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Research article

Some fractional integral inequalities for the Katugampola integral operator

Ravi Shanker Dubey[2] **and Pranay Goswami**[1,*]

[1] School of Liberal Studies, Ambedkar University Delhi, 110006, Delhi

[2] Department of Mathematics, AMITY School of Applied Science, AMITY University Rajasthan, 303002, Jaipur

* **Correspondence:** Email: pranaygoswami83@gmail.com.

Abstract: In this paper, several new integral inequalities are established by using Katugampola integral operator.

Keywords: fractional integral inequalities; generalized fractional integral; synchronous functions
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1. Introduction and Preliminaries

For over a half century, fractional calculus played an important role in complex phenomena in engineering and the applied sciences . For easy application of fractional calculus mathematician gave many different definitions of fractional derivatives and integrals, out of which the most commonly used and invoked is the Riemann-Liouville operator. For detail see [1, 2].

Definition 1.1. *Let $f(x) \in L^1[a,b]$ and $\alpha > 1$ then the Riemann-Liouville fractional integral of order α is*

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{1.1}$$

It is clear that the right side is defined point-wise on [a, b], where $\Gamma(\alpha) = \int_0^\infty (\tau)^{\alpha-1} e^{\tau}\, d\tau$, is the well-known gamma function.

Definition 1.2. *Let $f : [a,b] \to \Re$, where $f(x) \in L^1[a,b]$ and $\alpha > 0$, then the Riemann-Hadamard fractional integral of order α of a function f is*

$$\left({}^{H}I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\tau)}{\tau\,[\ln(x) - \ln(\tau)]^{1-\alpha}} d\tau. \tag{1.2}$$

Definition 1.3. *([3],see also [4], [5]) Let* $f : [a, b] \to \mathfrak{R}$, *where* $f(x) \in L^1[a, b]$ *and* $\alpha > 0$, *and* $\rho > 0$, *the Riemann-Katugampola fractional integral of order* α *of a function f is*

$$\left(I_{a+}^{\alpha,\rho} f\right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1} f(\tau)}{[x^\rho - \tau^\rho]^{1-\alpha}} d\tau. \tag{1.3}$$

Remark 1.1. *In generalized fractional integral, if* $\alpha \in C$, $Re(\alpha) \geq 0$, $n = \lceil Re(\alpha) \rceil$*and* $\rho > 0$.*Then for* $x > 0$

1. $\lim_{\rho \to 1} \left({}^\rho I_{0+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha - 1} f(\tau) d\tau,$
2. $\lim_{\rho \to o^+} \left({}^\rho I_{0+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log \frac{x}{\tau}\right)^{\alpha - 1} \frac{f(\tau)}{\tau} d\tau.$

Definition 1.4. *A real valued function* $f(x)$, $x \geq 0$*is said to be in the space* C_μ, $\mu \in R$,*if there exist a real number* $p > \mu$*such that* $f(t) = t^p f_1(t)$, *where*$f_1(t) \in C([0, \infty))$.

In this paper we have considered the following functional

$$T(f, g) = \frac{1}{(b-a)} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)} \left(\int_a^b f(x)dx\right)\left(\frac{1}{(b-a)} \int_a^b g(x)dx\right), \tag{1.4}$$

The inequality (2.1) has various generalizations that have appeared in the literature; see ([6–14]). A number of inequalities have appeared in the literature; see ([6–14]). The main aim of this paper is to generalize the results of [10] by using the Katugampola's fractional integral operator and the procedure which have used is similar to the method describe in [10].

2. Main results

Definition 2.1. *Let* $f, g : [a, b] \to \mathfrak{R}$,. *A function f and g are two integral functions which are synchronous on* $[a, b]$ *if*

$$A(x, y) = (f(x) - f(y))(g(x) - g(y)) \geq 0;\ x, y \in [a, b]. \tag{2.1}$$

Theorem 2.1. *Let f and g be two functions synchronous on* $[0, \infty)$. *Then for all* $t > 0$, $\alpha > 0$, *we have*

$${}^\rho I_0^\alpha (fg)(x) \geq \frac{\Gamma(\alpha + 1)}{\rho^\alpha x^{\rho\alpha+\rho-1}} {}^\rho I_0^\alpha f(x) \, {}^\rho I_0^\alpha g(x), \tag{2.2}$$

