e_b})_S) = \mathrm{tr}\,(Ae_iAe_b) = -2\delta_{ib}a_ja_k \neq 0 \tag{2.32}$$

for i, j, k cyclic. Note that the inner product of two vertical vectors is also nonzero,

$$\mathbf{g}_\xi((\mathcal{R}_{e_a})_S, (\mathcal{R}_{e_b})_S) = -\mathrm{tr}\left(A^2e_ae_b\right) = \delta_{ab}(a_j^2 + a_k^2) \tag{2.33}$$

for a, j, k cyclic. In order for $(\mathbf{F}_i)_\xi$ to be the horizontal lift of $(\mathcal{R}_{e_i})_R$, it is necessary and sufficient that, for $b = 1, 2, 3$,

$$\begin{aligned} 0 &= \mathbf{g}_\xi((\mathbf{F}_i)_\xi, (\mathcal{R}_{e_b})_S) \\ &= \mathbf{g}_\xi((\mathcal{R}_{e_i})_R + A_i^a(q)(\mathcal{R}_{e_a})_S, (\mathcal{R}_{e_b})_S) \\ &= -2\delta_{ib}a_ja_k + A_i^b(q)(a_j^2 + a_k^2). \end{aligned} \tag{2.34}$$

The off-diagonal connection coefficients for the rotational vectors vanish, and the diagonal values are

$$A_i^i(q) = \frac{2a_ja_k}{(a_j^2 + a_k^2)} \qquad (i, j, k \ \text{ cyclic}). \tag{2.35}$$

Thus, the Riemannian connexion for which the horizontal space is perpendicular to the vertical space corresponds to irrotational flow.

2.6. Invariant connection

The falling cat connexion is the invariant connexion on the Lie group $\mathcal{P}$. Since $\mathfrak{g}$ is the algebra of antisymmetric matrices and $\mathfrak{m}$ is the vector space of symmetric matrices, the Lie algebra of the group $\mathcal{P}$ is a direct sum of vector spaces, $M_3(\mathbb{R}) = \mathfrak{g} \oplus \mathfrak{m}$. Moreover the vector space $\mathfrak{m}$ is invariant with respect to the adjoint group transformation, $Ad_g(\mathfrak{m}) \subset \mathfrak{m}$ for all $g \in G$. These two properties of $\mathfrak{m}$ are necessary and sufficient for

$$H_\xi^{\mathrm{FC}} = \left\{(\mathcal{L}_Y)_\xi = -(\mathcal{R}_{Ad_\xi Y})_\xi \in T_\xi\mathcal{P} \mid Y \in \mathfrak{m}\right\} \tag{2.36}$$

to be a horizontal subspace [21]. In order to see that, note that the vertical vectors can be expressed in left invariant form,

$$V_\xi = \left\{(\mathcal{R}_\Lambda)_S = -(\mathcal{L}_{S^t\Lambda S})_\xi \in T_\xi\mathcal{P} \mid \Lambda \in \mathfrak{g}\right\}. \tag{2.37}$$

The tangent space to the bundle at ξ is a direct sum of the horizontal and vertical subspaces, because every matrix is a linear combination of a symmetric matrix Y and an antisymmetric matrix $S^t\Lambda S$. The right invariance of the horizontal subspaces is a consequence of

$$(R_g)_*(\mathcal{R}_{Ad_\xi Y})_\xi = (\mathcal{R}_{Ad_\xi Y})_{\xi g^{-1}} = (\mathcal{R}_{Ad_{\xi g^{-1}}Ad_g Y})_{\xi g^{-1}} \in H^{\mathrm{FC}}_{\xi g^{-1}}, \tag{2.38}$$

since $Ad_gY \in \mathfrak{m}$ for all $g \in G$ and $Y \in \mathfrak{m}$. The assignment of the subspaces is smooth, so H_ξ^{FC} is indeed a horizontal subspace.

