Electronic
Research Archive

https://www.aimspress.com/journal/era

ERA, 34(7): 4325–4358.
DOI: 10.3934/era.2026192
Received: 19 January 2026
Revised: 29 April 2026
Accepted: 07 May 2026
Published: 20 May 2026

Research article

Discontinuous Galerkin method for hybrid–dimensional fracture models of the advection–diffusion equation

Shuangshuang Chen* **and Longchao Jin**

School of Mathematics, Statistics and Mechanics, Beijing University of Technology, Beijing 100124, China

* **Correspondence:** Email: Chenshuangshuang@bjut.edu.cn.

Abstract: This paper presents a numerical method for the advection-diffusion process in fractured porous media described by a kind of hybrid–dimensional fracture models, in which fractures are represented as $(n-1)$–dimensional interfaces embedded within the n–dimensional porous matrix. We develop a numerical approach that combines the discontinuous Galerkin (DG) method for spatial discretization with the backward Euler method for temporal discretization. Leveraging the advantage of the DG method in flexibly handling boundary conditions, the DG–backward Euler method is extended from the model with a single fracture to one with intersecting fractures. Theoretical analysis shows that the optimal order error estimate for the concentration in the discrete H^1–norm is obtained. Numerical experiments with a single fracture and intersecting fractures are all carried out to verify the accuracy of the theoretical results.

Keywords: hybrid–dimensional fracture model; advection–diffusion equation; discontinuous Galerkin method; backward–Euler method

1. Introduction

Modeling flows in fractured porous media has received increasing interest over recent decades, owing to its fundamental significance in addressing key environmental and energy issues, such as geothermal extraction [1, 2], geological carbon storage [3, 4], gas and oil exploitation [5–7], and contaminant migration in reservoirs [8]. In these applications, the flow dynamics are significantly influenced by the presence of fractures. Depending on the hydraulic properties, fractures exhibit diverse physical characteristics: They can serve as primary pathways for fluid flow or act as barriers, depending on their permeability levels. In addition, when considering fractures and the matrix together, the physical differences between them can be substantial, sometimes varying by several orders of magnitude. As a result, fractured porous media often exhibit strong heterogeneity. Moreover, fractures typically have much smaller thickness compared with the characteristic scale of the domain, and are often associated with significantly more complex physical processes than those in the surrounding matrix. This feature makes the modeling and numerical simulation of fluid flow in fractured porous media exceptionally complex.

When fractures have a small aperture, the equal-dimensional models are impractical to a certain extent, where fractures are represented as geological structures of equal dimension to the matrix, as capturing flow within such thin fractures would require an excessively large number of grid cells. To address this issue, it is a common choice of hybrid–dimensional fracture models to represent fractures as $(n-1)$–dimensional interfaces embedded in the n–dimensional domain, such as surfaces in the three-dimensional domain or lines in the two-dimensional case, overcoming the constraint of fracture width in mesh partitioning to enhance the flexibility and practicality [9, 10]. The hybrid–dimensional fracture model was initially developed for single–phase Darcy flow, presented in [9], where pressure was assumed to be continuous across the interfaces of the matrix and the fractures for fractures acting as drains. The model was subsequently generalized in [10] to include Robin–type coupling conditions, allowing for pressure discontinuities to represent fractures functioning either as drains or as barriers. The hybrid–dimensional fracture models of the Darcy–Brinkman equation [11, 12] and the Darcy–Forchheimer equation [13] were also established, as well as the two–phase Darcy flow [14–16]. The theoretical framework for identifying discontinuous interfaces from limited boundary measurements in elastic wave inverse problems [17–19] may inform future extensions of our model to the reconstruction of fracture networks.

Numerical methods developed for this kind of fracture models include finite element schemes with both matching and non–matching grids at the matrix–fracture interfaces [10, 13, 20–22], extended finite element methods [23], finite volume methods [24], and finite difference approximations [25]. The advection–diffusion equation encompasses two processes, advection and diffusion, and thus can more comprehensively and accurately reflect the migration laws of pollutants. It is particularly suitable for simulating the behavior of pollutants in natural water bodies such as rivers, lakes, and groundwater, as well as in sewage treatment systems. The hybrid-dimensional fracture model for the advection-diffusion equation was proposed in [26], where the flow in the surrounding matrix was assumed to be governed by the advection-diffusion equation, and a suitable reduced version of this equation was formulated on the surface modeling the fracture. Physically consistent coupling conditions were added to account for the exchange of flow between the fracture and the matrix.

This paper presents research on the numerical simulation of this model based on the discontinuous Galerkin (DG)–backward Euler method. The DG method [27] inherently ensures local mass conservation, flexibility in handling complex geometries, and high–order accuracy and offers greater flexibility in handling general mesh configurations compared with standard finite element methods, which has been extensively used in numerical fluid simulations, including applications such as convection–diffusion equations [28], transport problems [29], and miscible displacement in porous media [30]. For the Darcy flow in fractured porous media, a scheme combining the interior penalty technique and the hybridized DG method was proposed in [31], which made substantial contributions to theoretical analysis and adaptive algorithms, DG methods based on polytopic and polyhedral grids were proposed in [32–34], and the well-posedness for schemes and optimal order error estimates were analyzed. A novel residual–type a posteriori error estimator for staggered DG methods on general polygonal meshes was designed and analyzed in [35]. DG methods were also used to Darcy–Forchheimer flow in [36], and multicomponent fluid flow in [37, 38].

For hybrid-dimensional fracture models of Darcy flow, there has been extensive research, including the construction of numerical schemes and theoretical analyses, such as mixed finite element methods with matching grids [10] and nonmatching grids [20], DG methods [32, 33], finite difference methods [39], and finite volume methods [24]. However, for hybrid–dimensional fracture models based on the advection–diffusion equation, research has focused more on algorithm design, with less attention given to theoretical studies. Therefore, in this paper, we construct numerical algorithms and, more importantly, we complete the theoretical analysis of optimal order error estimates. We consider not only a single fracture but also the design, analysis, and implementation of numerical algorithms for intersecting fractures. For intersecting fracture models, the intersections can be treated as boundaries for each individual fracture. We are well aware that the DG method has the advantage of flexibly handling discontinuities and non-matching grids. In addition, it also possesses certain advantages in handling boundary conditions. By leveraging this advantage, we can extend both the algorithms' design and theoretical analysis to intersecting fracture models.

The outline of this paper is organized as follows. Section 2 introduces the hybrid–dimensional fracture model of the advection–diffusion equation with a single fracture. Section 3 is devoted to the discretization of the model using the DG-backward Euler method, and the optimal order error estimate for the concentration in the discrete H^1–norm is analyzed. In Section 4, the DG–backward Euler method is extended to the model with intersecting fractures, and a corresponding theoretical analysis is provided. In Section 5, numerical examples with both a single fracture and intersecting fractures are presented to demonstrate the accuracy of the proposed algorithm. Finally, Section 6 summarizes the main findings of this study.