Proof. Since the function f and g are synchronous on $[0, \infty)$, than for all $\tau \geq 0$, $\eta \geq 0$, we have

$$(f(\tau) - f(\eta))(g(\tau) - g(\eta)) \geq 0. \tag{2.3}$$

or

$$f(\tau) g(\tau) + f(\eta) g(\eta) \geq f(\tau) g(\eta) + g(\tau) f(\eta), \tag{2.4}$$

on multiplying both sides of (2.4) by $\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \frac{\tau^{\rho-1}}{(x^\rho - \tau^\rho)^{1-\alpha}}$ and then integrating with respect to τ on both sides of equation over $(0, x)$, we get

$$\begin{aligned} &\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho-1} f(\tau) g(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho-1} f(\eta) g(\eta)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau \\ &\quad \geq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho-1} f(\tau) g(\eta)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho-1} g(\tau) f(\eta)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau, \end{aligned} \tag{2.5}$$

or

$$
\begin{aligned}
&{}^{\rho}I_0^{\alpha}(fg)(x) + f(\eta)\,g(\eta)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{\tau^{\rho-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}d\tau \\
&\quad \geq g(\eta)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{\tau^{\rho-1}f(\tau)}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}d\tau + f(\eta)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{\tau^{\rho-1}g(\tau)}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}d\tau,
\end{aligned}
\tag{2.6}
$$

we get

$$
{}^{\rho}I_0^{\alpha}(fg) + f(\eta)\,g(\eta)\,{}^{\rho}I_0^{\alpha}(1) \;\geq g(\eta)\,{}^{\rho}I_0^{\alpha}(f) + f(\eta)\,{}^{\rho}I_0^{\alpha}(g)\,. \tag{2.7}
$$

Now again multiplying both sides of (2.7) by$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\eta^{\rho-1}}{(x^{\rho}-\eta^{\rho})^{1-\alpha}}$, than integrating both sides with respect to η over (0, x), we get

$$
\begin{aligned}
&{}^{\rho}I_0^{\alpha}(fg)(x)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{\eta^{\rho-1}}{(x^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta + {}^{\rho}I_0^{\alpha}(1)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{f(\eta)g(\eta)\eta^{\rho-1}}{(x^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta \\
&\quad \geq {}^{\rho}I_0^{\alpha}(f)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{g(\eta)\eta^{\rho-1}}{(x^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta + {}^{\rho}I_0^{\alpha}(g)\,\tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^x \tfrac{f(\eta)\eta^{\rho-1}}{(x^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta,
\end{aligned}
\tag{2.8}
$$

After simple calculation

$$
{}^{\rho}I_0^{\alpha}(1)\left({}^{\rho}I_0^{\alpha}(fg)(x) + {}^{\rho}I_0^{\alpha}(fg)(x)\right) \geq {}^{\rho}I_{a+}^{\alpha}(f)\,{}^{\rho}I_{a+}^{\alpha}(g) + {}^{\rho}I_{a+}^{\alpha}(f)\,{}^{\rho}I_{a+}^{\alpha}(g)\,,
$$

which gives

$$
{}^{\rho}I_0^{\alpha}(fg)(x) \;\geq \frac{{}^{\rho}I_{a+}^{\alpha}(f)\,{}^{\rho}I_{a+}^{\alpha}(g)}{{}^{\rho}I_0^{\alpha}(1)}, \tag{2.9}
$$

or

$$
{}^{\rho}I_0^{\alpha}(fg)(x) \geq \frac{\Gamma(\alpha+1)}{\rho^{\alpha}x^{\rho\alpha+\rho-1}}\,{}^{\rho}I_0^{\alpha}f(x)\,{}^{\rho}I_0^{\alpha}g(x)\,, \tag{2.10}
$$

and this ends the proof. □

Theorem 2.2. *Let f and g be two functions synchronous on* $[0,\infty)$ *,then for all* $x > 0$, $\alpha > 0, \beta > 0$, *we have*

$$
\frac{\rho^{\alpha}x^{\rho\alpha+\rho-1}}{\Gamma(\alpha+1)}\,{}^{\rho}I_0^{\alpha}(fg)(x) + \frac{\rho^{\beta}x^{\rho\beta+\rho-1}}{\Gamma(\beta+1)}\,{}^{\rho}I_0^{\beta}(fg)(x) \;\geq {}^{\rho}I_0^{\alpha}f(x)\,{}^{\rho}I_0^{\beta}g(x) + {}^{\rho}I_0^{\beta}f(x)\,{}^{\rho}I_0^{\alpha}g(x)\,. \tag{2.11}
$$

