Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(1): 26–42.
DOI:10.3934/Math.2019.1.26
Received: 22 October 2018
Accepted: 17 December 2018
Published: 21 December 2018

Research article

A study of fractional differential equations and inclusions involving generalized Caputo-type derivative equipped with generalized fractional integral boundary conditions

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Abstract: In this paper, we introduce a new kind of generalized fractional integral boundary conditions and develop the existence theory for a fractional differential equation involving generalized Caputo-type fractional derivative equipped with these conditions. We also study the inclusion case of the given problem. Examples are constructed to demonstrate the application of the obtained results.

Keywords: differential equations and inclusion; generalized Caputo derivative; fractional integral; existence; fixed point
Mathematics Subject Classification: 26A33, 34A60, 34B15

1. Introduction

We investigate a new class of boundary value problems for generalized Caputo-type fractional differential equations and inclusions supplemented with Katugampola type generalized fractional integral boundary conditions. Precisely, we study the following problems:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0^+}y(t) = f(t, y(t)), \quad t \in J := [0, T], \\ y(T) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0^+}y(\eta_i) + \kappa, \quad \delta y(0) = 0, \quad \eta_i \in (0, T), \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0^+}y(t) \in F(t, y(t)), \quad t \in J := [0, T], \\ y(T) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0^+}y(\eta_i) + \kappa, \quad \delta y(0) = 0, \quad \eta_i \in (0, T), \end{cases} \tag{1.2}$$

where ${}^{\rho}_{c}D^{\alpha}_{0^+}$ denotes the generalized Caputo-type fractional derivative of order $1 < \alpha \leq 2, \rho > 0$, ${}^{\rho}I^{\beta}_{0^+}$ is the Katugampola type fractional integral of order $\beta > 0, \rho > 0$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\sigma_i \in \mathbb{R}, i = 1, 2, \ldots, m, \kappa \in \mathbb{R}$, $\delta = t^{1-\rho}\frac{d}{dt}$, and $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

Here we emphasize that the problem considered in the present paper is motivated by Laskin's work [1] on the generalization of the Feynman and Wiener path integrals in the context of fractional quantum mechanics and fractional statistical mechanics. One can find more details in the articles [2, 3]. It is expected that the results obtained in this paper will provide more leverage in dealing with Feynman and Wiener path type integrals involving an index like $\rho > 0$ in (1.1), instead of a fixed choice $\rho = 1$ (Caputo fractional derivative case). Moreover, chaos for a fractional order differential equation involving two parameters (α and $\rho > 0$) becomes more complicated than the one containing Caputo fractional derivative of order α (generalized Caputo-type fractional derivative with $\rho = 1$); one can find more details in [4, 5]. It is worthwhile to notice that Katugampola fractional integral unifies the Riemann-Liouville and Hadamard integrals into a single integral [6]. Thus, our results are more general in the context of integral boundary conditions.

The topic of fractional-order differential equations and inclusions attracted significant attention in recent years and several results on fractional differential equations involving Riemann-Liouville, Caputo, Hadamard type derivatives, supplemented with a variety of boundary conditions, can be found in the related literature [7–9]. The interest in the subject owes to its extensive applications in various disciplines of science and engineering, for instance, see the papers [10–18], and the references cited therein. In a recent paper [19], the authors studied fractional differential equations involving Caputo-Katugampola derivative. In a more recent work [20], the authors studied a fractional order boundary value problem involving Katugampola-type generalized fractional derivative and generalized fractional integral.

We organize the rest of the paper as follows. Section 2 contains preliminary material related to our work. The existence and uniqueness results for the problem (1.1), obtained with the aid of the standard fixed point theorem, are presented in Section 3. The existence results for the inclusions problem (1.2) are derived in Section 4. Examples are provided to demonstrate the application of the main theorems.

2. Preliminaries

For $c \in \mathbb{R}, 1 \leq p \leq \infty$, let $X^p_c(a, b)$ denote the space of all complex-valued Lebesgue measurable functions ϕ on (a, b) endowed with the norm:

$$\|\phi\|_{X^p_c} = \Big(\int_a^b |x^c\phi(x)|^p \frac{dx}{x}\Big)^{1/p} < \infty.$$

We denote by $L^1(a,b)$ the space of all Lebesgue measurable functions φ on (a,b) equipped with the norm:

$$\|\varphi\|_{L^1} = \int_a^b |\varphi(x)|dx < \infty.$$

Let $\mathcal{G} = C(J,\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|y\| = \sup_{t\in[0,T]} |y(t)|$.
Recall that

$$AC^n(J,\mathbb{R}) = \{h : J \to \mathbb{R} : h, h', \ldots, h^{(n-1)} \in C(J,\mathbb{R}) \text{ and } h^{(n-1)} \text{is absolutely continuous}\}.$$

For $0 \le \epsilon < 1$, we define $C_{\epsilon,\rho}(J,\mathbb{R}) = \{f : J \to \mathbb{R} : (t^\rho - a^\rho)^\epsilon f(t) \in C(J,\mathbb{R})\}$ equipped with the norm $\|f\|_{C_{\epsilon,\rho}} = \|(t^\rho - a^\rho)^\epsilon f(t)\|_C$. Moreover, let us introduce $AC^n_\delta(J)$, which consists of the functions f that have absolutely continuous δ^{n-1}-derivative, where $\delta = t^{1-\rho}\frac{d}{dt}$. Thus we define spaces $AC^n_\delta(J,\mathbb{R}) = \left\{f : J \to \mathbb{R} : \delta^{n-1} f \in AC(J,\mathbb{R}),\ \delta = t^{1-\rho}\frac{d}{dt}\right\}$, and $C^n_{\delta,\epsilon}(J,\mathbb{R}) = \left\{f : J \to \mathbb{R} : \delta^{n-1} f \in C(J,\mathbb{R}), \delta^n f \in C_{\epsilon,\rho}(J,\mathbb{R}), \delta = t^{1-\rho}\frac{d}{dt}\right\}$ endowed with the norms $\|f\|_{C^n_\delta} = \sum_{k=0}^{n-1} \|\delta^k f\|_C$ and $\|f\|_{C^n_{\delta,\epsilon}} = \sum_{k=0}^{n-1} \|\delta^k f\|_C + \|\delta^n f\|_{C_{\epsilon,\rho}}$ respectively. Here we use the convention $C^n_{\delta,0} = C^n_\delta$.