2. The hybrid–dimensional fracture model with one single fracture

(a) **(b)**

Figure 1. The hybrid–dimensional fracture model with one single fracture in two–dimensional domain. (a) Domain divided into two sub–domains Ω_1 and Ω_2 by a thin region Ω_f. (b) Fracture γ modeled as one-dimensional domain.

Let $\Omega \subset \mathbb{R}^n$ (with $n = 2$ or 3) be a convex polygonal domain. Assume that a fracture Ω_f subdivides Ω into two connected matrix domains Ω_i, $i = 1, 2$,

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_f, \quad \text{and} \quad \Omega_1 \cap \Omega_f = \emptyset, \quad \Omega_2 \cap \Omega_f = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset.$$

Let $\Gamma = \partial\Omega$ denote the boundary of Ω, and Γ_i is the part of the boundary of Ω_i in common with the boundary of Ω, i.e., $\Gamma_i = \partial\Omega_i \cap \Gamma, i = 1, 2$. Given that the fracture width d is significantly smaller than the characteristic size of the domain, the model represents the fracture as an interface, which implements a hybrid-dimensional approach. Here, following from the detailed derivation in [10], we rewrite the fracture $\Omega_f = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_\gamma + \mu\frac{d}{2}\mathbf{n}\}$, where $\mathbf{x}_\gamma \in \gamma, \mu \in (-1, 1)$, and $\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$ is the unit vector normal to γ at the point $\mathbf{x}_\gamma$ directed from Ω_1 to Ω_2, as shown in Figure 1. Moving to the advection–diffusion problem, we use $c = (c_1, c_2, c_\gamma)$ to denote the concentration, defined as the volume fraction of the tracer within the porosity, and $\chi = (\chi_1, \chi_2, \chi_\gamma)$ denotes the total flux. The model is consequently described by the following set of equations for $t \in [0, T]$.

In the matrix Ω_i, we have

$$\begin{aligned} \phi_i\frac{\partial c_i}{\partial t} + \nabla\cdot\chi_i &= q_i && \text{in } \Omega_i, \quad i = 1, 2, \\ \chi_i &= -\mathbf{K}_i\nabla c_i + \mathbf{u}_i c_i && \text{in } \Omega_i, \quad i = 1, 2, \\ c_i &= c_{\partial,i} && \text{on } \Gamma_i, \quad i = 1, 2. \end{aligned} \tag{2.1}$$

In the fracture γ, we have

$$\begin{aligned} d\phi_\gamma\frac{\partial c_\gamma}{\partial t} + \nabla_\tau\cdot\chi_\gamma &= q_\gamma + \chi_1\cdot\mathbf{n}_1 + \chi_2\cdot\mathbf{n}_2 && \text{in } \gamma, \\ \chi_\gamma &= -\mathbf{K}_{\gamma,\tau}d\nabla_\tau c_\gamma + d\mathbf{u}_\gamma c_\gamma && \text{in } \gamma, \\ c_\gamma &= c_{\partial,\gamma} && \text{on } \partial\gamma. \end{aligned} \tag{2.2}$$

For the ransmission conditions in γ, we have

$$-\xi\chi_i\cdot\mathbf{n_i} + (1-\xi)\chi_{i+1}\cdot\mathbf{n}_{i+1} = 2\eta_\gamma\left(c_\gamma - c_i\right) \quad \text{in } \gamma, \quad i = 1, 2. \tag{2.3}$$

At the initial time $t = 0$, we have

$$\begin{aligned} c_i &= c_i^0 && \text{in } \Omega_i, \quad i = 1, 2, \\ c_\gamma &= c_\gamma^0 && \text{in } \gamma, \end{aligned} \tag{2.4}$$

where $q_i, i = 1, 2$, q_γ are the source/sink terms in the matrix and fracture. We use the notations ∇_τ and $\nabla_\tau\cdot$ for the tangential gradient and divergence operators along γ, respectively. We suppose that the parameters $\phi_i \in L^\infty(\Omega_i)$, $\phi_\gamma \in L^\infty(\gamma)$, $\mathbf{u}_i \in (C^1(\bar{\Omega}_i))^n$, and $\mathbf{u}_\gamma \in (C^1(\bar{\gamma}))^{n-1}$. The permeability tensor in the fracture is $\mathbf{K}_\gamma = diag(\mathbf{K}_{\gamma,\mathbf{n}}, \mathbf{K}_{\gamma,\tau})$, $\eta_\gamma = \frac{\mathbf{K}_{\gamma,\mathbf{n}}}{d}$, and $\frac{1}{2} < \xi \le 1$. The index i varies in $\mathbb{Z}/2\mathbb{Z}$, satisfying $2 + 1 = 1$. We suppose that the permeability tensors $\mathbf{K}_i, i = 1, 2$, and $\mathbf{K}_\gamma$ are symmetric and positive, and the constants $\bar{k} > \underline{k} > 0$ exist such that

$$\begin{aligned} \underline{k}|\zeta_i|^2 \le \zeta_i^\top\mathbf{K}_i\zeta_i \le \bar{k}|\zeta_i|^2, \quad \forall\zeta_i \in \mathbb{R}^n, \\ \underline{k} \le 2\eta_\gamma \le \bar{k}, \quad \underline{k}|\zeta_\gamma|^2 \le \zeta_\gamma^\top\mathbf{K}_{\gamma,\tau}d\zeta_\gamma \le \bar{k}|\zeta_\gamma|^2, \quad \forall\zeta_\gamma \in \mathbb{R}^{n-1}. \end{aligned} \tag{2.5}$$

For simplicity, assume homogeneous Dirichlet boundary conditions for concentration c in our subsequent analysis, that is, $c_{\partial,i} = 0$ and $c_{\partial,\gamma} = 0$. Note that C denotes a generic positive constant in this paper.

3. The DG-backward Euler method

First, we address the partition of the considered domain. Assume that $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ are non-degenerate triangular or tetrahedral meshes of Ω_1 and Ω_2, with h being the maximum element diameter, and they coincide on the interface γ to form a unique partition for γ, denoted by $\mathcal{T}_\gamma^h$. Let the global partition for the entire domain be $\mathcal{T}^h = \mathcal{T}_1^h \cup \mathcal{T}_2^h \cup \mathcal{T}_\gamma^h$. We define, respectively, the collections of all interior edges or faces for $\mathcal{T}_i^h$ and the set of all interior points or edges for $\mathcal{T}_\gamma^h$ as $\mathcal{E}_{o,i}^h$ and $\mathcal{E}_{o,\gamma}^h$, and similarly define the sets of exterior edges or faces on Γ_i and the exterior points or edges on $\partial\gamma$ by $\mathcal{E}_{\partial,i}^h$ and $\mathcal{E}_{\partial,\gamma}^h$. Set $\mathcal{E}_i^h = \mathcal{E}_{o,i}^h \cup \mathcal{E}_{\partial,i}^h$ and $\mathcal{E}_\gamma^h = \mathcal{E}_{o,\gamma}^h \cup \mathcal{E}_{\partial,\gamma}^h$.

