AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(2): 199–214
DOI:10.3934/math.2019.2.199
Received: 17 December 2018
Accepted: 19 February 2019
Published: 04 March 2019

Research article

New exact solitary wave solutions to the space-time fractional differential equations with conformable derivative

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Abstract: The exact wave solutions to the space-time fractional modified Benjamin-Bona-Mahony (mBBM) and space time fractional Zakharov-Kuznetsov Benjamin-Bona-Mahony (ZKBBM) equations are studied in the sense of conformable derivative. The existence of chain rule and the derivative of composite functions permit the nonlinear fractional differential equations (NLFDEs) to convert into the ordinary differential equation using wave transformation. The wave solutions of these equations are examined by means of the expanding and effective two variable $(G'/G, 1/G)$-expansion method. The solutions are obtained in the form of hyperbolic, trigonometric and rational functions containing parameters. The method is efficient, convenient, accessible and is the generalization of the original (G'/G)-expansion method.

Keywords: Exact solution; space time fractional modified BBM equation; space time fractional ZKBBM equation; conformable fractional derivative; solitary wave solution
Mathematics Subject Classification: 35C25, 35C07, 35C08, 35Q20, 76B25

1. Introduction

The investigation of exact solutions to nonlinear fractional differential equations (NLFDEs) plays an active role in nonlinear sciences in the perspective of the physical phenomena of real world can effectively be modeled by the making use of the theory of derivatives and integrals of fractional order. NLFDEs have recently proved to be valuable tools to the modeling of many physical phenomena and have expanded the attention of many studies due to their frequent entrance in various applications, such as optical fibers, plasma physics, solid state physics, control theory, chemical kinematics, signal processing, fractional dynamics, fluid flow and other areas. The NLFDEs are also used in modeling of many chemical progressions, mathematical biology and many other problems in physics and engineering. In the last two decades, numerous techniques have been suggested to examine the exact solution to NLFDEs. As a consequence, a variety of methods have been established to get the exact solution to NLFDEs such as the variational iteration method [1–5], the Adomian's decomposition method [6–8], the differential transformation method [9,10], the Laplace perturbation method [11], the homotopy perturbation method [12–14] the finite element method [15], the (G'/G)-expansion method [16–19], the improve Bernoulli sub-equation method [20–22], the Hirota bilinear method [23], the extended sinh-Gordon expansion method [24], the exp-function method [25–27], the fractional sub-equation method [28–30], the first integral method [31,32], the two variable$(G'/G\,,1/G)$-expansion method [33–35] and so forth [36–38].

Alzaidy [39] derived the exact and analytical solution to space-time fractional mBBM and space-time fractional ZKBBM equation by using fractional sub-equation method. Ege et al. [40] established the analytical and approximate solution to the space time fractional mBBm and space time fractional potential Kadomtsev-Petviashvili equation by the modified Kudryashov method. Recently, Bekir et al. [41] studied the exact solution to the space time fractional mKdV and space time fractional mBBM equation by the first integral method. On the other hand, Song et al. [42] investigated the exact traveling wave solution to the space time fractional ZKBBM equation by Bifurcation method. Aksoy et al. [43] derived the analytical solution to the time fractional Fokas equation, fractional ZKBBM equation, and fractional couple Burgers equation by means of the exponential rational function method. In addition, Ekici et al. [44] studied the solitary wave solution and periodic wave solution to the space-time fractional ZKBBM by a fractional sub-equation method. To our comprehension, the space-time fractional mBBM and fractional ZKBBM equation have not been studied by means of recently established two variables $(G'/G\,,1/G)$-expansion method. So, the goal of this study is to establish new solutions to the above-mentioned equation by means of the proposed method. It is noteworthy to observe that, the two variable $(G'/G,1/G)$-expansion method is the generalization of (G'/G)-expansion method. The main idea of this method is that the solutions to NLFDEs are represented as a polynomial in two variables (G'/G) and $(1/G)$, wherein $G=G(\xi)$ satisfies the second order ODE $G''(\xi)+\lambda G(\xi)=\mu$, where λ and μ are constants. The objective of this article is to establish further general and some fresh close form solitary wave solution to the space-time fractional mBBM equation and the space-time fractional ZKBBM equation by means of the two variable $(G'/G,1/G)$-expansion method. The results obtained in this article have been associated with those existing in the literature and shown that the achieved solutions are new and further general. We have to end with determined that the solution might bring up their prominence throughout the contribution and be recorded in the literature.

The rest of this article is organized as follows: In section 2, we have introduced preliminaries and fundamental tools, in section 3, we have described the two variable $(G'/G,1/G)$-expansion method, in section 4, we have established the exact solution to the space-time fractional mBBM equation and the space time fractional ZKBBM equation by the proposed method. In section 5, we investigate results and discussion and in the last section, the conclusions are drawn.