Definition 2.1. *[6] The generalized fractional integral of order $\alpha > 0$ and $\rho > 0$ of $f \in X^p_c(a,b)$ for $-\infty < a < t < b < \infty$, is defined by*

$$({}^\rho I^\alpha_{a^+} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds. \tag{2.1}$$

Note that the integral in (2.1) is called the left-sided fractional integral. Similarly we can define right-sided fractional integral ${}^\rho I^\alpha_{b^-} f$ as

$$({}^\rho I^\alpha_{b^-} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s)ds. \tag{2.2}$$

Definition 2.2. *[21] The generalized fractional derivative, associated with the generalized fractional integrals (2.1) and (2.2) for $0 \le a < t < b < \infty$, are defined by*

$$\begin{aligned} ({}^\rho D^\alpha_{a^+} f)(t) &= \Big(t^{1-\rho}\frac{d}{dt}\Big)^n ({}^\rho I^{n-\alpha}_{a^+} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\Big(t^{1-\rho}\frac{d}{dt}\Big)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} f(s)ds \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} ({}^\rho D^\alpha_{b^-} f)(t) &= \Big(-t^{1-\rho}\frac{d}{dt}\Big)^n ({}^\rho I^{n-\alpha}_{b^-} f)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\Big(-t^{1-\rho}\frac{d}{dt}\Big)^n \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{\alpha-n+1}} f(s)ds, \end{aligned} \tag{2.4}$$

if the integrals in (2.3) and (2.4) exist.

Definition 2.3. *[22] For $\alpha \geq 0$ and $f \in AC^n_\delta[a,b]$, the generalized Caputo-type fractional derivatives ${}^\rho_c D^\alpha_a$ and ${}^\rho_c D^\alpha_b$ are defined via the above generalized fractional derivatives as follows*

$$ {}^\rho_c D^\alpha_{a^+} f(x) = {}^\rho D^\alpha_{a^+} \Big[f(t) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} \Big(\frac{t^\rho - a^\rho}{\rho} \Big)^k \Big](x), \quad \delta = x^{1-\rho} \frac{d}{dx}, \tag{2.5} $$

$$ {}^\rho_c D^\alpha_{b^-} f(x) = {}^\rho D^\alpha_{b^-} \Big[f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k f(b)}{k!} \Big(\frac{b^\rho - t^\rho}{\rho} \Big)^k \Big](x), \quad \delta = x^{1-\rho} \frac{d}{dx}, \tag{2.6} $$

where $n = [\alpha] + 1$.

Lemma 2.1. *[22] Let $\alpha \geq 0, n = [\alpha] + 1$ and $f \in AC^n_\delta[a,b]$, where $0 < a < b < \infty$. Then,*

(1) for $\alpha \notin \mathbb{N}$,

$$ {}^\rho_c D^\alpha_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \Big(\frac{t^\rho - s^\rho}{\rho} \Big)^{n-\alpha-1} \frac{(\delta^n f)(s) ds}{s^{1-\rho}} = {}^\rho I^{n-\alpha}_{a^+} (\delta^n f)(t), \tag{2.7} $$

$$ {}^\rho_c D^\alpha_{b^-} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b \Big(\frac{s^\rho - t^\rho}{\rho} \Big)^{n-\alpha-1} \frac{(-1)^n (\delta^n f)(s) ds}{s^{1-\rho}} = {}^\rho I^{n-\alpha}_{b^-} (\delta^n f)(t); \tag{2.8} $$

(2) for $\alpha \in \mathbb{N}$,

$$ {}^\rho_c D^\alpha_{a^+} f = \delta^n f, \quad {}^\rho_c D^\alpha_{b^-} f = (-1)^n \delta^n f. \tag{2.9} $$

Lemma 2.2. *[22] Let $f \in AC^n_\delta[a,b]$ or $C^n_\delta[a,b]$ and $\alpha \in \mathbb{R}$. Then*

$$ {}^\rho I^\alpha_{a^+} {}^\rho_c D^\alpha_{a^+} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\delta^k f)(a)}{k!} \Big(\frac{x^\rho - a^\rho}{\rho} \Big)^k, $$

$$ {}^\rho I^\alpha_{b^-} {}^\rho_c D^\alpha_{b^-} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k (\delta^k f)(a)}{k!} \Big(\frac{b^\rho - x^\rho}{\rho} \Big)^k. $$

In particular, for $0 < \alpha \leq 1$, we have

$$ {}^\rho I^\alpha_{a^+} {}^\rho_c D^\alpha_{a^+} f(x) = f(x) - f(a), $$

$$ {}^\rho I^\alpha_{b^-} {}^\rho_c D^\alpha_{b^-} f(x) = f(x) - f(b). $$

Next we define a solution for the problem (1.1).

Definition 2.4. *A function $y \in AC^2_\delta([0,T],\mathbb{R})$ is said to be a solution of (1.1) if y satisfies the equation ${}^\rho_c D^\alpha y(t) = f(t, y(t))$ on $[0,T]$, and the conditions $y(T) = \sum_{i=1}^m \sigma_i \, {}^\rho I^\beta y(\eta_i) + \kappa, \ \delta y(0) = 0$.*

Relative to the problem (1.1), we consider the following lemma.

Lemma 2.3. *Let $h \in C(0,T) \cap L^1(0,T)$, $y \in AC^2_\delta(J)$ and*

$$ \Omega = 1 - \sum_{i=1}^m \sigma_i \frac{\eta_i^{\rho\beta}}{\rho^\beta \Gamma(\beta+1)} \neq 0. \tag{2.10} $$

Then the integral solution of the linear boundary value problem (BVP):

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0^+}y(t) = h(t), \quad t \in J := [0, T], \\ y(T) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0^+}y(\eta_i) + \kappa, \quad \delta y(0) = 0, \quad \eta_i \in (0, T), \end{cases} \tag{2.11}$$

is given by

$$y(t) = {}^{\rho}I^{\alpha}_{0^+}h(t) + \frac{1}{\Omega}\Big\{ -{}^{\rho}I^{\alpha}_{0^+}h(T) + \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\alpha+\beta}_{0^+}h(\eta_i) + \kappa \Big\}. \tag{2.12}$$