The relation

$$\dot{a}_i\left(\frac{\partial}{\partial a_i}\right)_A = -(\mathcal{R}_{R^tA^{-1}\dot{A}R})_\xi = (\mathcal{L}_{S^tA^{-1}\dot{A}S})_\xi \tag{2.39}$$

shows that the vibrational vectors are horizontal ($S^tA^{-1}\dot{A}S$ is symmetric), but the rotational vectors are not horizontal since

$$(\mathcal{R}_{e_i})_R = -(\mathcal{R}_{R^te_iR})_\xi = (\mathcal{L}_{S^tA^{-1}e_iAS})_\xi \tag{2.40}$$

and $S^tA^{-1}e_iAS$ is not symmetric. If the matrix $S^tA^{-1}e_iAS$ is expressed as a sum of symmetric X_s and antisymmetric X_a parts, i.e., $X_s = (S^tA^{-1}e_iAS - S^tAe_iA^{-1}S)/2$, $X_a = (S^tA^{-1}e_iAS + S^tAe_iA^{-1}S)/2$, the angular momentum may be written as a sum of horizontal and vertical vectors

$$(\mathcal{R}_{e_i})_R = (\mathcal{L}_{X_s})_\xi + (\mathcal{L}_{X_a})_\xi \in H_\xi^{\mathrm{FC}} \oplus V_\xi. \tag{2.41}$$

The horizontal lifts of the angular momentum vectors are

$$\begin{aligned}(\mathbf{F}_i)_\xi &= (\mathcal{R}_{e_i})_R + A_i^a(q)(\mathcal{R}_{e_a})_S \\ &= (\mathcal{L}_{X_s})_\xi + \left[(\mathcal{L}_{X_a})_\xi - A_i^a(q)(\mathcal{L}_{S^te_aS})_\xi\right],\end{aligned} \tag{2.42}$$

where $(\mathcal{L}_{X_s})_\xi$ is the horizontal lift and the two vertical vectors in the square brackets must cancel. Therefore, the invariant connexion is given by

$$A_i^a(q)e_a = (A^{-1}e_iA + Ae_iA^{-1})/2, \tag{2.43}$$

or the connection coefficients are diagonal and

$$A_i^i(q) = \frac{(a_j^2 + a_k^2)}{2a_ja_k}, \tag{2.44}$$

where i, j, k are cyclic.

3. Part 2: Quantum Riemann ellipsoids

A modern formulation of classical Riemann ellipsoids, as presented here, is in terms of the differential geometry of the principal G-bundle $\mathcal{P}$, where the structure or gauge group is the vorticity group G≅SO(3) and the base manifold is the space of all ellipsoids. A time-dependent curve in the base manifold describes the rotation and vibration of the ellipsoid. The connection determines the horizontal lift of the ellipsoid's trajectory to the bundle and its additional vortex degrees of freedom. The physical origin of the lift depends on the response of the particles or fluid to the ellipsoid's motion. The response may be a trivial rigid body motion or complex irrotational flow or something in-between.

The framework for the quantum theory of Riemann ellipsoids is the associated vector bundle, E = $\mathcal{P}\times_\rho$V for ρ an irreducible representation on the vector space V of the gauge group G $\subset \mathcal{P}$. Wave functions are sections of the associated bundle.

A non-negative integer C labels the inequivalent (2C+1)-dimensional irreducible representations ρ of the vorticity group G ≅ SO(3). The Hilbert space $\mathcal{H}_\rho$ of bundle sections consists of functions $\Psi : \mathcal{P} \to V$ of type-ρ,

$$\Psi(\xi g) = \rho(g^{-1})\Psi(\xi), \text{ for all } \xi \in \mathrm{P}, g \in \mathrm{G}, \tag{3.1}$$

which are square-integrable $\int \|\Psi(\xi)\|^2 d\mu(\xi) < \infty$, where $d\mu(\xi)$ denotes the Haar measure on $\mathcal{P}$. The representation of $x \in \mathcal{P}$ is

$$(\pi_\rho(x)\Psi)(\xi) = \Psi(x^{-1}\xi), \text{ for all } \xi \in \mathrm{P}. \tag{3.2}$$

When the representation ρ is trivial (C=0), the bundle sections are the wave functions of the nuclear Bohr-Mottelson model [10,27,28]. However, nature allows any irreducible representation of the gauge vorticity group and all associated bundles are physical.

3.1. Laplacian on the associated G-bundle

The Laplacian determines the kinetic energy and the moment of inertia. It is the key dynamical element of the quantum theory of Riemann ellipsods.