2. Preliminaries and fundamental tools

Let $f: [0,\infty) \to \mathbb{R}$, be a function. The α-order "conformable derivative" of f is defined by [45]:

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0, \alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$. The following theorems point out some properties which are satisfied with the conformable derivative.

Theorem 1. Let $\alpha \in (0,1]$ and suppose f, g be α-differentiable at a point $t > 0$. Then

- $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$, for all $c, d \in \mathbb{R}$.
- $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
- $T_\alpha(c) = 0$, for all constant function $f(t) = c$.
- $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
- $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
- In addition, if f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Some more properties involving the chain rule, Gronwall's inequality, some integration techniques, Laplace transform, Tailor series expansion and exponential function with respect to the conformable fractional derivative are explained in the study [46].

Theorem 2. Let f be an α-differentiable function in conformable differentiable and suppose that g is also differentiable and defined in the range of f. Then

$$T_\alpha(f \circ g)(t) = t^{1-\alpha} v'(t) u'(v(t)). \tag{2.2}$$

3. The two variable $(G'/G, 1/G)$-expansion method

Suppose the second order differential equation

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{3.1}$$

and let us consider the following relations

$$\phi = G'/G,\ \psi = 1/G. \tag{3.2}$$

Thus, it is obtained

$$\phi' = -\phi^2 + \mu\psi - \lambda,\ \psi' = -\phi\psi. \tag{3.3}$$

The solution to the mentioned equation in (3.1) depends on λ for which its values as $\lambda < 0, \lambda > 0$ and $\lambda = 0$.

Remark 1: If $\lambda < 0$, the general solution to equation (3.1) is:

$$\mathrm{G}(\xi) = \mathrm{A}_1 \sinh\left(\sqrt{-\lambda}\,\xi\right) + \mathrm{A}_2 \cosh\left(\sqrt{-\lambda}\,\xi\right) + \frac{\mu}{\lambda}, \tag{3.4}$$

where A_1 and A_2 are arbitrary constants. Accordingly, it yields

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma+\mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{3.5}$$

where $\sigma = A_1^2 - A_2^2$.
Remark 2: If $\lambda > 0$, the general solution to equation (3.1) is:

$$G(\xi) = A_1 \sin\left(\sqrt{\lambda}\,\xi\right) + A_2 \cos\left(\sqrt{\lambda}\,\xi\right) + \frac{\mu}{\lambda}, \tag{3.6}$$

where A_1 and A_2 are arbitrary constants. Consequently, it provides

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma-\mu^2}(\phi^2 - 2\mu\psi + \lambda), \tag{3.7}$$

where $\sigma = A_1^2 + A_2^2$.
Remark 3: If $\lambda = 0$, the general solution to equation (3.1) is:

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{3.8}$$

where $A_1$and A_2 are arbitrary constants. As a result, it delivers

$$\psi^2 = \frac{1}{A_1^2-2\mu A_2}(\phi^2 - 2\mu\psi). \tag{3.9}$$

Let us suppose the general nonlinear fractional differential equation (NLFDE) of the form

$$P\left(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta, \ldots \;\ldots\; \ldots\right) = 0, 0 < \alpha \le 1, 0 < \beta \le 1, \tag{3.10}$$

where u is an unidentified function of spatial derivative x and temporal derivative t and P is a polynomial of u and its partial fractional derivatives wherein the maximum number of derivatives and the nonlinear terms are involved.
Step 1: We suppose the traveling wave transformation in the subsequent

$$\xi = k\frac{x^\beta}{\beta} + c\frac{t^\alpha}{\alpha}, u(x,t) = u(\xi), \tag{3.11}$$

where c and k are nonzero arbitrary constants.
Using this wave transformation, we can rewrite the equation (3.10) as:

$$R(u, u', u'', u''', \ldots \;\ldots\; \ldots) = 0, \tag{3.12}$$

where prime stands for the derivative of $u$with respect to ξ.
Step 2: Assume that the solution of (3.12) can be revealed as a polynomial in two variables ϕ and ψ as the following form:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{3.13}$$

where in $a_i(i = 0,1,2, \ldots)$ and $b_i(i = 0,1,2, \ldots\ )$ are constants to be evaluated afterword.
Step 3: The homogeneous balancing between the highest order derivatives and the nonlinear terms appearing in equation (3.12) determined the positive integer N.

Step 4: Setting (3.13) into (3.12) together with (3.3) and (3.5), the equation (3.12) can be reduced into a polynomial in ϕ and ψ, where the degree of ψ is not greater than one. Equating the coefficients of this polynomial of like power to zero give a system of algebraic equations which can be solved by using the computer algebra, like Maple or Mathematica yields the values of a_i, b_i, μ, A_1, A_2 and λ where $\lambda < 0$, which provide hyperbolic function solutions.
Step 5: Similarly, we examine the values of a_i, b_i, μ, A_1, A_2 and λ when $\lambda > 0$ and $\lambda = 0$, yield the trigonometric and rational function solutions respectively.