Proof. Applying ${}^{\rho}I^{\alpha}_{0^+}$ on both sides of the fractional differential equation in (2.11) and using Lemma 2.2, we get

$$y(t) = {}^{\rho}I^{\alpha}_{0^+}h(t) + c_1 + c_2\frac{t^{\rho}}{\rho} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^{\rho} - s^{\rho})^{\alpha-1}h(s)ds + c_1 + c_2\frac{t^{\rho}}{\rho}, \tag{2.13}$$

for some $c_1, c_2 \in \mathbb{R}$. Taking δ-derivative of (2.13), we get

$$\delta y(t) = {}^{\rho}I^{\alpha-1}_{0^+}h(t) + c_2 = \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^t s^{\rho-1}(t^{\rho} - s^{\rho})^{\alpha-2}h(s)ds + c_2. \tag{2.14}$$

Using the boundary condition $\delta y(0) = 0$ in (2.14), we get $c_2 = 0$. Applying the generalized integral ${}^{\rho}I^{\beta}_{0^+}$ to (2.13) after inserting the value of c_2 in it, we get

$${}^{\rho}I^{\beta}_{0^+}y(t) = {}^{\rho}I^{\alpha+\beta}_{0^+}h(t) + c_1\frac{t^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}. \tag{2.15}$$

Making use of the first boundary condition $y(T) = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\beta}y(\eta_i) + \kappa$ in (2.15), we get

$${}^{\rho}I^{\alpha}_{0^+}h(T) + c_1 = \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\alpha+\beta}_{0^+}h(\eta_i) + \sum_{i=1}^{m} \sigma_i c_1 \frac{\eta_i^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)} + \kappa,$$

which, on solving for c_1 together with (2.10), yields

$$c_1 = \frac{1}{\Omega}\Big\{ -{}^{\rho}I^{\alpha}_{0^+}h(T) + \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\alpha+\beta}_{0^+}h(\eta_i) + \kappa \Big\}.$$

Substituting the values of c_1 and c_2 in (2.13), we obtain the solution (2.12). The converse follows by direct computation. The proof is completed. □

3. Main results for the problem (1.1)

Using Lemma 2.3, we define an operator $\mathcal{N} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{N}y(t) \;=\; {}^{\rho}I^{\alpha}_{0^+}f(t, y(t)) + \frac{1}{\Omega}\Big\{ -{}^{\rho}I^{\alpha}_{0^+}f(T, y(T)) + \sum_{i=1}^{m} \sigma_i {}^{\rho}I^{\alpha+\beta}_{0^+}f(\eta_i, y(\eta_i)) + \kappa \Big\}. \tag{3.1}$$

In the following, for brevity, we set the notation:

$$\Lambda = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Big\{\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \sum_{i=1}^{m}|\sigma_i|\frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big\}. \tag{3.2}$$

Our first existence result for the problem (1.1) relies on Leray-Schauder nonlinear alternative [23].

Theorem 3.1. *Assume that*

(A_1) $|f(t,y)| \leq p(t)\psi(\|y\|), \forall (t,y) \in [0,T]\times\mathbb{R}$, *where* $p \in L^1([0,T],\mathbb{R}^+)$ *and* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing function;*

(A_2) *we can find a positive constant* W *satisfying the inequality:*

$$\frac{W}{\psi(W)\Big({}^{\rho}I_{0^+}^{\alpha}p(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I_{0^+}^{\alpha}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}p(\eta_i) + |\kappa|\Big\}\Big)} > 1.$$

Then there exists at least one solution for the problem (1.1) on $[0,T]$.

Proof. Consider the operator $\mathcal{N} : \mathcal{G} \to \mathcal{G}$ defined by (3.1) and show that it is continuous and completely continuous. We establish in four steps.

(i) ***$\mathcal{N}$ is continuous.*** Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\mathcal{G}$. Then

$$\begin{aligned}|\mathcal{N}(y_n)(t) - \mathcal{N}(y)(t)| &\leq {}^{\rho}I_{0^+}^{\alpha}|f(t,y_n(t)) - f(t,y(t))| + \frac{1}{\Omega}\Bigg\{{}^{\rho}I_{0^+}^{\alpha}|f(T,y_n(T)) - f(T,y(T))| \\ &\quad + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}|f(\eta_i,y_n(\eta_i)) - f(\eta_i,y(\eta_i))|\Bigg\} \leq \Lambda\|f(\cdot,y_n) - f(\cdot,y)\|.\end{aligned}$$

Since f is a continuous function, therefore, we have

$$\|\mathcal{N}(y_n) - \mathcal{N}(y)\| \leq \Lambda\|f(\cdot,y_n) - f(\cdot,y)\| \to 0, \text{ as } n \to \infty.$$

(ii) ***The operator $\mathcal{N}$ maps bounded sets into bounded sets in $\mathcal{G}$.***

For any $\bar{r} > 0$, it is indeed enough to show that there exists a positive constant ℓ such that $\|\mathcal{N}(y)\| \leq \ell$ for $y \in B_{\bar{r}} = \{y \in \mathcal{G} : \|y\| \leq \bar{r}\}$. By the assumption (A_1), for each $t \in J$, we have

$$\begin{aligned}|\mathcal{N}(y)(t)| &\leq {}^{\rho}I_{0^+}^{\alpha}|f(t,y(t))| + \frac{1}{|\Omega|}\Bigg\{{}^{\rho}I_{0^+}^{\alpha}|f(T,y(T))| + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}|f(\eta_i,y(\eta_i))| + |\kappa|\Bigg\} \\ &\leq {}^{\rho}I_{0^+}^{\alpha}p(T)\Omega(\|y\|) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I_{0^+}^{\alpha}p(T)\psi(\|y\|) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}p(\eta_i)\psi(\|y\|) + |\kappa|\Big\} \\ &\leq \psi(\|y\|)\Big({}^{\rho}I_{0^+}^{\alpha}p(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I_{0^+}^{\alpha}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I_{0^+}^{\alpha+\beta}p(\eta_i) + |\kappa|\Big\}\Big).\end{aligned}$$

Thus

$$\|\mathcal{N}(y)\| \leq \psi(\bar{r})\Big({}^{\rho}I^{\alpha}_{0^+}p(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}p(\eta_i) + |\kappa|\Big\}\Big) := \ell.$$

(iii) ***$\mathcal{N}$ maps bounded sets into equicontinuous sets of $\mathcal{G}$.***

Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$, $B_{\bar{r}}$ be a bounded set of $\mathcal{G}$ as in (ii) and let $y \in B_{\bar{r}}$. Then

$$\begin{aligned}
|\mathcal{N}(y)(t_2) - \mathcal{N}(y)(t_1)| &\leq \Big|{}^{\rho}I^{\alpha}_{0^+}f(t_2, y(t_2)) - {}^{\rho}I^{\alpha}_{0^+}f(t_1, y(t_1))\Big| \\
&\leq \frac{\rho^{1-\alpha}\psi(\bar{r})}{\Gamma(\alpha)}\Big|\int_0^{t_1}\Big[\frac{s^{\rho-1}}{(t_2^{\rho}-s^{\rho})^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^{\rho}-s^{\rho})^{1-\alpha}}\Big]p(s)ds + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^{\rho}-s^{\rho})^{1-\alpha}}p(s)ds\Big| \\
&\to 0 \text{ as } t_1 \longrightarrow t_2, \text{ independent of } y.
\end{aligned}$$

From the steps $(i)-(iii)$, we deduce by the Arzelá-Ascoli theorem that $\mathcal{N} : \mathcal{G} \longrightarrow \mathcal{G}$ is completely continuous.