In this article, we utilize the traditional Lebesgue spaces $L^\ell(D)$ and the Sobolev spaces $W^{k,\ell}(D)$ with $1 \le \ell \le \infty$ over a domain D. In particular, $W^{k,2}(D)$ is denoted by $H^k(D)$ with $H^0(D) = L^2(D)$, and the corresponding norms are denoted by $\|\cdot\|_{W^{k,\ell}(D)}$ and $\|\cdot\|_{H^k(D)}$, as well as seminorms $|\cdot|_{W^{k,\ell}(D)}$ and$|\cdot|_{H^k(D)}$. To simplify the notation, we also denote $\|\cdot\|_{W^{k,\ell}(D)}$ and $|\cdot|_{W^{k,\ell}(D)}$ by $\|\cdot\|_{k,\ell,D}$ and $|\cdot|_{k,\ell,D}$, respectively, and $\|\cdot\|_{H^k(D)}$ and $|\cdot|_{H^k(D)}$ by $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$. For $k > 0$, the broken Sobolev space is given by $H^k\left(\mathcal{T}^h\right) = H^k\left(\mathcal{T}_1^h\right) \times H^k\left(\mathcal{T}_2^h\right) \times H^k\left(\mathcal{T}_\gamma^h\right)$ with

$$
\begin{aligned}
H^k\left(\mathcal{T}_i^h\right) &:= \left\{z_i \in L^2\left(\Omega_i\right) : z_i|_K \in H^k(K), \forall K \in \mathcal{T}_i^h\right\}, \quad i = 1, 2, \\
H^k\left(\mathcal{T}_\gamma^h\right) &:= \left\{z_\gamma \in L^2(\gamma) : z_\gamma\big|_K \in H^k(K), \forall K \in \mathcal{T}_\gamma^h\right\}.
\end{aligned}
$$

Let $e \in \mathcal{E}_{o,i}^h$ or $\mathcal{E}_{o,\gamma}^h$ be shared by two elements K_e^1 and K_e^2, i.e., $e = K_e^1 \bigcap K_e^2$, with $\mathbf{n}_e$ being a unit normal vector exterior to K_e^1. Then we define the average and jump values for $z \in H^k\left(\mathcal{T}^h\right)\left(k > \frac{1}{2}\right)$ on e by

$$
\{z\} := \frac{1}{2}\left(\left(z|_{K_e^1}\right)\Big|_e + \left(z|_{K_e^2}\right)\Big|_e\right), \quad [z] := \left(z|_{K_e^1}\right)\Big|_e - \left(z|_{K_e^2}\right)\Big|_e .
$$

For $e \in \mathcal{E}_{\partial,i}^h$ or $\mathcal{E}_{\partial,\gamma}^h, \{z\} := \left(z|_{K_e}\right)\Big|_e, [z] := \left(z|_{K_e}\right)\Big|_e$, with $e \subset \partial K_e$, and $\mathbf{n}_e$ coincides with the outward normal vector on Γ_i or $\partial\gamma$.

In the matrix $\Omega_i, i = 1, 2$, multiplying the first equation in (2.1) by a test function $z_i \in H^1\left(\mathcal{T}_i^h\right)$, integrating within each element $K \in \mathcal{T}_i^h$, and using Green's formulation, we have

$$
\int_K \phi_i \frac{\partial c_i}{\partial t} z_i - \int_K \chi_i \cdot \nabla z_i + \int_{\partial k} \chi_i \cdot \mathbf{n}_{\partial k} z_i = \int_K q_i z_i, \tag{3.1}
$$

where $\mathbf{n}_{\partial K}$ is the outward normal vector on ∂K. By subtracting and adding the two conditions in (2.3), we obtain

$$
\begin{aligned}
\chi_1 \cdot \mathbf{n}_1 - \chi_2 \cdot \mathbf{n}_2 &= 2\eta_\gamma \left(c_1 - c_2\right), \\
\chi_1 \cdot \mathbf{n}_1 + \chi_2 \cdot \mathbf{n}_2 &= \frac{2\eta_\gamma}{2\xi - 1}\left(c_1 + c_2 - 2c_\gamma\right).
\end{aligned} \tag{3.2}
$$

Subtracting and adding the two equations in (3.2) then shows that

$$
\chi_i \cdot \mathbf{n}_i = \frac{2\eta_\gamma}{2\xi - 1}\left(\xi c_i + (1 - \xi) c_{i+1} - c_\gamma\right), \quad \text{in } \gamma, \quad i = 1, 2. \tag{3.3}
$$

Using the second equation in (2.1) and the second equation in (3.3), we find that

$$
\begin{aligned}
&\int_K \phi_i \frac{\partial c_i}{\partial t} z_i + \int_K \mathbf{K}_i \nabla c_i \cdot \nabla z_i - \int_{\partial K \setminus \gamma} \mathbf{K}_i \nabla c_i \cdot \mathbf{n}_{\partial K} z_i - \int_K \mathbf{u}_i c_i \cdot \nabla z_i + \int_{\partial K \setminus \gamma} \mathbf{u}_i c_i \cdot \mathbf{n}_{\partial K} z_i \\
&\quad + \int_{\partial K \bigcap \gamma} \frac{2\eta_\gamma}{2\xi - 1}\left(\xi c_i + (1 - \xi) c_{i+1} - c_\gamma\right) z_i = \int_K q_i z_i, \quad \forall K \in \mathcal{T}_i^h.
\end{aligned}
$$

Summing the equation above over all elements in $\mathcal{T}_i^h$, we obtain the following for any $z_i \in H^1\left(\mathcal{T}_i^h\right)$:

$$\begin{aligned}&\int_{\Omega_i} \phi_i \frac{\partial c_i}{\partial t} z_i + \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{K}_i \nabla c_i \cdot \nabla z_i - \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{K}_i \nabla c_i \cdot \mathbf{n}_e\}[z_i] - \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{u}_i c_i \cdot \nabla z_i \\ &\quad + \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{u}_i c_i \cdot \mathbf{n}_e\}[z_i] + \int_\gamma \frac{2\eta_\gamma}{2\xi - 1}\left(\xi c_i + (1-\xi) c_{i+1} - c_\gamma\right) z_i = \int_{\Omega_i} q_i z_i.\end{aligned} \tag{3.4}$$