4. Formulation of exact solutions

4.1. The exact wave solutions to the space-time fractional mBBM equation

In this subsection, we investigate more general and new closed form wave solutions to the space time fractional mBBM equation by means of the two variables $(G'/G, 1/G)$-expansion method. We introduce the space time fractional mBBM equation [39] as follows:

$$D_t^\alpha u + D_x^\alpha u - vu^2 D_x^\alpha u + D_x^{3\alpha} u = 0, \tag{4.1}$$

where v is a nonzero constant. If spatial derivative term $D_x^\alpha u$ is removed then the equation (4.1) transformed into mKdV equation. This equation was first derived to designate an approximation for surface long waves in nonlinear dispersive media. It is also describe the acoustic waves in inharmonic crystals, the hydro-magnetic waves in cold plasma and acoustic gravity waves in compressible fluids.

For the mBBM equation (4.1), we present the following transformation:

$$\xi = k\frac{x^\alpha}{\alpha} + c\frac{t^\alpha}{\alpha}, u(x,t) = u(\xi), \tag{4.2}$$

where c is the speed of the traveling wave. Using the transformation (4.2), the equation (4.1) reduces to the following integer order ordinary differential equation (ODE):

$$(c+k)u' - vku^2u' + k^3u''' = 0. \tag{4.3}$$

Integrating equation (4.3) with zero constant (see [47] for details), we attain

$$(c+k)u - \frac{vku^3}{3} + k^3u'' = 0. \tag{4.4}$$

Balancing the highest order derivative u'' and the nonlinear term u^3, yields $N = 1$. So, the solution to (4.3) converted to the following form:

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \tag{4.5}$$

where a_0, a_1 and b_1 are constants to be determined.
Case 1: For $\lambda < 0$, setting equation (4.5) into (4.4) along with (3.3) and (3.5) yields a set of algebraic equations and solving these equations by using symbolic computation software Maple, we obtained the following results:

$$a_1 = \pm b_1\sqrt{-\frac{\lambda}{\mu^2+\lambda^2\sigma}},\ k = \pm b_1\sqrt{-\frac{2v\lambda}{3\mu^2+3\lambda^2\sigma}},\ a_0 = 0 \quad \text{and}$$

$$c = \mp\frac{1}{3}\frac{\sqrt{-\frac{2v\lambda}{3\mu^2+3\lambda^2\sigma}}\times b_1(3\mu^2-vb_1^2\lambda^2+3\lambda^2\sigma)}{\mu^2+\lambda^2\sigma}.$$

Substituting these values into (4.5), we find the solution to the mBBM equation as the form:

$$u_{1_1}(x,t) = \pm\, b_1\sqrt{-\frac{\lambda}{\mu^2+\lambda^2\sigma}}\times\frac{\sqrt{-\lambda}\left(A_1\cosh\left(\sqrt{-\lambda}\,\xi\right)+A_2\sinh\left(\sqrt{-\lambda}\xi\right)\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}+\frac{b_1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}, \quad (4.6)$$

where $\xi = \pm b_1\sqrt{-\frac{2v\lambda}{3\mu^2+3\lambda^2\sigma}}\frac{x^\alpha}{\alpha}\mp\frac{1}{3}\frac{\sqrt{-\frac{2v\lambda}{3\mu^2+3\lambda^2\sigma}}\times b_1(3\mu^2-vb_1^2\lambda^2+3\lambda^2\sigma)}{\mu^2+\lambda^2\sigma}\frac{t^\alpha}{\alpha}$ and $\sigma = A_1^2 - A_2^2$.

Since A_1 and A_2 are integral constants, one may choose arbitrarily their values. Therefore, if we choose $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$ in (4.6), we obtain the following solitary wave solution

$$u_{1_2}(x,t) = \pm\frac{b_1}{\sqrt{\sigma}}\times\tanh\left(\pm\sqrt{-\lambda}\left(b_1\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha}+\frac{1}{3}\frac{\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2+3\lambda^2\sigma)}{\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right)+b_1$$

$$\times\, sech\left(\pm\sqrt{-\lambda}\left(b_1\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha}+\frac{1}{3}\frac{\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2+3\lambda^2\sigma)}{\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right). \quad (4.7)$$

Again if $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$ in (4.6), we obtain the solitary wave solution.

$$u_{1_3}(x,t) = \pm\frac{b_1}{\sqrt{\sigma}}\times\coth\left(\pm\sqrt{-\lambda}\left(b_1\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha}+\frac{1}{3}\frac{\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2+3\lambda^2\sigma)}{\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right)+b_1$$

$$\times\, cosech\left(\pm\sqrt{-\lambda}\left(b_1\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha}+\frac{1}{3}\frac{\sqrt{-\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2+3\lambda^2\sigma)}{\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right). \quad (4.8)$$