(iv) ***There exists an open set $V \subseteq \mathcal{G}$ with $y \neq \nu\mathcal{N}(y)$ for $\nu \in (0,1)$ and $y \in \partial V$.***

Let $y \in \mathcal{G}$ be a solution of $y - \nu\mathcal{N}y = 0$ for $\nu \in [0,1]$. Then, for $t \in [0,T]$, we obtain

$$\begin{aligned}
|y(t)| &= |\nu(\mathcal{N}y)(t)| \\
&\leq {}^{\rho}I^{\alpha}_{0^+}|f(t,y(t))| + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}|f(T,y(T))| + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}|f(\eta_i, y(\eta_i))| + |\kappa|\Big\} \\
&\leq {}^{\rho}I^{\alpha}_{0^+}p(T)\psi(\|y\|) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}p(T)\psi(\|y\|) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}p(\eta_i)\psi(\|y\|) + |\kappa|\Big\} \\
&\leq \psi(\|y\|)\Big({}^{\rho}I^{\alpha}_{0^+}p(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}p(\eta_i) + |\kappa|\Big\}\Big),
\end{aligned}$$

which, on taking the norm for $t \in [0,T]$, implies that

$$\frac{\|y\|}{\psi(\|y\|)\Big({}^{\rho}I^{\alpha}_{0^+}p(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}p(\eta_i) + |\kappa|\Big\}\Big)} \leq 1.$$

By the assumption (A_2), there exists a positive constant W such that $\|y\| \neq W$. Next we define $V = \{y \in \mathcal{G} : \|y\| < W\}$ and note that the operator $\mathcal{N} : \overline{V} \to \mathcal{G}$ is continuous and completely continuous. By the choice of V, there does not exist any $y \in \partial V$ satisfying $y = \nu\mathcal{N}(y)$ for some $\nu \in (0,1)$. In consequence, by the nonlinear alternative of Leray-Schauder type [23], we deduce that there exists a fixed point $y \in \overline{V}$ for the operator $\mathcal{N}$, which is a solution of the problem (1.1). □

In our next result, we make use of Banach contraction mapping principle to establish the uniqueness of solutions for the problem (1.1).

Theorem 3.2. *Suppose that*

(A_3) *there exists a nonnegative constant L such that*

$$|f(t,u) - f(t,v)| \leq L\|u - v\|, \quad \textit{for } t \in [0,T] \textit{ and every } u, v \in \mathbb{R}.$$

Then the problem (1.1) has a unique solution on $[0, T]$ *if*

$$L\Lambda < 1, \tag{3.3}$$

where Λ *is defined by (3.2).*

Proof. Consider the operator $\mathcal{N} : \mathcal{G} \to \mathcal{G}$ associated with the problem (1.1) defined by (3.1). With Λ given by (3.2), we fix

$$r \geq \frac{\Lambda f_0 + |\kappa|/|\Omega|}{1 - L\Lambda}, \quad f_0 = \sup_{t\in[0,T]} |f(t, 0)|,$$

and show that $FB_r \subset B_r$, where $B_r = \{y \in \mathcal{G} : \|y\| \leq r\}$. For $y \in B_r$, using (A_3) , we get

$$\begin{aligned} |\mathcal{N}(y)(t)| &\leq {}^{\rho}I^{\alpha}_{0^+}[|f(t, y(t)) - f(t, 0)| + |f(t, 0)|] + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}[|f(T, y(T)) - f(T, 0)| + |f(T, 0)|] \\ &\quad + \sum_{i=1}^{m} |\sigma_i|\, {}^{\rho}I^{\alpha+\beta}_{0^+}[|f(\eta_i, y(\eta_i)) - f(\eta_i, 0)| + |f(\eta_i, 0)|] + |\kappa|\Big\} \\ &\leq (L\|y\| + f_0)\left[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)} + \frac{1}{|\Omega|}\left\{\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha + 1)} + \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha + \beta + 1)}\right\}\right] + \frac{|\kappa|}{|\Omega|} \\ &\leq (Lr + f_0)\Lambda + \frac{|\kappa|}{|\Omega|} \leq r, \end{aligned}$$

which, on taking the norm for $t \in [0, T]$, yields $\|\mathcal{N}(y)\| \leq r$. This shows that $\mathcal{N}$ maps B_r into itself. Now we show that the operator $\mathcal{N}$ is a contraction. Let $y, u \in \mathcal{G}$. Then we get

$$\begin{aligned} |\mathcal{N}(y)(t) - \mathcal{N}(u)(t)| &\leq {}^{\rho}I^{\alpha}_{0^+}|f(t, y(t)) - f(t, u(t))| + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}|f(T, y(T)) - f(T, u(T))| \\ &\quad + \sum_{i=1}^{m} |\sigma_i|\, {}^{\rho}I^{\alpha+\beta}_{0^+}|f(\eta_i, y(\eta_i)) - f(\eta_i, u(\eta_i))|\Big\} \\ &\leq L\Lambda\|y - u\|. \end{aligned}$$

Consequently we obtain $\|\mathcal{N}(y) - \mathcal{N}(u)\| \leq L\Lambda\|y - u\|$, which shows that $\mathcal{N}$ is a contraction by means of (3.3). Thus the contraction mapping principle applies and the operator $\mathcal{N}$ has a unique fixed point. This shows that there exists a unique solution for the problem (1.1) on $[0, T]$. □

Now we prove the uniqueness of solutions for the problem (1.1) by applying a fixed point theorem for nonlinear contractions due to Boyd and Wong [24].