Similarly, in the fracture γ, we have the following for any $z_\gamma \in H^1\left(\mathcal{T}_\gamma^h\right)$:

$$\begin{aligned}&\int_\gamma d\phi_\gamma \frac{\partial c_\gamma}{\partial t} z_\gamma + \sum_{K\in\mathcal{T}_\gamma^h} \int_K \mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \nabla_\tau z_\gamma - \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] - \sum_{K\in\mathcal{T}_\gamma^h} \int_K d\mathbf{u}_\gamma c_\gamma \cdot \nabla_\tau z_\gamma \\ &\quad + \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{d\mathbf{u}_\gamma c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] + \int_\gamma \frac{2\eta_\gamma}{2\xi - 1}\left(c_1 + c_2 - 2c_\gamma\right) z_\gamma = \int_\gamma q_\gamma z_\gamma.\end{aligned} \tag{3.5}$$

Based on (3.4) and (3.5), we define the following bilinear forms:

$$\begin{aligned}A(c,z) =& \sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{K}_i \nabla c_i \cdot \nabla z_i - \sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{K}_i \nabla c_i \cdot \mathbf{n}_e\}[z_i] + \nu \sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{K}_i \nabla z_i \cdot \mathbf{n}_e\}[c_i] \\ &+ \sum_{K\in\mathcal{T}_\gamma^h} \int_K \mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \nabla_\tau z_\gamma - \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] + \nu \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau z_\gamma \cdot \mathbf{n}_e\}[c_\gamma] \\ &+ \sigma J(c,z) + \sum_{i=1,2} \int_\gamma \frac{2\eta_\gamma}{2\xi - 1}\left(\xi c_i + (1-\xi) c_{i+1} - c_\gamma\right)(z_i - z_\gamma), \\ B(c,z) =& -\sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{u}_i c_i \cdot \nabla z_i + \sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{u}_i c_i \cdot \mathbf{n}_e\}[z_i] - \sum_{K\in\mathcal{T}_\gamma^h} \int_K d\mathbf{u}_\gamma c_\gamma \cdot \nabla_\tau z_\gamma \\ &+ \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{d\mathbf{u}_\gamma c_\gamma \cdot \mathbf{n}_e\}[z_\gamma].\end{aligned} \tag{3.6}$$

In the form above, we take $\nu \in \{-1, 0, 1\}$, σ as a positive constant, and the interior penalty term

$$J(c,z) = \sum_{i=1,2} J_i(c_i, z_i) + J_\gamma\left(c_\gamma, z_\gamma\right),$$

where

$$J_i(c_i, z_i) = \sum_{e\in\mathcal{E}_i^h} \frac{1}{|e|} \int_e [c_i][z_i], \quad i = 1, 2,$$

and

$$J_\gamma\left(c_\gamma, z_\gamma\right) = \begin{cases} \sum\limits_{e\in\mathcal{E}_\gamma^h} \frac{1}{|e|} \int_e [c_\gamma][z_\gamma], & \text{two-dimensional fracture,} \\ \sum\limits_{e\in\mathcal{E}_{o,\gamma}^h} \frac{1}{2\left(|K_e^1| + |K_e^2|\right)}[c_\gamma][z_\gamma] + \sum\limits_{e\in\mathcal{E}_{\partial,\gamma}^h} \frac{1}{|K_e|}[c_\gamma][z_\gamma], & \text{one-dimensional fracture.}\end{cases}$$

For the exact solution $c = (c_1, c_2, c_\gamma)$ of the hybrid-dimensional fracture model with one single fracture in (2.1)–(2.4), the following equality holds:

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i \frac{\partial c_i}{\partial t} z_i + \int_\gamma d\phi_\gamma \frac{\partial c_\gamma}{\partial t} z_\gamma + A(c,z) + B(c,z) = \sum_{i=1,2} \int_{\Omega_i} q_i z_i + \int_\gamma q_\gamma z_\gamma, \tag{3.7}$$

for any $z = (z_1, z_2, z_\gamma) \in H^1(\mathcal{T}^h), t \in (0, T]$.

As concerns the time discretization, a backward Euler method is used. We divide the time interval $[0, T]$ into N equal subintervals of length Δt, $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$. Use $t^j = j\Delta t$ and ψ^j to denote the function ψ at time t^j.

Let $\mathbb{P}_k(K)$ denote the space of polynomials of total degree less than or equal to k in the element K, and the discontinuous finite element space can be defined as $\mathcal{D}_k(\mathcal{T}^h) = \mathcal{D}_k(\mathcal{T}_1^h) \times \mathcal{D}_k(\mathcal{T}_2^h) \times \mathcal{D}_k(\mathcal{T}_\gamma^h)$ with

$$\begin{aligned} \mathcal{D}_k(\mathcal{T}_i^h) &:= \{z_i \in L^2(\Omega_i) : z_i|_K \in \mathbb{P}_k(K),\ \forall K \in \mathcal{T}_i^h\}, \quad i = 1, 2, \\ \mathcal{D}_k(\mathcal{T}_\gamma^h) &:= \{z_\gamma \in L^2(\gamma) : z_\gamma|_K \in \mathbb{P}_k(K),\ \forall K \in \mathcal{T}_\gamma^h\}. \end{aligned}$$

We then approximate the concentration in $\mathcal{D}_k(\mathcal{T}^h)$, and the DG–backward Euler method discretization formation is described as follows.

Algorithm 1 (DG–backward Euler method for the model with one single fracture)**.** For $j = 1, 2, ..., N$, given $c^{h,j-1} = \left(c_1^{h,j-1}, c_2^{h,j-1}, c_\gamma^{h,j-1}\right)$, find $c^{h,j} = \left(c_1^{h,j}, c_2^{h,j}, c_\gamma^{h,j}\right) \in \mathcal{D}_k(\mathcal{T}^h)$ such that for any $z = (z_1, z_2, z_\gamma) \in \mathcal{D}_k(\mathcal{T}^h)$, we have

$$\begin{aligned} \sum_{i=1,2} \int_{\Omega_i} \phi_i \frac{c_i^{h,j} - c_i^{h,j-1}}{\Delta t} z_i + \int_\gamma d\phi_\gamma \frac{c_\gamma^{h,j} - c_\gamma^{h,j-1}}{\Delta t} z_\gamma + A(c^{h,j}, z) + B(c^{h,j}, z) &= \sum_{i=1,2} \int_{\Omega_i} q_i z_i + \int_\gamma q_\gamma z_\gamma, \\ \sum_{i=1,2} \int_{\Omega_i} c_i^{h,0} z_i + \int_\gamma c_\gamma^{h,0} z_\gamma = \sum_{i=1,2} \int_{\Omega_i} c_i^0 z_i + \int_\gamma c_\gamma^0 z_\gamma&. \end{aligned} \tag{3.8}$$