Case 2: In a similar manner, substituting (4.5) into (4.4) along side with (3.3) and (3.7) yield a set of algebraic equations for a_0, a_1, b_1, c and k and solving these equations, we obtain the following solutions:

$$a_0 = 0,\ a_1 = \pm b_1\sqrt{\frac{\lambda}{\lambda^2\sigma-\mu^2}},\ b_1 = b_1,\ k = \pm b_1\sqrt{\frac{2v\lambda}{3\lambda^2\sigma-3\mu^2}},$$

$$c = \pm\frac{1}{3}\frac{\sqrt{\frac{2v\lambda}{3\lambda^2\sigma-3\mu^2}}\times b_1(3\mu^2-vb_1^2\lambda^2-3\lambda^2\sigma)}{\mu^2-\lambda^2\sigma}.$$

When we substituted these values into equation (4.5), we found the following solution to mBBM equation:

$$u_{1_4}(x,t) = \pm\, b_1\sqrt{\frac{\lambda}{\mu^2-\lambda^2\sigma}} \times \frac{\sqrt{\lambda}(A_1\cos(\sqrt{\lambda}\,\xi)-A_2\sin(\sqrt{\lambda}\xi))}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}} + \frac{b_1}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}, \tag{4.9}$$

where $\xi = b_1\sqrt{\frac{2v\lambda}{3\lambda^2\sigma-3\mu^2}}\frac{x^\alpha}{\alpha} \pm \frac{1}{3}\frac{\sqrt{\frac{2v\lambda}{3\lambda^2\sigma-3\mu^2}}\times b_1(3\mu^2-vb_1^2\lambda^2-3\lambda^2\sigma)}{\mu^2-\lambda^2\sigma}\frac{t^\alpha}{\alpha}$ and $\sigma = A_1^2 + A_2^2$.

If we set $A_1 = 0, A_2 \neq 0$ (or $A_1 \neq 0, A_2 = 0$) and $\mu = 0$ in (4.9), we obtain the solitary wave solution

$$u_{1_5}(x,t) = \pm\frac{b_1}{\sqrt{\sigma}} \times \tan\left(\pm\sqrt{\lambda}\left(b_1\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha} + \frac{1}{3}\frac{\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2-3\lambda^2\sigma)}{-\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right) + b_1$$

$$\times\, sec\left(\pm\sqrt{\lambda}\left(b_1\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha} + \frac{1}{3}\frac{\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2-3\lambda^2\sigma)}{-\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right), \tag{4.10}$$

$$u_{1_6}(x,t) = \pm\frac{b_1}{\sqrt{\sigma}} \times \cot\left(\pm\sqrt{\lambda}\left(b_1\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha} + \frac{1}{3}\frac{\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2-3\lambda^2\sigma)}{-\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right) + b_1$$

$$\times\, cosec\left(\pm\sqrt{\lambda}\left(b_1\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\frac{x^\alpha}{\alpha} + \frac{1}{3}\frac{\sqrt{\frac{2v\lambda}{3\lambda^2\sigma}}\times b_1(-vb_1^2\lambda^2-3\lambda^2\sigma)}{-\lambda^2\sigma}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.11}$$

Case 3: In the similar fashion when $\lambda = 0$, using(4.5) and (4.4) along with(3.3) and (3.9), we will find a group of algebraic equations, whose solutions are as follows:

$$a_0 = 0,\, a_1 = \pm b_1\sqrt{\frac{1}{A_1^2-2\mu A_2}},\, b_1 = b_1,\, c = \mp b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}} \text{ and } k = \pm b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}.$$

Substituting these values into equation (4.5), we attained the solution to equation (4.3)

$$u_{1_7}(x,t) = \pm b_1\sqrt{\frac{1}{A_1^2-2\mu A_2}} \times \frac{\mu(b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{x^\alpha}{\alpha}\mp b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{t^\alpha}{\alpha})+A_1}{+\frac{\mu}{2}(b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{x^\alpha}{\alpha}\mp b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{t^\alpha}{\alpha})^2+A_1\times(b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{x^\alpha}{\alpha}\mp b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{t^\alpha}{\alpha})+A_2} +$$
$$b_1 \times \frac{1}{\frac{\mu}{2}(b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{x^\alpha}{\alpha}\mp b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{t^\alpha}{\alpha})^2+A_1\times(b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{x^\alpha}{\alpha}\mp b_1\sqrt{\frac{2v}{3A_1^2-6\mu A_2}}\frac{t^\alpha}{\alpha})+A_2}. \tag{4.12}$$

It is remarkable to see that the traveling wave solution $u_{1_1-u_{1_7}}$ of the space-time fractional mBBM equation are new and further general and have not been investigated in the previous study.

These solutions forces are convenient to characterize the hydromagnetic waves in cold plasma, acoustic waves in inharmonic crystals and acoustic-gravity waves incompressible fluids.