Definition 3.1. *A mapping* $\mathcal{H} : E \to E$ *is called a nonlinear contraction if we can find a continuous nondecreasing function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(0) = 0$, $\phi(\xi) < \xi$ *for all* $\xi > 0$ *and* $\|\mathcal{H}y - \mathcal{H}u\| \leq \phi(\|y - u\|), \forall y,\ u \in E$ *(E is a Banach space).*

Lemma 3.1. *(Boyd and Wong) [24] Let E be a Banach space and let* $N : E \to E$ *be a nonlinear contraction. Then N has a unique fixed point in E.*

Theorem 3.3. *Assume that*

(A_4) $|f(t, y) - f(t, u)| \leq g(t)\dfrac{|y - u|}{G^* + |y - u|}$, $t \in [0, T]$, $y, u \geq 0$, *where* $g : [0, T] \to \mathbb{R}^+$ *is continuous and*

$$G^* = {}^{\rho}I^{\alpha}_{0^+}g(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}g(T) + \sum_{i=1}^{m}|\sigma_i|^{\rho}I^{\alpha+\beta}_{0^+}g(\eta_i)\Big\}. \tag{3.4}$$

Then the problem (1.1) has a unique solution on $[0, T]$.

Proof. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$, defined by

$$\Psi(\xi) = \frac{G^*\xi}{G^* + \xi}, \ \ \forall \xi \geq 0.$$

Let $y, u \in \mathcal{G}$. Then

$$|f(s, y(s)) - f(s, u(s))| \leq \frac{g(s)}{G^*}\Psi(\|y - u\|),$$

so that

$$\begin{aligned} |\mathcal{N}(y)(t) - \mathcal{N}(u)(t)| &\leq {}^{\rho}I^{\alpha}_{0^+}\Big(g(t)\frac{|y(t) - u(t)|}{G^* + |y(t) - u(t)|}\Big) \\ &\quad + \frac{1}{|\Omega|}\bigg\{{}^{\rho}I^{\alpha}_{0^+}\Big(g(T)\frac{|y(T) - u(T)|}{G^* + |y(T) - u(T)|}\Big) + \sum_{i=1}^{m}|\sigma_i|^{\rho}I^{\alpha+\beta}_{0^+}\Big(g(\eta_i)\frac{|y(\eta_i) - u(\eta_i)|}{G^* + |y(\eta_i) - u(\eta_i)|}\Big)\bigg\} \\ &\leq \frac{|y(t) - u(t)|}{G^* + |y(t) - u(t)|}\bigg\{{}^{\rho}I^{\alpha}_{0^+}g(T) + \frac{1}{|\Omega|}\bigg\{{}^{\rho}I^{\alpha}_{0^+}g(T) + \sum_{i=1}^{m}|\sigma_i|^{\rho}I^{\alpha+\beta}_{0^+}g(\eta_i)\bigg\}\bigg\}, \end{aligned}$$

for $t \in [0, T]$. By the condition (3.4), we deduce that $\|\mathcal{N}(y) - \mathcal{N}(u)\| \leq \Psi(\|y - u\|)$ and hence $\mathcal{N}$ is a nonlinear contraction. Thus it follows from the fixed point theorem due to Boyd and Wong [24] that the operator $\mathcal{N}$ has a unique fixed point in $\mathcal{G}$, which is indeed a unique solution of problem (1.1). □

Example 3.1. *Let us consider the following boundary value problem*

$$\begin{cases} {}^{1/3}_{c}D^{5/4}_{0^+}y = f(t, y), \ t \in [0, 2], \\ y(2) = 2\,{}^{1/3}I^{3/4}y(1/2) + 1/2\,{}^{1/3}I^{3/4}y(3/2) + 1/4, \quad \delta y(0) = 0, \end{cases} \tag{3.5}$$

where $\rho = 1/3$, $\alpha = 5/4$, $\sigma_1 = 2$, $\sigma_2 = 1/2$, $\beta = 3/4$, $\eta_1 = 1/2$, $\eta_2 = 3/2$, $\kappa = 1/4$, $T = 2$ *and* $f(t, y(t))$ *will be fixed later.*

Using the given data, we find that $|\Omega| = 4.543695998$ and $\Lambda = 7.572001575$, where Ω and Λ are given by (2.10) and (3.2) respectively.

For illustrating Theorem 3.1, we take

$$f(t, y) = \frac{(1 + t)}{30}\Big(\frac{|y|}{|y| + 1} + y + \frac{1}{8}\Big), \tag{3.6}$$

and find that $p(t) = \frac{(1+t)}{30}$ and $\psi(\|y\|) = \|y\| + \frac{9}{8}$. By condition (A_2), we have $W > 0.7066246467$. Thus, the hypothesis of Theorem 3.1 holds true, which implies that the problem (3.5) has at least one solution.

Furthermore, for the uniqueness results, Theorem 3.2 can be illustrated by choosing

$$f(t, y) = \frac{\tan^{-1} y + e^{-t}}{2\sqrt{81 + \sin t}}. \tag{3.7}$$

Clearly the condition (A_3) is satisfied with $L = 1/18$. Also

$$L\Lambda \approx 0.4206667542 < 1.$$

Obviously all the conditions of Theorem 3.2 hold and consequently the problem (3.5) with $f(t, y)$ given by (3.7) has a unique solution on $[0, 2]$ by the conclusion of Theorem 3.2.

Finally, for illustrating Theorem 3.3, we take

$$f(t, y) = t\left(\frac{|y|}{|y| + 11} + \frac{1}{8}\right). \tag{3.8}$$

Here we choose $g(t) = (1 + t)$ and find that

$$G^* = {}^{\rho}I^{\alpha}_{0^+}g(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}g(T) + \sum_{i=1}^{m}|\sigma_i|{}^{\rho}I^{\alpha+\beta}_{0^+}g(\eta_i)\Big\} \approx 9.923097014,$$

and

$$|f(t, y) - f(t, u)| = t\left(\frac{|y| - |u|}{11 + |y| + |u| + \frac{|y||u|}{11}}\right) \le \frac{(1 + t)|y - u|}{9.923097014 + |y - u|}.$$

So, the conclusion of Theorem 3.3 applies to the problem (3.5) with $f(t, y)$ given by (3.8).

4. Existence results for the multivalued problem (1.2)

In this section, we present existence results for the problem (1.2).