The norms of interest are given below. For $z = \left(z_1, z_2, z_\gamma\right) \in H^k\left(\mathcal{T}^h\right)$, we have

$$|||z|||^2 = ||z||_{1,\mathcal{T}^h}^2 + J(z,z) + \sum_{i=1,2} \left\|z_i - z_\gamma\right\|_{0,\gamma}^2, \tag{3.9}$$

with

$$||z||_{1,\mathcal{T}^h}^2 = \sum_{i=1,2} ||z_i||_{1,\mathcal{T}_i^h}^2 + \left\|z_\gamma\right\|_{1,\mathcal{T}_\gamma^h}^2, \quad ||z_i||_{1,\mathcal{T}_i^h}^2 = \sum_{K \in \mathcal{T}_i^h} |z_i|_{1,K}^2, \quad \left\|z_\gamma\right\|_{1,\mathcal{T}_\gamma^h}^2 = \sum_{K \in \mathcal{T}_\gamma^h} \left|z_\gamma\right|_{1,K}^2,$$

in which the terms $||z_i - z_\gamma||_{0,\gamma}, i = 1, 2$, are added in the definition of $||| \cdot |||$-norm, in order to obtain the coercivity and continuity of the form $A(\cdot, \cdot)$ and then obtain the optimal order error estimate for the concentration.

Meanwhile, we define

$$||z||_{2,\mathcal{T}^h}^2 = \sum_{i=1,2} ||z_i||_{2,\mathcal{T}_i^h}^2 + \left\|z_\gamma\right\|_{2,\mathcal{T}_\gamma^h}^2, \tag{3.10}$$

with $||z_i||_{2,\mathcal{T}_i^h}^2 = \sum_{K \in \mathcal{T}_i^h} |z_i|_{2,K}^2$, $\left\|z_\gamma\right\|_{2,\mathcal{T}_\gamma^h}^2 = \sum_{K \in \mathcal{T}_\gamma^h} |z_\gamma|_{2,K}^2$, and the L^2-norm in the entire domain is defined by

$$||z||_{0,\Omega}^2 = \sum_{i=1,2} ||z_i||_{0,\Omega_i}^2 + ||z_\gamma||_{0,\gamma}^2.$$

Lemma 1. *(See [40]) If we let $K \in \mathcal{T}^h$ and $z \in \mathbb{P}_k(K)$, then a constant C independent of z, k and h exists such that*

$$\|z\|_{0,\partial K} \le Ch^{-\frac{1}{2}}\|z\|_{0,K}.$$

Lemma 2. *(See [41]) If we let $K \in \mathcal{T}^h$ and $z \in H^1(K)$, then a constant C independent of z and h exists such that*

$$\|z\|_{0,e}^2 \le C\left(h^{-1}\|z\|_{0,K}^2 + h\|\nabla z\|_{0,K}^2\right), \quad \forall e \subset \partial K.$$

Lemma 3. *For any $c, z \in \mathcal{D}_k(\mathcal{T}^h)$, a constant C exists, and $\alpha > 0$ satisfies*

$$A(c,z) \le C|||c|||\ |||z|||, \tag{3.11}$$

$$A(c,c) \ge \alpha|||c|||^2. \tag{3.12}$$

Proof. By using the definition of $A(c,z)$, we have

$$\begin{aligned}
A(c,z) =& \sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\int_K \mathbf{K}_i\nabla c_i\cdot\nabla z_i + \sum_{K\in\mathcal{T}_\gamma^h}\int_K \mathbf{K}_{\gamma,\tau}d\nabla_\tau c_\gamma\cdot\nabla_\tau z_\gamma - \sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}\int_e\{\mathbf{K}_i\nabla c_i\cdot\mathbf{n}_e\}[z_i]\\
&-\sum_{e\in\mathcal{E}_\gamma^h}\int_e\{\mathbf{K}_{\gamma,\tau}d\nabla_\tau c_\gamma\cdot\mathbf{n}_e\}[z_\gamma] + \nu\sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}\int_e\{\mathbf{K}_i\nabla z_i\cdot\mathbf{n}_e\}[c_i]\\
&+\nu\sum_{e\in\mathcal{E}_\gamma^h}\int_e\{\mathbf{K}_{\gamma,\tau}d\nabla_\tau z_\gamma\cdot\mathbf{n}_e\}[c_\gamma]+\sigma J(c,z)+\sum_{i=1,2}\int_\gamma\frac{2\eta_\gamma}{2\xi-1}\left(\xi c_i+(1-\xi)c_{i+1}-c_\gamma\right)(z_i-z_\gamma)\\
:=& I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.
\end{aligned} \tag{3.13}$$

By using the Cauchy–Schwarz inequality and the assumption of $\mathbf{K}_i$, I_1 can be bounded as

$$\begin{aligned}
I_1 &\le \bar{k}\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\|\nabla c_i\|_{0,K}\|\nabla z_i\|_{0,K} \le \bar{k}\left(\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\|\nabla c_i\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\|\nabla z_i\|_{0,K}^2\right)^{\frac{1}{2}}\\
&\le \bar{k}\left(\sum_{i=1,2}\|c_i\|_{1,\mathcal{T}_i^h}^2\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\|z_i\|_{1,\mathcal{T}_i^h}^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.14}$$

Similarly, I_2 can be bounded as

$$I_2 \le \bar{k}\|c_\gamma\|_{1,\mathcal{T}_\gamma^h}\|z_\gamma\|_{1,\mathcal{T}_\gamma^h}. \tag{3.15}$$