4.2. The exact wave solutions to the space-time fractional ZKBBM equation

In this subsection, we establish the general and some fresh solutions to the space-time fractional ZKBBM equation over the two variable $(G'/G\,,1/G)$-expansion method. This equation is significant in many physical phenomena and it arises as an explanation of gravity water waves in the long-wave regime. We consider the space-time fractional ZKBBM equation [39] as follows:

$$D_t^{\alpha}u + D_x^{\alpha}u - 2auD_x^{\alpha}u - bD_t^{\alpha}(D_x^{2\alpha}u) = 0.\ t > 0, 0 < \alpha \le 1, \tag{4.13}$$

where a and b are arbitrary constants. If we removed the spatial derivative term $D_x^{\alpha}u$, the equation (4.13) converted to BBM equation.

For the space-time fractional ZKBBM equation, we present the following transformation:

$$\xi = k\frac{x^{\alpha}}{\alpha} + c\frac{t^{\alpha}}{\alpha}, u(x,t) = u(\xi), \tag{4.14}$$

where c is the velocity of the traveling wave. Using traveling wave variable (4.14) equation (4.13) reduces to the following ODE for $u = u(\xi)$:

$$(c+k)u' - 2akuu' - bck^2u''' = 0. \tag{4.15}$$

Integrating (4.15) once with respect to ξ with zero constant, we obtain

$$(c+k)u - aku^2 - bck^2u'' = 0. \tag{4.16}$$

Balancing the highest order derivatives and nonlinear terms, we obtained $N = 2$. Therefore, the solution to equation (4.16) is of the following form:

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + b_1\psi(\xi) + b_2\phi(\xi)\psi(\xi). \tag{4.17}$$

Case 1: For $\lambda < 0$, inserting equation (4.17) into (4.16) along with (3.3) and (3.5) yields a set of algebraic equations whose solutions from the symbolic computation software Maple are as follows:

Set 1: $a_0 = \frac{3bk^2\lambda}{(bk^2\lambda+1)a}$, $a_1 = 0$, $a_2 = \frac{3k^2b}{(bk^2\lambda+1)a}$, $b_1 = -\frac{3k^2b\mu}{(bk^2\lambda+1)a}$, $b_2 = \frac{\pm\sqrt{\frac{-9\mu^2-9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda+1)a}$ and

$c = -\frac{k}{bk^2\lambda+1}$,

Set 2: $a_0 = \frac{2bk^2\lambda}{(bk^2\lambda-1)a}$, $a_1 = 0$, $a_2 = -\frac{3k^2b}{(bk^2\lambda-1)a}$, $b_1 = \frac{3k^2b\mu}{(bk^2\lambda-1)a}$, $b_2 = \frac{\pm\sqrt{\frac{-9\mu^2-9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda-1)a}$ and

$c = \frac{k}{bk^2\lambda-1}$,

where k is an arbitrary constant.

For Set 1, we obtain the following general solution to the space time fractional ZKBBM equation:

$$u_{2_1}(x,t) = \frac{3bk^2\lambda}{(bk^2\lambda+1)a} + \frac{3k^2b}{(bk^2\lambda+1)a} \times \left(\frac{A_1.\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2.\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)^2 \frac{3k^2b\mu}{(bk^2\lambda+1)a} \times \frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}} \pm \frac{\sqrt{\frac{-9\mu^2-9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda+1)a} \times \frac{A_1.\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2.\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}} \times \frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}, \tag{4.18}$$

where $\xi = k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}$ and $\sigma = A_1^2 - A_2^2$.

Here A_1 and A_2 are constants of integration. Therefore, one can freely select their values. If we choose $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$ in (4.18), we attain the solitary wave solution

$$u_{2_2}(x,t) = \frac{3bk^2\lambda}{(bk^2\lambda+1)a} - \frac{3k^2b\lambda}{(bk^2\lambda+1)a} \times tanh^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right) \pm \frac{3\lambda bk^2\sqrt{\sigma}}{(bk^2\lambda+1)a} \times tanh\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right) \times sech\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.19}$$

Again if we set $A_1 \neq 0, A_2 = 0$ and $\mu = 0$ in (4.18), we obtain the solitary wave solution

$$u_{2_3}(x,t) = \frac{3bk^2\lambda}{(bk^2\lambda+1)a} - \frac{3k^2b\lambda}{(bk^2\lambda+1)a} \times coth^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right) \pm \frac{3\lambda bk^2\sqrt{\sigma}}{(bk^2\lambda+1)a} \times coth\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right) \times cosech\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.20}$$

Again for Set 2, we obtain the following general solution to the space time fractional ZKBBM equation:

$$u_{2_4}(x,t) = \frac{2bk^2\lambda}{(bk^2\lambda-1)a} - \frac{3k^2b}{(bk^2\lambda-1)a} \times \left(\frac{A_1.\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2.\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}\right)^2 + \frac{3k^2b\mu}{(bk^2\lambda-1)a} \times \frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}} \pm \frac{\sqrt{\frac{-9\mu^2-9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda-1)a} \times \frac{A_1.\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\xi\right)+A_2.\sqrt{-\lambda}\sinh\left(\sqrt{-\lambda}\xi\right)}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}} \times \frac{1}{A_1\sinh\left(\sqrt{-\lambda}\xi\right)+A_2\cosh\left(\sqrt{-\lambda}\xi\right)+\frac{\mu}{\lambda}}, \tag{4.21}$$

where $\xi = k\frac{x^\alpha}{\alpha} + \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}$ and $\sigma = A_1^2 - A_2^2$.