Definition 4.1. *A function $y \in AC^2_{\delta}([0, T], \mathbb{R})$ is called a solution of the problem (1.2) if $y(T) = \sum_{i=1}^{m} \sigma_i\ {}^{\rho}I^{\beta}_{0^+}y(\eta_i) + \kappa, \delta y(0) = 0$ and there exists function $v \in L^1([0, T], \mathbb{R})$ such that $v(t) \in F(t, y(t))$ a.e. on $[0, T]$ and*

$$y(t) = {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{-{}^{\rho}I^{\alpha}_{0^+}v(T) + \sum_{i=1}^{m}\sigma_i\,{}^{\rho}I^{\alpha+\beta}_{0^+}v(\eta_i) + \kappa\Big\}.$$

4.1. The Carathéodory case

Here we prove an existence result for the problem (1.2) by applying nonlinear alternative for Kakutani maps [23] when F has convex values and is of Carathéodory type.

Theorem 4.1. *Assume that*

- (B_1) *$F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory, where $\mathcal{P}_{cp,c}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y$ is compact and convex$\}$;*
- (B_2) *there exist a continuous nondecreasing function $\varphi : [0, \infty) \to (0, \infty)$ and a function $p \in L^1([0, T], \mathbb{R}^+)$ such that*

$$\|F(t, y)\|_{\mathcal{P}} := \sup\{|x| : x \in F(t, y)\} \le p(t)\varphi(\|y\|) \ \text{for each}\ (t, y) \in [0, T] \times \mathbb{R};$$

(B_3) *there exists a constant* $\widehat{W} > 0$ *satisfying*

$$\frac{\widehat{W}}{\varphi(\widehat{W})\Big({}^{\rho}I^{\alpha}_{0^+}p(T) + \frac{1}{\Omega}\Big({}^{\rho}I^{\alpha}_{0^+}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}p(\eta_i) + \kappa\Big)\Big)} > 1.$$

Then there exists at least one solution for the problem (1.2) on $[0, T]$.

Proof. Define an operator $\mathcal{M} : C([0, T], \mathbb{R}) \longrightarrow \mathcal{P}(C([0, T], \mathbb{R}))$ by

$$\mathcal{M}(y) = \{h \in C([0, T], \mathbb{R}) : h(t) = \mathcal{F}(y)(t)\}, \tag{4.1}$$

where

$$\mathcal{F}(y)(t) \quad = {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{ - {}^{\rho}I^{\alpha}_{0^+}v(T) + \sum_{i=1}^{m}\sigma_i\,{}^{\rho}I^{\alpha+\beta}_{0^+}v(\eta_i) + \kappa\Big\},$$

for $v \in S_{F,y}$. Here $S_{F,y}$ denotes the set of selections of F and is defined by

$$S_{F,y} := \{v \in L^1([0, T], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. on } [0, T]\},$$

for each $y \in C([0, T], \mathbb{R})$. Notice that the fixed points of the operator $\mathcal{M}$ are solutions of the problem (1.2).

To show that $\mathcal{M}$ satisfies the assumptions of Leray-Schauder nonlinear alternative [23], we split the proof in several steps.

Step 1. $\mathcal{M}(y)$ is convex for each $y \in C([0, T], \mathbb{R})$ as $S_{F,y}$ is convex (F has convex values).

Step 2. Let $B_r = \{y \in C([0, T], \mathbb{R}) : \|y\| \leq r\}$ be a bounded ball in $C([0, T], \mathbb{R})$, where r is a positive number. Then, for each $h \in \mathcal{M}(y)$, $y \in B_r$, there exists $v \in S_{F,y}$ such that

$$h(t) \quad = {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{ - {}^{\rho}I^{\alpha}_{0^+}v(T) + \sum_{i=1}^{m}\sigma_i\,{}^{\rho}I^{\alpha+\beta}_{0^+}v(\eta_i) + \kappa\Big\}$$

with

$$\|h\| \leq \varphi(r)\Big({}^{\rho}I^{\alpha}_{0^+}p(T) + \frac{1}{|\Omega|}\Big\{{}^{\rho}I^{\alpha}_{0^+}p(T) + \sum_{i=1}^{m}|\sigma_i|\,{}^{\rho}I^{\alpha+\beta}_{0^+}p(\eta_i) + |\kappa|\Big\}\Big) := \ell_1.$$

This shows that $\mathcal{M}$ maps bounded sets (balls) into bounded sets in $C([0, T], \mathbb{R})$.

Step 3. In order to show that $\mathcal{M}$ maps bounded sets into equicontinuous sets of $C([0, T], \mathbb{R})$, we take $t_1, t_2 \in (0, T]$, $t_1 < t_2$, and $y \in B_r$. Then we find that

$$|h(t_2) - h(t_1)| \quad \leq \quad \frac{\rho^{1-\alpha}\varphi(r)}{\Gamma(\alpha)}\Big|\int_0^{t_1}\Big[\frac{s^{\rho-1}}{(t_2^{\rho} - s^{\rho})^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^{\rho} - s^{\rho})^{1-\alpha}}\Big]p(s)ds + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t^{\rho} - s^{\rho})^{1-\alpha}}p(s)ds\Big|,$$

which tends to zero independently of $y \in B_r$ as $t_2 - t_1 \to 0$. In view of the foregoing steps, it follows by the Arzelá-Ascoli theorem that $\mathcal{M} : C([0, T], \mathbb{R}) \to \mathcal{P}(C([0, T], \mathbb{R}))$ is completely continuous.

Step 4. In our next step, we show that $\mathcal{M}$ is upper semi-continuous (u.s.c.). Since $\mathcal{M}$ is completely continuous, it is enough to establish that it has a closed graph (see [25, Proposition 1.2]). For that, let

$y_n \to y_*, h_n \in \mathcal{M}(y_n)$ and $h_n \to h_*$. Then we have to show that $h_* \in \mathcal{M}(y_*)$. Associated with $h_n \in \mathcal{M}(y_n)$, we can find $v_n \in S_{F,y_n}$ such that for each $t \in [0, T]$,

$$h_n(t) = {}^\rho I^\alpha_{0^+} v_n(t) + \frac{1}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v_n(s) v_n(T) + \sum_{i=1}^m \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v_n(\eta_i) + \kappa \Big\}.$$

Next, for each $t \in [0, T]$, we establish that there exists $v_* \in S_{F,y_*}$ satisfying

$$h_*(t) = {}^\rho I^\alpha_{0^+} v_*(t) + \frac{1}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v_*(T) + \sum_{i=1}^m \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v_*(\eta_i) + \kappa \Big\}.$$

Consider the linear operator $\Theta : L^1([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ given by

$$v \mapsto \Theta v(t) = {}^\rho I^\alpha_{0^+} v(t) + \frac{1}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v(T) + \sum_{i=1}^m \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v(\eta_i) + \kappa \Big\}.$$