The exterior covariant derivative d_∇ determines a Laplacian $\triangle$ that acts on vector-valued functions ψ on the base manifold $Q \simeq \mathcal{P}/\mathrm{G}$:

$$d_\nabla \star d_\nabla \psi = \triangle(\psi)\,\omega. \tag{3.3}$$

In this equation defining $\triangle$, the volume element ω and Hodge star $\star$ refer to the base manifold and its Riemannian geometry. A connexion one-form A on the base manifold determines the exterior covariant derivative. Hence the calculation of $\triangle$ requires choices for several geometrical ingredients. In this section and the next, the Ehresmann connexion is not restricted to Riemannian, and $\triangle$ is not generally the Laplace-Beltrami operator.

3.2. Riemannian geometry on Q

A basis of six vector fields for a chart $\mathrm{U} \subset Q$ is $\{f_s; s = 1, 2, \ldots, 6\}$, where $f_\alpha = l_\alpha$, the angular momentum projected on the body-fixed frame, and $f_{3+\alpha} = t_\alpha$, the vibrational momentum, $(\alpha = 1, 2, 3)$.

The Riemannian geometry on Q is inherited from the Riemannian geometry on $\mathcal{P}$ in a natural way. Using the connexion on $\mathcal{P}$, two tangent vectors on Q may be lifted to horizontal vectors on $\mathcal{P}$. The Riemannian geometry on $\mathcal{P}$ determines the inner product of the two lifted vectors. This inner product on $\mathcal{P}$ is set equal to the new inner product on Q. In the case of the Riemannian connexion, the horizontal lift of l_α is D_α, and t_α is already horizontal. Hence the Riemannian metric for the orthogonal basis of vector fields $\{f_s\}$ on Q is

$$g(f_s, f_s) = B_s, \tag{3.4}$$

where, for $\alpha = 1, 2, 3$, $B_\alpha = g(D_\alpha, D_\alpha) = \mathcal{I}^{\mathrm{I}}_\alpha$ are the irrotational flow moments of inertia and $B_{3+\alpha} = g(t_\alpha, t_\alpha) = a^2_\alpha$. The corresponding dual basis of one-forms $\{f^s\}$ is also orthogonal,

$$g(f^s, f^s) = B_s^{-1}. \tag{3.5}$$

3.3. Volume element and Hodge star on $\mathcal{P}/G$

The Haar measure vol on $\mathcal{P}$ determines the Haar measure on the coset space $\mathcal{P}$ /G. The scaling property is resolved in a way similar to that for $\mathcal{P}$ because only the space $\mathcal{D}$ is involved. Thus the volume element on Q is

$$\omega = h f^1 \wedge f^2 \wedge \ldots \wedge f^6. \tag{3.6}$$

The Riemannian geometry and volume element on Q determines the Hodge star on this space,

$$\star f^s = \frac{h}{B_s} \hat{f}^s, \tag{3.7}$$

where $\hat{f}^s = (-1)^{q-1} f^1 \wedge \ldots f^{q-1} \wedge f^{q+1} \wedge \ldots \wedge f^6$. Note that $f^{q'} \wedge \hat{f}^s = \delta_{q'q} h^{-1} \omega$.

3.4. Connexion one-form on Q

A connexion one-form A defined on the base manifold is a Lie algebra-valued form,

$$A = -\sum_{s\beta} A_s^\beta(q)\dot{\rho}(\mathfrak{e}_\beta)f^s, \tag{3.8}$$

where s ranges over the basis of vector fields on the base manifold, β runs over a basis $\mathfrak{e}_\beta$ of the Lie algebra of the structure group, and the connexion coefficients $A_s^\beta(q)$ are smooth functions on the base manifold $q \in \mathcal{P}/\mathrm{G}$. The connexion one-form consists of $(\dim V)^2$ one-forms,

$$A_{u'}^u = -\sum_{s\beta} A_s^\beta(q)\dot{\rho}(\mathfrak{e}_\beta)_{u'}^u f^s, \tag{3.9}$$

where u, u' range over a basis for the representation space V of $\dot{\rho}$.

For the Riemannian or irrotational flow connexion, the coefficients $A_s^\beta(m)$ are zero when either $q = 4, 5, 6$ or $\beta \neq q$,

$$A^{\mathrm{I}} = -\sum_{\alpha} A_\alpha^{\mathrm{I}}\dot{\rho}(\mathfrak{e}_\alpha)l^\alpha, \tag{3.10}$$

and the irrotational flow connexion coefficients are functions of a_1, a_2, a_3.