If we choose $A_1 = 0$, $A_2 \neq 0$ (or $A_1 \neq 0$, $A_2 = 0$) and $\mu = 0$ in (4.21), we obtain the solitary wave solution

$$u_{2_5}(x,t) = \frac{2bk^2\lambda}{(bk^2\lambda-1)a} + \frac{3k^2b\lambda}{(bk^2\lambda-1)a} \times tanh^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} + \frac{k}{bk^2\lambda+1}\frac{t^\alpha}{\alpha}\right)\right) \pm \frac{3\lambda bk^2\sqrt{\sigma}}{(bk^2\lambda-1)a}$$

$$\times tanh\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\times sech\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right). \quad (4.22)$$

$$u_{2_6}(x,t)=\frac{2bk^2\lambda}{(bk^2\lambda-1)a}+\frac{3k^2b\lambda}{(bk^2\lambda-1)a}\times coth^2\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\pm\frac{3\lambda bk^2\sqrt{\sigma}}{(bk^2\lambda+1)a}$$

$$\times coth\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\times cosech\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right). \quad (4.23)$$

Case 2: In the similar manner, when $\lambda>0$inserting (4.17) into (4.16) along with (3.3) and (3.7) yields a set of algebraic equation for $a_0, a_1, b_1, a_2, b_2, c, k$ and solving these equations we have achieved the following results:

Set 1: $a_0=-\frac{2bk^2\lambda}{(bk^2\lambda-1)a}$, $a_1=0$, $a_2=-\frac{3k^2b}{(bk^2\lambda-1)a}$, $b_1=\frac{3k^2b\mu}{(bk^2\lambda-1)a}$, $b_2=\frac{\pm\sqrt{\frac{-9\mu^2+9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda-1)a}$,

$c=\frac{k}{bk^2\lambda+1}$ and $k=k$.

Set 2: $a_0=\frac{3bk^2\lambda}{(bk^2\lambda+1)a}$, $a_1=0$, $a_2=\frac{3k^2b}{(bk^2\lambda+1)a}$, $b_1=\frac{3k^2b\mu}{(bk^2\lambda+1)a}$, $b_2=\frac{\pm\sqrt{\frac{-9\mu^2+9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda+1)a}$,

$c=-\frac{k}{bk^2\lambda+1}$ and $k=k$.

Now, we find the following exact solution to the space time fractional ZKBBM equation (4.13) for Set 1:

$$u_{2_7}(x,t)=$$

$$-\frac{2bk^2\lambda}{(bk^2\lambda-1)a}-\frac{3k^2b}{(bk^2\lambda-1)a}\times\left(\frac{A_1.\sqrt{\lambda}\cos(\sqrt{\lambda}\xi)-A_2.\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}\right)^2+\frac{3k^2b\mu}{(bk^2\lambda-1)a}\times\frac{1}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}\pm$$

$$\frac{\sqrt{\frac{-9\mu^2+9\lambda^2\sigma}{\lambda}}\times bk^2}{(bk^2\lambda-1)a}\times\frac{A_1.\sqrt{\lambda}\cos(\sqrt{\lambda}\xi)-A_2.\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}\times\frac{1}{A_1\sin(\sqrt{\lambda}\xi)+A_2\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}, \quad (4.24)$$

where $\xi=k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}$ and $\sigma=A_1^2+A_2^2$.

As A_1 and A_2 are integral constants, if we select $A_1=0$, $A_2\neq0$(or $A_1\neq0$, $A_2=0$) and $\mu=0$ into (4.24), we attain the solitary wave solution

$$u_{2_8}(x,t)=-\frac{2bk^2\lambda}{(bk^2\lambda-1)a}-\frac{3k^2b\lambda}{(bk^2\lambda-1)a}\times tan^2\left(\sqrt{\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\pm\frac{3\lambda bk^2\sqrt{\sigma}}{(bk^2\lambda-1)a}$$

$$\times tan\left(\sqrt{\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\times sec\left(\sqrt{\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right), \quad (4.25)$$

$$u_{2_9}(x,t)=-\frac{2bk^2\lambda}{(bk^2\lambda-1)a}-\frac{3k^2b\lambda}{(bk^2\lambda-1)a}\times cot^2\left(\sqrt{\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^2\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\pm\frac{3\lambda bk^2\sqrt{\sigma}}{(bk^2\lambda-1)a}$$

$$\times cot\left(\sqrt{\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\times cosec\left(\sqrt{\lambda}\left(k\frac{x^{\alpha}}{\alpha}+\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right). \quad (4.26)$$