From Lemma 1 and the Cauchy–Schwarz inequality, I_3 can be estimated as

$$\begin{aligned}
I_3 &= \sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{\partial,i}} \int_e \mathbf{K}_i \nabla c_i \cdot \mathbf{n}_e|_{K_e} [z_i] + \frac{1}{2} \sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{o,i}} \int_e \left(\mathbf{K}_i \nabla c_i \cdot \mathbf{n}_e|_{K^1_e} + \mathbf{K}_i \nabla c_i \cdot \mathbf{n}_e|_{K^2_e} \right) [z_i] \\
&\leq \bar{k} \sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{\partial,i}} \|\nabla c_i \cdot \mathbf{n}_e|_{K_e}\|_{0,e} \|[z_i]\|_{0,e} + \frac{\bar{k}}{2} \sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{o,i}} (\left\|\nabla c_i \cdot \mathbf{n}_e|_{K^1_e}\right\|_{0,e} + \left\|\nabla c_i \cdot \mathbf{n}_e|_{K^2_e}\right\|_{0,e}) \|[z_i]\|_{0,e} \\
&\leq C \left(\sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{\partial,i}} h^{-1} \|\nabla c_i\|^2_{0,K_e} \right)^{\frac{1}{2}} \left(\sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{\partial,i}} \|[z_i]\|^2_{0,e} \right)^{\frac{1}{2}} \\
&\quad + C \left(\sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{o,i}} h^{-1} \left(\|\nabla c_i\|^2_{0,K^1_e} + \|\nabla c_i\|^2_{0,K^2_e} \right) \right)^{\frac{1}{2}} \left(\sum_{i=1,2} \sum_{e \in \mathcal{E}^h_{o,i}} \|[z_i]\|^2_{0,e} \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{i=1,2} \sum_{K \in \mathcal{T}^h_i} \|\nabla c_i\|^2_{0,K} \right)^{\frac{1}{2}} \left(\sum_{i=1,2} \sum_{e \in \mathcal{E}^h_i} h^{-1} \|[z_i]\|^2_{0,e} \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{i=1,2} \|c_i\|^2_{1,\mathcal{T}^h_i} \right)^{\frac{1}{2}} \left(\sum_{i=1,2} J_i(z_i, z_i) \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

Similarly, $I_4 - I_6$ can be bounded as

$$\begin{aligned}
I_4 &\leq C \|c_\gamma\|_{1,\mathcal{T}^h_\gamma} J^{\frac{1}{2}}_\gamma \left(z_\gamma, z_\gamma \right), \\
I_5 &\leq C \left(\sum_{i=1,2} \|z_i\|^2_{1,\mathcal{T}^h_i} \right)^{\frac{1}{2}} \left(\sum_{i=1,2} J_i(c_i, c_i) \right)^{\frac{1}{2}}, \\
I_6 &\leq C \|z_\gamma\|_{1,\mathcal{T}^h_\gamma} J^{\frac{1}{2}}_\gamma \left(c_\gamma, c_\gamma \right).
\end{aligned} \tag{3.17}$$

By using the Cauchy–Schwarz inequality, we have

$$I_7 \leq C \left(\sum_{i=1,2} J_i(c_i, c_i) \right)^{\frac{1}{2}} \left(\sum_{i=1,2} J_i(z_i, z_i) \right)^{\frac{1}{2}} + J^{\frac{1}{2}}_\gamma \left(c_\gamma, c_\gamma \right) J^{\frac{1}{2}}_\gamma \left(z_\gamma, z_\gamma \right). \tag{3.18}$$

By using the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}
I_8 &= \sum_{i=1,2} \int_\gamma \frac{2\eta_\gamma}{2\xi - 1} \left(\xi \left(c_i - c_\gamma \right) + (1 - \xi) \left(c_{i+1} - c_\gamma \right) \right) \left(z_i - z_\gamma \right) \\
&\leq C \sum_{i=1,2} \|c_i - c_\gamma\|_{0,\gamma} \|z_i - z_\gamma\|_{0,\gamma}.
\end{aligned} \tag{3.19}$$

Moreover, (3.14)–(3.19) yields

$$A(c, z) \leq C |||c||| \, |||z|||. \tag{3.20}$$

When both terms in (3.13) assume c, we have

$$\begin{aligned}A(c,c) =& \sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\int_K \mathbf{K}_i\nabla c_i\cdot\nabla c_i + \sum_{K\in\mathcal{T}_\gamma^h}\int_K \mathbf{K}_{\gamma,\tau}d\nabla_\tau c_\gamma\cdot\nabla_\tau c_\gamma\\ &+(\nu-1)\sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}\int_e\{\mathbf{K}_i\nabla c_i\cdot\mathbf{n}_e\}[c_i] + (\nu-1)\sum_{e\in\mathcal{E}_\gamma^h}\int_e\{\mathbf{K}_{\gamma,\tau}d\nabla_\tau c_\gamma\cdot\mathbf{n}_e\}[c_\gamma]\\ &+\sigma J(c,c) + \sum_{i=1,2}\int_\gamma\frac{2\eta_\gamma}{2\xi-1}\left(\xi c_i + (1-\xi)c_{i+1} - c_\gamma\right)(c_i - c_\gamma)\\ :=& L_1 + L_2 + L_3 + L_4 + L_5 + L_6.\end{aligned} \tag{3.21}$$

By using the the hypothesis of $\mathbf{K}_i$, we have

$$L_1 \geq \underline{k}\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}||\nabla c_i||_{0,K}^2 \geq \underline{k}\sum_{i=1,2}||c_i||_{1,\mathcal{T}_i^h}^2. \tag{3.22}$$

Similarly, we have

$$L_2 \geq \underline{k}||c_\gamma||_{1,\mathcal{T}_\gamma^h}^2. \tag{3.23}$$

Similar to I_3, L_3 can be bounded as

$$L_3 \geq -C\left(\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}||\nabla c_i||_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}h^{-1}||[c_i]||_{0,e}^2\right)^{\frac{1}{2}} \geq -\varepsilon\sum_{i=1,2}||c_i||_{1,\mathcal{T}_i^h}^2 - C\sum_{i=1,2}J_i(c_i,c_i). \tag{3.24}$$

Similarly, we have

$$L_4 \geq -\varepsilon||c_\gamma||_{1,\mathcal{T}_\gamma^h}^2 - CJ_\gamma\left(c_\gamma,c_\gamma\right). \tag{3.25}$$

The last term can be estimated as follows, according to Cauchy–Schwarz inequality and $1/2 < \xi < 1$, and we refer to [10] for the detailed proof.

$$L_6 \geq \frac{\underline{k}}{2\xi-1}(2\xi-1)\sum_{i=1,2}||c_i - c_\gamma||_{0,\gamma}^2. \tag{3.26}$$

From (3.22)–(3.26), it can be seen that

$$A(c,c) \geq \left(\underline{k}-\varepsilon\right)\sum_{i=1,2}||c_i||_{1,\mathcal{T}_i^h}^2 + \left(\underline{k}-\varepsilon\right)||c_\gamma||_{1,\mathcal{T}_\gamma^h}^2 + (\sigma - C)J(c,c) + \underline{k}\sum_{i=1,2}||c_i - c_\gamma||_{0,\gamma}^2. \tag{3.27}$$

The proof is finished by setting $\varepsilon = \frac{\underline{k}}{2}$ and a large enough σ.