The covariant derivative in the direction of f_s of a section ψ of E is another section of E,

$$\nabla_s\psi = f_s\psi + A(f_s)\psi. \tag{3.11}$$

In the first term of this expression, the vector field f_s is applied to each of the components of the vector-valued function ψ, and, in the second term, the matrix $A(f_s)$ is applied to the column vector ψ.

The exterior covariant derivative d_∇ maps vector-valued p-forms to vector-valued $(p+1)$-forms. When ψ is a 0-form, the covariant derivative determines its exterior covariant derivative

$$d_\nabla\psi = (\nabla_s\psi)f^s. \tag{3.12}$$

For the Riemannian connexion,

$$d_\nabla\psi = \sum_{\alpha=1}^{3}\left\{\left((l_\alpha - A_\alpha^{\mathrm{I}}\dot{\rho}(\mathfrak{e}_\alpha))\psi\right)l^\alpha + t_\alpha(\psi)t^\alpha\right\}. \tag{3.13}$$

Any E-valued p-form is a sum of monomials $\eta \wedge \mu$, where η is a bundle section or E-valued 0-form and μ is an ordinary p-form on the base manifold. The exterior covariant derivative of a monomial is

$$d_\nabla(\eta \wedge \mu) = d_\nabla\eta \wedge \mu + \eta \wedge d\mu. \tag{3.14}$$

3.5. Laplacian $\triangle$

If ψ is a bundle section, then Eq.(3.12) gives its exterior covariant derivative. The Hodge star of the resulting 1-form is a 5-form,

$$\star\, d_\nabla\psi = (\nabla_s\psi)\frac{h}{B_s}\hat{f}^s. \tag{3.15}$$

Because $d\hat{f}^s = 0$, the exterior covariant derivative of this 5-form is the 6-form,

$$\begin{aligned} d_\nabla \star d_\nabla \psi &= \nabla_{s'}\left((\nabla_s \psi)\frac{h}{B_s}\right) f^{s'} \wedge \hat{f}^s \\ &= h^{-1}\nabla_s\left((\nabla_s \psi)\frac{h}{B_s}\right)\omega. \end{aligned} \tag{3.16}$$

Hence the de Rham Laplacian is

$$\triangle = \sum_{\alpha=1}^{3}\left\{\frac{\nabla_\alpha^2}{\mathcal{I}_\alpha^1} + t_\alpha \frac{1}{a_\alpha^2} t_\alpha + t_\alpha(\ln h) t_\alpha\right\}. \tag{3.17}$$

Although the Riemannian connexion determines $\triangle$, the derivation applies to any Ehresmann connexion. A summary of the construction of the de Rham Laplacian for the associated bundle E = $\mathcal{P} \times_\rho$ G is as follows:

1. Choose a connexion 1-form A, Eq. (3.8).
2. Define the Riemannian metric on the base manifold by horizontally lifting vector fields on the base manifold to the Riemannian manifold $\mathcal{P}$.
3. Choose the volume element ω on the coset space $\mathcal{P}$ /G and find the corresponding Hodge star on the base manifold.
4. Determine the exterior covariant derivative d_∇.
5. Evaluate the de Rham Laplacian on bundle sections ψ, $d_\nabla \star d_\nabla \psi = (\triangle\psi)\omega$.

4. Conclusion

The nonholonomic constraints to irrotational flow and the "falling cat" problem correspond to the Riemannian connection and the invariant connection, respectively. Littlejohn and Reinsch [29] reviewed the relationship between gauge theory and traditional physics approaches to nonholonomic constraints, especially in atomic and molecular science, while Massa and Pagani [30] and Bates and Sniatycki [31] provide mathematical overviews of the nonholonomic problem.

The concept of a horizontal lift is physically natural. It says that a many-body system responds to rotations and vibrations (described by a curve γ in the base manifold) by internal vortex motions (described by a horizontally-lifted curve $\tilde{\gamma}$ in the bundle). This response is determined typically by a nonholonomic constraint that depends ultimately on the nature of the forces between the particles. The constraint that the tangent to the lifted curve lies in a horizontal subspace is equivalent to a bundle connection.