Again, we find the following exact solution to the space time fractional ZKBBM equation for Set 2:

$$u_{2_{10}}(x,t)=$$

$$\frac{3bk^{2}\lambda}{(bk^{2}\lambda+1)a}-\frac{3k^{2}b}{(bk^{2}\lambda+1)a}\times\left(\frac{A_{1}.\sqrt{\lambda}\cos(\sqrt{\lambda}\xi)-A_{2}.\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{A_{1}\sin(\sqrt{\lambda}\xi)+A_{2}\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}\right)^{2}+\frac{3k^{2}b\mu}{(bk^{2}\lambda+1)a}\times\frac{1}{A_{1}\sin(\sqrt{\lambda}\xi)+A_{2}\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}\pm$$

$$\frac{\sqrt{\frac{-9\mu^{2}+9\lambda^{2}\sigma}{\lambda}}\times bk^{2}}{(bk^{2}\lambda+1)a}\times\frac{A_{1}.\sqrt{\lambda}\cos(\sqrt{\lambda}\xi)-A_{2}.\sqrt{\lambda}\sin(\sqrt{\lambda}\xi)}{A_{1}\sin(\sqrt{\lambda}\xi)+A_{2}\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}}\times\frac{1}{A_{1}\sin(\sqrt{\lambda}\xi)+A_{2}\cos(\sqrt{\lambda}\xi)+\frac{\mu}{\lambda}},$$

(4.27)

where $\xi=k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}$ and $\sigma=A_{1}^{2}+A_{2}^{2}$.

If we set $A_{1}=0,\ A_{2}\neq 0$ (or $A_{1}\neq 0,\ A_{2}=0$) and $\mu=0$ into (4.27), we find the solitary wave solution:

$$u_{2_{11}}(x,t)=\frac{3bk^{2}\lambda}{(bk^{2}\lambda+1)a}-\frac{3k^{2}b\lambda}{(bk^{2}\lambda+1)a}\times tan^{2}\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\pm\frac{3\lambda bk^{2}\sqrt{\sigma}}{(bk^{2}\lambda+1)a}\times tan\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\times sec\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right).$$

(4.28)

$$u_{2_{12}}(x,t)=\frac{3bk^{2}\lambda}{(bk^{2}\lambda+1)a}-\frac{3k^{2}b\lambda}{(bk^{2}\lambda+1)a}\times cot^{2}\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\pm\frac{3\lambda bk^{2}\sqrt{\sigma}}{(bk^{2}\lambda+1)a}\times cot\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right)\times cosec\left(\sqrt{-\lambda}\left(k\frac{x^{\alpha}}{\alpha}-\frac{k}{bk^{2}\lambda+1}\frac{t^{\alpha}}{\alpha}\right)\right).$$

(4.29)

Case 3: Finally when $\lambda=0$, substituting equation (4.17) into (4.16) along with (3.3) and (3.9) yields a set of algebraic equations for $a_{0},a_{1},b_{1},a_{2},b_{2},c,k$ and whose solution are as follows:

$$a_{0}=0,\ a_{1}=0,\ a_{2}=\frac{3bk^{2}}{a},\ b_{1}=-\frac{3bk^{2}\mu}{a},\ b_{2}=\pm\frac{\sqrt{9A_{1}^{2}+18\mu A_{2}}\times bk^{2}}{a},\ c=-k \text{ and } k=k.$$

Inserting these values into equation (4.16), we attain the rational function solution to the space time fractional ZKBBM equation (4.13) as follows:

$$u_{2_{13}}(x,t)=\frac{3bk^{2}}{a}\times\left(\frac{\mu\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)+A_{1}}{\frac{\mu}{2}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)^{2}+A_{1}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)+A_{2}}\right)^{2}-\frac{3bk^{2}\mu}{a}\times\frac{1}{\frac{\mu}{2}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)^{2}+A_{1}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)+A_{2}}\pm$$

$$\frac{\sqrt{9A_{1}^{2}+18\mu A_{2}}\times bk^{2}}{a}\times\frac{\mu\xi+A_{1}}{\frac{\mu}{2}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)^{2}+A_{1}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)+A_{2}}\times\frac{1}{\frac{\mu}{2}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)^{2}+A_{1}\times\left(k\frac{x^{\alpha}}{\alpha}-k\frac{t^{\alpha}}{\alpha}\right)+A_{2}}. \quad (4.30)$$

5. Results and discussion

It is remarkable to observe that some of the obtained solutions demonstrate good similarity with earlier established solutions. A comparison between Mohyud-Din et al. [48] solutions and our obtained solutions is presented in the following Table 1:

Table 1. Comparison between Mohyud-Din et al. [48] solutions and the obtained solutions.