Notice that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$. Thus we deduce by the closed graph theorem [26] that $\Theta \circ S_F$ is a closed graph operator. Furthermore, we have $h_n(t) \in \Theta(S_{F,y_n})$. As $y_n \to y_*$, we have

$$h_*(t) = {}^\rho I^\alpha_{0^+} v_*(t) + \frac{1}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v_*(T) + \sum_{i=1}^m \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v_*(\xi) + \kappa \Big\}, \text{ for some } v_* \in S_{F,y_*}.$$

Step 5. Finally, we show the existence of an open set $U \subseteq C([0, T], \mathbb{R})$ such that $y \notin \lambda \mathcal{M}(y)$ for any $\lambda \in (0, 1)$ and all $y \in \partial U$. For that we take $\lambda \in (0, 1)$ and $y \in \lambda \mathcal{M}(y)$. Then there exists $v \in L^1([0, T], \mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in [0, T]$, we have

$$y(t) = \lambda {}^\rho I^\alpha_{0^+} v(t) + \frac{\lambda}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v(T) + \sum_{i=1}^m \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v(\eta_i) + \kappa \Big\}.$$

As in the second step, one can obtain

$$\begin{aligned} |y(t)| &\le {}^\rho I^\alpha_{0^+} |v(T)| + \frac{1}{|\Omega|}\Big\{ {}^\rho I^\alpha_{0^+} |v(T)| + \sum_{i=1}^m |\sigma_i| \, {}^\rho I^{\alpha+\beta}_{0^+} |v(\eta_i)| + |\kappa| \Big\} \\ &\le \varphi(\|y\|)\Big({}^\rho I^\alpha_{0^+} p(T) + \frac{1}{|\Omega|}\Big\{ {}^\rho I^\alpha_{0^+} p(T) + \sum_{i=1}^m |\sigma_i| \, {}^\rho I^{\alpha+\beta}_{0^+} p(\eta_i) + |\kappa| \Big\}\Big), \end{aligned}$$

which implies that

$$\frac{\|y\|}{\varphi(\|y\|)\bigg({}^\rho I^\alpha_{0^+} p(T) + \frac{1}{|\Omega|}\Big\{ {}^\rho I^\alpha_{0^+} p(T) + \sum_{i=1}^m |\sigma_i| \, {}^\rho I^{\alpha+\beta}_{0^+} p(\eta_i) + |\kappa| \Big\}\bigg)} \le 1.$$

By the hypothesis (B_3), we can find $\widehat{W}$ such that $\|y\| \ne \widehat{W}$. Setting $\mathcal{Y} = \{y \in C(J, \mathbb{R}) : \|y\| < \widehat{W}\}$, we notice that the operator $\mathcal{M} : \overline{\mathcal{Y}} \to \mathcal{P}(C(J, \mathbb{R}))$ is compact multi-valued, u.s.c. with convex closed values. From the choice of $\mathcal{Y}$, there does not exist any $y \in \partial \mathcal{Y}$ satisfying $y \in \lambda \mathcal{M}(y)$ for some $\lambda \in (0, 1)$. In consequence, we deduce by the nonlinear alternative of Leray-Schauder type [23] that $\mathcal{M}$ has a fixed point $y \in \overline{\mathcal{Y}}$ which is a solution of the problem (1.2). This completes the proof. □

4.2. The Lipschitz case

Consider a mapping $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ defined by

$$H_d(S, V) = \max\{\sup_{s \in S} d(a, V), \sup_{v \in V} d(S, v)\},$$

where $d(S, v) = \inf_{s \in S} d(s; v)$, $d(s, V) = \inf_{v \in V} d(s; v)$ and (X, d) is a metric space induced from the normed space $(X; \|\cdot\|)$. Note that $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space (see [27]), where $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$.

The following result, dealing with the existence of solutions for the problem (1.2) with nonconvex valued right hand side of the inclusion, relies on Covitz and Nadler's fixed point theorem for multivalued maps [28].

Theorem 4.2. *Assume that*

(C_1) $F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, y) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $y \in \mathbb{R}$, *where* $\mathcal{P}_{cp}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \textit{ is compact}\}$;

(C_2) $H_d(F(t, y), F(t, \bar{y})) \leq \mu(t)|y - \bar{y}|$ *for almost all* $t \in [0, T]$ *and* $y, \bar{y} \in \mathbb{R}$ *with* $\mu \in C([0, T], \mathbb{R}^+)$ *and* $d(0, F(t, 0)) \leq \mu(t)$ *for almost all* $t \in [0, T]$.

Then the problem (1.2) has at least one solution on $[0, T]$ *provided that*

$$\vartheta = \|\mu\|\Lambda < 1, \tag{4.2}$$

where Λ *is given by (3.2).*

Proof. By the assumption (C_1), the set $S_{F,y}$ is nonempty for each $y \in C([0, T], \mathbb{R})$ and F has a measurable selection by Theorem III.6 in [29]. Now we proceed to show that the operator $\mathcal{M} : (C[0, T], \mathbb{R}) \to \mathcal{P}_{cl}(C([0, T], \mathbb{R}))$ $(\mathcal{P}_{cl}(C([0, T], \mathbb{R})) = \{Y \in P(C([0, T], \mathbb{R})) : Y \text{ is closed}\})$ is a contraction so that Covitz and Nadler's Theorem [28] is applicable.

In the first step, we show that $\mathcal{M}(y) \in \mathcal{P}_{cl}((C[0, T], \mathbb{R}))$ for each $y \in C([0, T], \mathbb{R})$. Let $\{u_n\}_{n \geq 0} \in \mathcal{M}(y)$ with $u_n \to u$ $(n \to \infty)$ in $C([0, T], \mathbb{R})$. Then $u \in C([0, T], \mathbb{R})$ and there exists $v_n \in S_{F,y_n}$ satisfying

$$u_n(t) = {}^\rho I^\alpha_{0^+} v_n(t) + \frac{1}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v_n(T) + \sum_{i=1}^{m} \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v_n(\eta_i) + \kappa \Big\} \text{ for each } t \in [0, T].$$

In view of the compact values of F, we pass onto a subsequence (if necessary) to find that v_n converges to v in $L^1([0, T], \mathbb{R})$. For $v \in S_{F,y}$ and for each $t \in [0, T]$, we have

$$u_n(t) \to u(t) = {}^\rho I^\alpha_{0^+} v(t) + \frac{1}{\Omega}\Big\{ - {}^\rho I^\alpha_{0^+} v(T) + \sum_{i=1}^{m} \sigma_i \, {}^\rho I^{\alpha+\beta}_{0^+} v(\eta_i) + \kappa \Big\}.$$

Thus $u \in \mathcal{M}(y)$.