Remark 1. Under the assumptions that the velocity fields in the matrix and fracture are all divergence-free, that is, $\nabla\cdot\mathbf{u}_i = 0, i = 1,2$, and $\nabla_\tau\cdot\mathbf{u}_\gamma = 0$, the fluid in the matrix cannot penetrate the interface γ, that is, $\mathbf{u}_i\cdot\mathbf{n}_i = 0, i = 1,2$, on γ, and the diffusion coefficient $\mathbf{K}_i, i = 1,2$, and $\mathbf{K}_\gamma$ satisfy assumptions in (2.5), the bilinear form of the diffusion part vanishes identically for $z = c^h$.

$$
\begin{aligned}
B\left(c^{h,j}, c^{h,j}\right) = & -\sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{u}_i c_i^{h,j} \cdot \nabla c_i^{h,j} + \sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{u}_i c_i^{h,j} \cdot \mathbf{n}_e\}[c_i^{h,j}] \\
& - \sum_{K\in\mathcal{T}_\gamma^h} \int_K d\mathbf{u}_\gamma c_\gamma^{h,j} \cdot \nabla_\tau c_\gamma^{h,j} + \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{d\mathbf{u}_\gamma c_\gamma^{h,j} \cdot \mathbf{n}_e\}[c_\gamma^{h,j}].
\end{aligned} \tag{3.28}
$$

For the first term on the right-hand side of the equation above, by using Green's formulation, we have

$$
\begin{aligned}
& -\sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{u}_i c_i^{h,j} \cdot \nabla c_i^{h,j} \\
= & -\frac{1}{2} \sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_K \mathbf{u}_i \cdot \nabla \left(c_i^{h,j}\right)^2 \\
= & \frac{1}{2} \sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_K \nabla \cdot \mathbf{u}_i \left(c_i^{h,j}\right)^2 - \frac{1}{2} \sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \int_{\partial K} \mathbf{u}_i \cdot \mathbf{n}_{\partial K} \left(c_i^{h,j}\right)^2 \\
= & -\frac{1}{2} \sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \mathbf{u}_i \cdot \mathbf{n}_e \left[\left(c_i^{h,j}\right)^2\right],
\end{aligned} \tag{3.29}
$$

where we use $\nabla \cdot \mathbf{u}_i = 0$ in Ω_i and $\mathbf{u}_i \cdot \mathbf{n}_i = 0$ on γ. Through a straightforward computation, the second term can be expressed as

$$
\sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \int_e \{\mathbf{u}_i c_i^{h,n} \cdot \mathbf{n}_e\}[c_i^{h,n}] = \frac{1}{2} \sum_{i=1,2} \sum_{e\in\mathcal{E}_i^h} \int_e \mathbf{u}_i \cdot \mathbf{n}_e \left[\left(c_i^{h,n}\right)^2\right]. \tag{3.30}
$$

Similar results hold for the third and fourth terms on the right-hand side of (3.28). Then, by using the second result in Lemma 3, the following result can be obtained in the absence of source/sink terms:

$$
\begin{aligned}
& \sum_{i=1,2} \int_{\Omega_i} \phi_i \frac{c_i^{h,j} - c_i^{h,j-1}}{\Delta t} c^{h,j} + \int_\gamma d\phi_\gamma \frac{c_\gamma^{h,j} - c_\gamma^{h,j-1}}{\Delta t} c_\gamma^{h,j} \\
= & \sum_{i=1,2} \frac{1}{2\Delta t} \left(\|\phi_i^{\frac{1}{2}} c_i^{h,j}\|_{0,\Omega_i}^2 - \|\phi_i^{\frac{1}{2}} c_i^{h,j-1}\|_{0,\Omega_i}^2 \right) + \frac{1}{2\Delta t} \left(\|\phi_\gamma^{\frac{1}{2}} c_\gamma^{h,j}\|_{0,\gamma}^2 - \|\phi_i^{\frac{1}{2}} c_\gamma^{h,j-1}\|_{0,\gamma}^2 \right) \le 0.
\end{aligned} \tag{3.31}
$$

Under the assumption of ϕ_i and ϕ_γ, we can easily get

$$
\|c^{h,j}\|_{0,\Omega}^2 \le \|c^{h,j-1}\|_{0,\Omega}^2, \tag{3.32}
$$

which indicates that the proposed scheme is L^2 stable.

Lemma 4. *For any $c \in H^1\left(\mathcal{T}^h\right)$ and $z \in D_k(\mathcal{T}^h)$, there is a constant $C > 0$ independent of h, such that*

$$
|A\,(c, z)| \le C\left(|||c||| + h\|c\|_{2,\mathcal{T}^h} + h^{-1}\|c\|_{0,\Omega}\right)|||z|||. \tag{3.33}
$$

Proof. From the definition (3.6), we know

$$\begin{aligned}
A(c,z) = & \sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\int_K \mathbf{K}_i\nabla c_i\cdot\nabla z_i + \sum_{K\in\mathcal{T}_\gamma^h}\int_K \mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma\cdot\nabla_\tau z_\gamma - \sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}\int_e\{\mathbf{K}_i\nabla c_i\cdot\mathbf{n}_e\}[z_i] \\
& - \sum_{e\in\mathcal{E}_\gamma^h}\int_e \{\mathbf{K}_{\gamma,\tau}d\nabla_\tau c_\gamma\cdot\mathbf{n}_e\}[z_\gamma] + \nu \sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}\int_e\{\mathbf{K}_i\nabla z_i\cdot \mathbf{n}_e\}[c_i] \\
& + \nu\sum_{e\in\mathcal{E}_\gamma^h}\int_e\{\mathbf{K}_{\gamma,\tau}d\nabla_\tau z_\gamma\cdot\mathbf{n}_e\}[c_\gamma] + \sigma J(c,z) + \sum_{i=1,2}\int_\gamma \frac{2\eta_\gamma}{2\xi-1}\left(\xi c_i+(1-\xi)c_{i+1}-c_\gamma\right)\left(z_i-z_\gamma\right) \\
:= & \, O_1+O_2+O_3+O_4+O_5+O_6+O_7+O_8.
\end{aligned} \tag{3.34}$$

By using the hypothesis of $\mathbf{K}_i$ and the Cauchy–Schwarz inequality, we get

$$O_1 \le \bar{k}\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\|\nabla c_i\|_{0,K}\|\nabla z_i\|_{0,K} \le \bar{k}\left(\sum_{i=1,2}\|c_i\|^2_{1,\mathcal{T}_i^h}\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\|z_i\|^2_{1,\mathcal{T}_i^h}\right)^{\frac{1}{2}}. \tag{3.35}$$

Similarly,

$$O_2 \le \bar{k}\|c_\gamma\|_{1,\mathcal{T}_\gamma^h}\|z_\gamma\|_{1,\mathcal{T}_\gamma^h}. \tag{3.36}$$