Mohyud-Din et al. solutions [48]	Obtained solutions
If $\mu = D = 1$ and $C = 0$ the solution $U_{31}(\xi)$ becomes: $U_{31}(\xi) = -\frac{2bk^2\lambda}{(4bk^2\lambda - 1)a} - \frac{6k^2b\lambda}{(4bk^2\lambda - 1)a} \times tan^2(\sqrt{\lambda}\xi)$	If $A_1 = \mu = \sigma = 0$ and $A_2 = 1$ then our solution $u_{2_8}(x,t)$ becomes: $u_{2_8}(x,t) = -\frac{2bk^2\lambda}{(bk^2\lambda - 1)a} - \frac{3k^2b\lambda}{(bk^2\lambda - 1)a} \times tan^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} + \frac{k}{bk^2\lambda + 1}\frac{t^\alpha}{\alpha}\right)\right)$
If $\mu = C = 1$ and $D = 0$ the solution $U_{31}(\xi)$ becomes: $U_{31}(\xi) = -\frac{2bk^2\lambda}{(4bk^2\lambda - 1)a} + \frac{6k^2b\lambda}{(4bk^2\lambda - 1)a} \times cot^2(\sqrt{\lambda}\xi)$	If $A_2 = \mu = \sigma = 0$ and $A_1 = 1$ then our solution $u_{2_9}(x,t)$ becomes: $u_{2_9}(x,t) = -\frac{2bk^2\lambda}{(bk^2\lambda - 1)a} - \frac{3k^2b\lambda}{(bk^2\lambda - 1)a} \times cot^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} + \frac{k}{bk^2\lambda + 1}\frac{t^\alpha}{\alpha}\right)\right)$
If $\mu = D = 1$ and $C = 0$ then the solution $U_{41}(\xi)$ becomes: $U_{41}(\xi) = \frac{6bk^2\lambda}{(4bk^2\lambda + 1)a} - \frac{6k^2b\lambda}{(4bk^2\lambda + 1)a} tan^2(\sqrt{\lambda}\xi)$	If $A_1 = \mu = \sigma = 0$ and $A_2 = 1$ then obtain solution $u_{2_{11}}(x,t)$ becomes: $u_{2_{11}}(x,t) = \frac{3bk^2\lambda}{(bk^2\lambda + 1)a} - \frac{3k^2b\lambda}{(bk^2\lambda + 1)a} \times tan^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda + 1}\frac{t^\alpha}{\alpha}\right)\right)$
If $\mu = C = 1$ and $D = 0$ then the solution $U_{41}(\xi)$ becomes: $\frac{6bk^2\lambda}{(4bk^2\lambda + 1)a} - \frac{6k^2b\lambda}{(4bk^2\lambda + 1)a} cot^2(\sqrt{\lambda}\xi)$	If $A_2 = \mu = \sigma = 0$ and $A_1 = 1$ then obtain solution $u_{2_{12}}(x,t)$ becomes: $u_{2_{12}}(x,t) = \frac{3bk^2\lambda}{(bk^2\lambda + 1)a} - \frac{3k^2b\lambda}{(bk^2\lambda + 1)a} \times cot^2\left(\sqrt{-\lambda}\left(k\frac{x^\alpha}{\alpha} - \frac{k}{bk^2\lambda + 1}\frac{t^\alpha}{\alpha}\right)\right)$

The trigonometric function solutions referred to the above table is similar and if we set definite values of the arbitrary constants they are identical. It is substantial to understand that the traveling wave solutions $u_{2_2}(x,t)$, $u_{2_3}(x,t)$, $u_{2_5}(x,t)$, $u_{2_6}(x,t)$, and $u_{2_2}(x,t)$ of the fractional ZKBBM equation are all new and very much important which were not originate in the previous work. This diffusion equation is significant in various physical phenomena. Itarises as an explanation of gravity water waves in the long-wave regime that creates outstanding model in physics and engineering.

6. Conclusion

In this study, we have obtained some new and further general solitary wave solutions to two nonlinear space time fractional differential equation, namely, the time fractional mBBM equation and the fractional ZKBBM equation in terms of hyperbolic, trigonometric and rational function solution involving parameters. It is remarkable to see that our achieve solutions through the suggested method are more new and further general compared to the existing literature. The obtained solutions to these equations are capable to investigate the mathematical model of gravity water waves in the long-wave regime, the acoustic waves in inharmonic crystals, hydromagnetic waves in cold plasma and acoustic-gravity waves incompressible fluids. The competence of the two variables$(G'/G, 1/G)$-expansion method is consistent, reliable and very much attractive. Since each nonlinear equation has its own inconsistent characteristic, the future research might be how to recommend method is compatible for revealing the solutions to other NLFDEs.

Acknowledgement

The authors would like to express their deepest appreciation to the reviewers for their valuable suggestions and comments to improve the article.

Conflict of interest

The authors declare no conflict of interest.

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