The connections corresponding to rigid rotation, irrotational flow, and the falling cat were shown to be natural geometrical or group-theoretical concepts. Although not mathematically natural, other choices of connection coefficients define nonholonomic constraint forces that are not excluded by physical law. For example, the S-type Riemann ellipsoids are a sequence of special case solutions for which the angular momentum, Kelvin circulation, and the angular and vortex velocity vectors are aligned with a principal axis, say the 1-axis [2, 3]. This sequence is indexed by a continuous real parameter f restricted to the interval $-2 \le f \le 0$. There is only one horizontal lift to consider and the connection coefficient is

$$A_1^1(q) = -\frac{f a_2 a_3}{(a_2^2 + a_3^2)}. \tag{4.1}$$

At $f = 0$, the connection yields rigid rotation, and, at $f = -2$, it is irrotational flow. The S-type ellipsoids are the simplest models that allow for a continuous interpolation between rigid rotation and irrotational flow. This connection has no natural geometrical or group-theoretic significance – but it does model a variety of rotating physical systems.

An unsolved basic science problem is to determine the connection from the interactions among the particles that form a rotating system. A complete theory of collective rotation requires equations that incorporate these interactions into the gauge theory and whose unique solution are the connection coefficients. They must involve a coordinate independent object and the curvature form is the obvious candidate.

The quantum theory introduces interesting new physics. For rigid rotation, the connection 1-form $A = 0$. The base manifold vector fields l_α, t_α are then horizontal. Hence the metric on the base manifold differs from Eqs. (3.4, 3.5) by $B_\alpha = \mathcal{I}_\alpha^{\mathrm{R}}$ instead of the irrotational flow inertia. §3.3 gives the volume element and Hodge star. The result is the rigid rotor Laplacian,

$$\triangle_R = \sum_{\alpha=1}^{3} \left\{ \frac{l_\alpha^2}{\mathcal{I}_\alpha^{\mathrm{R}}} + t_\alpha \frac{1}{a_\alpha^2} t_\alpha + t_\alpha (\ln h) t_\alpha \right\}. \tag{4.2}$$

More generally, suppose r is a parameter, $0 \le r \le 1$, and the connection one-form is $A^{\mathrm{r}} = (1-r)\,A^{\mathrm{I}}$. For $r = 0$, the Laplacian corresponds to irrotational flow and, for $r = 1$, rigid rotation. The horizontal lift of l_α is $D_\alpha = l_\alpha - (1-r)\,A_\alpha^{\mathrm{I}} c_\alpha$, and the metric in this case is an interpolation between the rigid and irrotational inertias,

$$\mathcal{I}_\alpha^{\mathrm{r}} = B_\alpha = g(D_\alpha, D_\alpha) = r^2 \mathcal{I}_\alpha^{\mathrm{R}} + (1-r^2)\mathcal{I}_\alpha^{\mathrm{I}}. \tag{4.3}$$

The Laplacian for this connection differs from the irrotational flow and rigid rotor by the interpolated inertia and by the covariant derivative $\nabla_\alpha = l_\alpha - A_\alpha^{\mathrm{r}}\,\dot{\rho}(\mathfrak{e}_\alpha)$,

$$\triangle_r = \sum_{\alpha=1}^{3} \left\{ \frac{\nabla_\alpha^2}{\mathcal{I}_\alpha^{\mathrm{r}}} + t_\alpha \frac{1}{a_\alpha^2} t_\alpha + t_\alpha (\ln h) t_\alpha \right\}. \tag{4.4}$$

For deformed nuclei, the experimental moment of inertia is about five times the irrotational flow inertia and half the rigid rotor inertia. Thus the interpolating parameter is about $r = 2/3$.

The covariant derivative ∇ is a generalization of the electromagnetic covariant derivative for which the connection is the vector potential. Thus the Laplacian of this paper introduces a so-called "magnetic" interaction into the Bohr-Mottelson nuclear collective model. This interaction is velocity-dependent, i.e., it depends on the angular momentum. Because the connection coefficients depend on the axes lengths, the interaction also depends on the deformation.

An interesting contrast between particle physics and collective physics concerns the Riemannian geometry of the base manifold. In the standard model of particle physics, the base manifold is Minkowski space and the metric on it is fixed. In the collective model, the Riemannian geometry of the base manifold depends on the Riemannian geometry of the bundle $\mathcal{P}$ and on the connection. Tangent vectors to the base manifold of ellipsoids must be lifted horizontally to the bundle where the Riemannian metric is well-defined.

Conflict of interest

The author declares there is no conflicts of interest in this paper.

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