Now, for each $y, \bar{y} \in C([0, T], R)$, we establish that there exists $\vartheta < 1$ (defined by (4.2)) satisfying

$$H_d(\mathcal{M}(y), \mathcal{M}(\bar{y})) \leq \vartheta \|y - \bar{y}\|.$$

Let $y, \bar{y} \in C([0,T],\mathbb{R})$ and $h_1 \in \mathcal{M}(y)$. Then there exists $v_1(t) \in F(t, y(t))$ satisfying

$$h_1(t) = {}^{\rho}I_{0^+}^{\alpha} v_1(t) + \frac{1}{\Omega}\Big\{ - {}^{\rho}I_{0^+}^{\alpha} v_1(T) + \sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} v_1(\eta_i) + \kappa\Big\},$$

for each $t \in [0,T]$. By (C_2), we have

$$H_d(F(t,y), F(t,\bar{y})) \le \mu(t)|y(t) - \bar{y}(t)|.$$

Therefore, we can find $w \in F(t, \bar{y}(t))$ satisfying

$$|v_1(t) - w| \le \mu(t)|y(t) - \bar{y}(t)|, \quad t \in [0,T].$$

Introduce $\mathcal{U} : [0,T] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{U}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le \mu(t)|y(t) - \bar{y}(t)|\}.$$

As $\mathcal{U}(t) \cap F(t, \bar{y}(t))$ is measurable (Proposition III.4 [29]), we can find a measurable selection $v_2(t)$ for $\mathcal{U}$ such that $v_2(t) \in F(t, \bar{y}(t))$ satisfying $|v_1(t) - v_2(t)| \le \mu(t)|y(t) - \bar{y}(t)|$ for each $t \in [0,T]$.

Define

$$h_2(t) = {}^{\rho}I_{0^+}^{\alpha} v_2(t) + \frac{1}{\Omega}\Big\{ - {}^{\rho}I_{0^+}^{\alpha} v_2(T) + \sum_{i=1}^{m} \sigma_i \, {}^{\rho}I_{0^+}^{\alpha+\beta} v_2(\eta_i) + \kappa\Big\},$$

for each $t \in [0,T]$. Then

$$\begin{aligned} |h_1(t) - h_2(t)| &\le {}^{\rho}I_{0^+}^{\alpha}|v_1(t) - v_2(t)| + \frac{1}{|\Omega|}\Big\{{}^{\rho}I_{0^+}^{\alpha}|v_1(T) - v_2(T)| + \sum_{i=1}^{m} |\sigma_i| \, {}^{\rho}I_{0^+}^{\alpha+\beta}|v_1(\eta_i) - v_2(\eta_i)|\Big\} \\ &\le \|\mu\|\left[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left\{\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\right\}\right]\|y - \bar{y}\|. \end{aligned}$$

Hence

$$\|h_1 - h_2\| \le \|\mu\|\left[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left\{\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\right\}\right]\|y - \bar{y}\|.$$

Analogously, switching the roles of y and $\bar{y}$, we can obtain

$$H_d(\mathcal{M}(y), \mathcal{M}(\bar{y})) \le \|\mu\|\left[\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left\{\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \sum_{i=1}^{m} |\sigma_i| \frac{\eta_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\right\}\right]\|y - \bar{y}\|.$$

So $\mathcal{M}$ is a contraction. Thus, by Covitz and Nadler's fixed point theorem [28], the operator $\mathcal{M}$ has a fixed point y, which corresponds to a solution of (1.2). □

Example 4.1. *Consider the following boundary value problem*

$$\begin{cases} {}^{1/3}_{\ c}D_{0^+}^{5/4} y \in F(t,y), \ t \in [0,2], \\ y(2) = 2\, {}^{1/3}I^{3/4} y(1/2) + 1/2 \ {}^{1/3}I^{3/4} y(3/2) + 1/4, \quad \delta y(0) = 0, \end{cases} \tag{4.3}$$

where $F(t,y)$ will be fixed later.

For illustrating Theorem 4.1, we take

$$F(t,y)=\left[\frac{e^{-t}}{\sqrt{900+t}}\left(\sin y+\frac{1}{2}\right),\ \frac{(1+t)}{30}\left(\frac{|y|}{|y|+1}+y+\frac{1}{8}\right)\right]. \tag{4.4}$$

Using the given data, we find $p(t)=\frac{(1+t)}{30}, \varphi(\|y\|)=\|y\|+\frac{9}{8}$, and by condition (B_3), we have $\widehat{W}>0.7066246467$. Thus all conditions of Theorem 4.1 are satisfied and consequently, there exists at least one solution for the problem (4.3) with $F(t,y)$ given by (4.4) on $[0,2]$.

In order to demonstrate the application of Theorem 4.2, let us choose

$$F(t,y)=\left[\frac{e^{-t}}{\sqrt{900+t}}\left(\tan^{-1}y+\frac{1}{2}\right),\ \frac{(1+t)}{30}\left(\frac{|y|}{|y|+1}+\frac{1}{8}\right)\right]. \tag{4.5}$$

Clearly

$$H_d(F(t,y),F(t,\bar{y}))\leq\frac{(t+1)}{30}\|y-\bar{y}\|.$$

Letting $\mu(t)=\frac{(t+1)}{30}$, it is easy to check that $d(0,F(t,0))\leq\mu(t)$ holds for almost all $t\in[0,2]$ and $\vartheta\approx 0.7572001575<1$ (ϑ is given by 4.2). As the hypotheses of Theorem 4.2 are satisfied, we conclude that the problem (4.3) with $F(t,y)$ given by (4.5) has at least one solution on $[0,2]$.

5. Conclusion

We have developed the existence theory for fractional differential equations and inclusions involving Caputo-type generalized fractional derivative equipped with generalized fractional integral boundary conditions (in the sense of Katugampola). Standard fixed point theorems for single-valued and multi-valued maps are employed to obtain the desired results, which are well illustrated with the aid of examples. Our results are new in the given configuration and contribute significantly to the existing literature on the topic.

Acknowledgments

The authors gratefully acknowledge the referees for their useful comments on their paper.

Conflict of Interest

The authors declare that they have no conflict of interests.

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