Proof. We can prove Theorem 2.2 by the similar method which we have used in Theorem 2.1. We can say

$$
\begin{aligned}
&\tfrac{(x-\rho)^{\beta-1}}{\Gamma(\beta)}\,{}^{\rho}I_0^{\alpha}(fg)(x) + {}^{\rho}I_0^{\alpha}(1)\,\tfrac{(x-\rho)^{\beta-1}}{\Gamma(\beta)}f(\tau)\,g(\tau) \\
&\quad \geq \tfrac{(x-\tau)^{\beta-1}}{\Gamma(\beta)}g(\tau)\;{}^{\rho}I_0^{\alpha}(f)(x) + \tfrac{(x-\tau)^{\beta-1}}{\Gamma(\beta)}f(\tau)\,{}^{\tau}I_0^{\alpha}(f)(x)\;,
\end{aligned}
\tag{2.12}
$$

on integrating (2.12) over $(0, x)$, with respect to τ we get

$$
\begin{aligned}
&\tfrac{{}^{\rho}I_0^{\alpha}(fg)(x)}{\Gamma(\beta)}\int_0^x (x-\tau)^{\beta-1}\,d\tau + \tfrac{{}^{\rho}I_0^{\alpha}(1)}{\Gamma(\beta)}\int_0^x (x-\tau)^{\beta-1}\,f(\tau)\,g(\tau)\;d\tau \\
&\quad \geq \tfrac{{}^{\rho}I_0^{\alpha}(f)(x)}{\Gamma(\beta)}\int_0^x (x-\tau)^{\beta-1}\,g(\tau)\;d\tau \;+ \tfrac{{}^{\rho}I_0^{\alpha}(f)(x)}{\Gamma(\beta)}\int_0^x (x-\tau)^{\beta-1}\,f(\tau)\,d\tau
\end{aligned}
\tag{2.13}
$$

we get the required result. □

Remark 2.1. *Theorem 2.2 can also be proved by putting* $\alpha = \beta$ *in Theorem 2.1.*

Remark 2.2. *The inequalities defined in Theorem 2.2 and Theorem 2.1 for* $\alpha = \beta$ *are reversed if the functions are asynchronous on* $[0,\infty)$ *, i.e.* $(f(x) - f(y))\,(g(x) - g(y)) \leq 0$, *for any* $x, y \in [0,\infty)$.

Theorem 2.3. *Let $f_1, f_2 ..., f_n$ be n positive increasing functions on $[0, \infty)$ .Then for any $x > 0, \alpha > 0$,we have*

$$^{\rho}I_0^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(x) \geq \frac{\prod_{i=1}^{n} {}^{\rho}I_0^{\alpha}(f_i)(x)}{\left({}^{\rho}I_0^{\alpha}(1)\right)^{n-1}}. \tag{2.14}$$

Proof. We can prove this Theorem by mathematical induction.
For $n = 1$, we have

$$^{\rho}I_0^{\alpha}(f_1)(x) \geq {}^{\rho}I_0^{\alpha}(f_1)(x), \tag{2.15}$$

for all $x > 0, \alpha > 0$,
for $n = 2$,on using Theorem 2.1,we have

$$^{\rho}I_0^{\alpha}(f_1 f_2)(x) \geq \frac{{}^{\rho}I_0^{\alpha}(f_1)(x)\,{}^{\rho}I_0^{\alpha}(f_2)(x)}{\left({}^{\rho}I_0^{\alpha}(1)\right)}, \tag{2.16}$$

for all $x > 0, \alpha > 0$. Now, suppose that the above relation is true for $n - 1$

$$^{\rho}I_0^{\alpha}\left(\prod_{i=1}^{n-1} f_i\right)(x) \geq \frac{\prod_{i=1}^{n-1} {}^{\rho}I_0^{\alpha}(f_i)(x)}{\left({}^{\rho}I_0^{\alpha}(1)\right)^{n-2}}, \tag{2.17}$$

for all $x > 0, \alpha > 0$.
Since $(f_i)$where $i = 1, 2, ..., n$ are positive increasing functions, then their product $\left(\prod_{i=1}^{n-1} f_i\right)(x)$ is also increasing function.
Now let us consider $\prod_{i=1}^{n-1} f_i = g$ and $f_n = f$.Then from Theorem 2.1, we can say that

$$^{\rho}I_0^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(x) = {}^{\rho}I_0^{\alpha}\left(\prod_{i=1}^{n-1} f_i . f_n\right)(x) \geq \left({}^{\rho}I_0^{\alpha}(1)\right)^{-1}\left({}^{\rho}I_0^{\alpha}\left(\prod_{i=1}^{n-1}(f_i)\right)(x)\right)\left({}^{\rho}I_0^{\alpha}(f_n)(x)\right). \tag{2.18}$$

On using the relation (2.17), we get

$$^{\rho}I_0^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(x) \geq \left({}^{\rho}I_0^{\alpha}(1)\right)^{-1}\left(\left({}^{\rho}I_0^{\alpha}(1)\right)^{2-n}\left(\prod_{i=1}^{n-1} {}^{\rho}I_0^{\alpha}(f_i)\right)(x)\right)\left({}^{\rho}I_0^{\alpha}(f_n)(x)\right). \tag{2.19}$$

On that way we can say that the above defined theorem is true for all values of n.

□

Theorem 2.4. *Let us consider f and g are two functions defined on $[0, \infty)$,such that f is increasing, g is differentiable and there exists a real number$p = \inf_{t \geq 0} g'(x)$. Then we get the inequality*

$$^{\rho}I_0^{\alpha}(fg)(x) \geq \left({}^{\rho}I_0^{\alpha}(1)\right)^{-1} {}^{\rho}I_0^{\alpha}(f)(x)\,{}^{\rho}I_0^{\alpha}(g)(x) - \frac{px}{\alpha+1}\,{}^{\rho}I_0^{\alpha}(f)(x) + p\,{}^{\rho}I_0^{\alpha}(xf(x)), \tag{2.20}$$

for all $t > 0, \alpha > 0$,

Proof. Let us suppose the function $h(x) = g(x) - px$, where h is differentiable and it is increasing on$[0, \infty)$. Thus, from Theorem (2.1), we can say

$$\begin{aligned} &{}^{\rho}I_0^{\alpha}\left((g-px)f(x)\right) \geq \left({}^{\rho}I_0^{\alpha}(1)\right)^{-1}{}^{\rho}I_0^{\alpha}(f)(x)\left({}^{\rho}I_0^{\alpha}(g)(x) - p^{\rho}I_0^{\alpha}(x)\right) \\ &\geq \left({}^{\rho}I_0^{\alpha}(1)\right)^{-1\rho}I_0^{\alpha}(f)(x)^{\rho}I_0^{\alpha}(g)(x) - \tfrac{p\left({}^{\rho}I_0^{\alpha}(1)\right)^{-1}x^{\alpha+1}{}^{\rho}}{\Gamma\alpha+2}I_0^{\alpha}(x) \\ &\geq \left({}^{\rho}I_0^{\alpha}(1)\right)^{-1\rho}I_0^{\alpha}(f)(x)^{\rho}I_0^{\alpha}(g)(x) - \tfrac{px}{\alpha+1}{}^{\rho}I_0^{\alpha}(x). \end{aligned} \tag{2.21}$$

Hence we get

$${}^{\rho}I_0^{\alpha}(fg)(x) \geq ({}^{\rho}I_0^{\alpha}(1))^{-1}\,{}^{\rho}I_0^{\alpha}(f)(x)\,{}^{\rho}I_0^{\alpha}(g)(x) - \frac{px}{\alpha+1}{}^{\rho}I_0^{\alpha}(f)(x) + p^{\rho}I_0^{\alpha}(xf(x)). \tag{2.22}$$

□

3. Corollaries and consequences

Upon setting $\rho \to 1$ in above Theorems,we have the following corollaries for Reimann-Liuville

Corollary 3.1. *Let f and g be two functions synchronous on* $[0, \infty)$. *Then for all* $t > 0$, $\alpha > 0$, *we have*

$$I_0^{\alpha}(fg)(x) \geq \frac{\Gamma(\alpha+1)}{x^{\alpha}}I_0^{\alpha}f(x)\,I_0^{\alpha}g(x). \tag{3.1}$$

Corollary 3.2. *Let* $f_1, f_2 ..., f_n$ *be n positive increasing functions on* $[0, \infty)$.*Then for any* $x > 0, \alpha > 0$,*we have*

$$I_0^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(x) \geq \left(\frac{\Gamma(\alpha+1)}{x^{\alpha}}\right)^{n-1}\prod_{i=1}^{n} I_0^{\alpha}(f_i)(x). \tag{3.2}$$

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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