By using the hypothesis of $\mathbf{K}_i$, the Cauchy–Schwarz inequality, and Lemma 2, we have

$$\begin{aligned}
O_3 = & -\sum_{i=1,2}\sum_{e\in\mathcal{E}_{\partial,i}^h}\int_e(\mathbf{K}_i\nabla c_i\cdot\mathbf{n}_e)|_{K_e}[z_i] - \frac{1}{2}\sum_{i=1,2}\sum_{e\in\mathcal{E}_{o,i}^h}\int_e\left((\mathbf{K}_i\nabla c_i\cdot\mathbf{n}_e)|_{K_e^1}+(\mathbf{K}_i\nabla c_i\cdot\mathbf{n}_e)|_{K_e^2}\right)[z_i] \\
\le & \, \bar{k}\sum_{i=1,2}\sum_{e\in\mathcal{E}_{\partial,i}^h}\left\|\nabla c_i|_{K_e}\right\|_{0,e}\|[z_i]\|_{0,e} + \frac{\bar{k}}{2}\sum_{i=1,2}\sum_{e\in\mathcal{E}_{o,i}^h}\left(\left\|\nabla c_i|_{K_e^1}\right\|_{0,e}+\left\|\nabla c_i|_{K_e^2}\right\|_{0,e}\right)\|[z_i]\|_{0,e} \\
\le & \, C\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_{\partial,i}^h}h\left(h|c_i|^2_{2,K_e}+h^{-1}|c_i|^2_{1,K_e}\right)\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_{\partial,i}^h}h^{-1}\|[z_i]\|^2_{0,e}\right)^{\frac{1}{2}} \\
& + C\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_{o,i}^h}h\left(h|c_i|^2_{2,K_e^1}+h^{-1}|c_i|^2_{1,K_e^1}\right)\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_{o,i}^h}h^{-1}\|[z_i]\|^2_{0,e}\right)^{\frac{1}{2}} \\
& + C\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_{o,i}^h}h\left(h|c_i|^2_{2,K_e^2}+h^{-1}|c_i|^2_{1,K_e^2}\right)\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_{o,i}^h}h^{-1}\|[z_i]\|^2_{0,e}\right)^{\frac{1}{2}} \\
\le & \, C\left(\sum_{i=1,2}\left(h^2\|c_i\|^2_{2,\mathcal{T}_i^h}+\|c_i\|^2_{1,\mathcal{T}_i^h}\right)\right)^{\frac{1}{2}}\left(\sum_{i=1,2}J_i\left(z_i,z_i\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.37}$$

Similarly, we have

$$O_4 \le C\left(h^2\|c_\gamma\|^2_{2,\mathcal{T}_\gamma^h}+\|c_\gamma\|^2_{1,\mathcal{T}_\gamma^h}\right)^{\frac{1}{2}}J_\gamma^{\frac{1}{2}}\left(z_\gamma,z_\gamma\right). \tag{3.38}$$

Similar to the estimate of O_3, we have

$$\begin{aligned}
O_5 &= \nu \sum_{i=1,2} \sum_{e\in\mathcal{E}^h_{\partial,i}} \int_e (\mathbf{K}_i \nabla z_i \cdot \mathbf{n}_e)\,|_{K_e} c_i|_{K_e} \\
&\quad + \frac{\nu}{2} \sum_{i=1,2} \sum_{e\in\mathcal{E}^h_{o,i}} \int_e \Big((\mathbf{K}_i \nabla z_i \cdot \mathbf{n}_e)\,|_{K_e^1} + (\mathbf{K}_i \nabla z_i \cdot \mathbf{n}_e)\,|_{K_e^2} \Big) \Big(c_i|_{K_e^1} - c_i|_{K_e^2} \Big) \\
&\le \bar{k} \sum_{i=1,2} \sum_{e\in\mathcal{E}^h_{\partial,i}} \|\nabla z_i|_{K_e}\|_{0,e} \|c_i|_{K_e}\|_{0,e} \\
&\quad + \frac{\bar{k}}{2} \sum_{i=1,2} \sum_{e\in\mathcal{E}^h_{o,i}} \Big(\|\nabla z_i|_{K_e^1}\|_{0,e} + \|\nabla z_i|_{K_e^2}\|_{0,e} \Big) \Big(\|c_i|_{K_e^1}\|_{0,e} + \|c_i|_{K_e^2}\|_{0,e} \Big) \qquad (3.39) \\
&\le C \left(\sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \|\nabla z_i\|^2_{0,K} \right)^{\frac{1}{2}} \left(\sum_{i=1,2} \sum_{K\in\mathcal{T}_i^h} \Big(\|\nabla c_i\|^2_{0,K} + h^{-2} \|c_i\|^2_{0,K} \Big) \right)^{\frac{1}{2}} \\
&\le C \left(\sum_{i=1,2} \|z_i\|^2_{1,\mathcal{T}_i^h} \right)^{\frac{1}{2}} \left(\sum_{i=1,2} \Big(\|c_i\|^2_{1,\mathcal{T}_i^h} + h^{-2} \|c_i\|^2_{0,\Omega_i} \Big) \right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we have

$$O_6 \le C \|z_\gamma\|_{1,\mathcal{T}_\gamma^h} \Big(\|c_\gamma\|^2_{1,\mathcal{T}_\gamma^h} + h^{-2} \|c_\gamma\|^2_{0,\gamma} \Big)^{\frac{1}{2}}. \qquad (3.40)$$

We obtain the estimate for O_7 by using the Cauchy–Schwarz inequality and Lemma 2 as follows:

$$\begin{aligned}
J_i(c_i, z_i) &\le C h^{-1} \sum_{e\in\mathcal{E}_i^h} \|[c_i]\|_{0,e} \|[z_i]\|_{0,e} \\
&\le C h^{-1} \sum_{e\in\mathcal{E}^h_{\partial,i}} \|c_i|_{K_e}\|_{0,e} \|[z_i]\|_{0,e} + C h^{-1} \sum_{e\in\mathcal{E}^h_{o,i}} \Big(\|c_i|_{K_e^1}\|_{0,e} + \|c_i|_{K_e^2}\|_{0,e} \Big) \|[z_i]\|_{0,e} \\
&\le C h^{-1} \left(\sum_{e\in\mathcal{E}^h_{\partial,i}} h \|\nabla c_i\|^2_{0,K_e} + h^{-1} \|c_i\|^2_{0,K_e} \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}^h_{\partial,i}} \|[z_i]\|^2_{0,e} \right)^{\frac{1}{2}} \\
&\quad + C h^{-1} \left(\sum_{e\in\mathcal{E}^h_{o,i}} \Big(h \|\nabla c_i\|^2_{0,K_e^1} + h^{-1} \|c_i\|^2_{0,K_e^1} + h \|\nabla c_i\|^2_{0,K_e^2} + h^{-1} \|c_i\|^2_{0,K_e^2} \Big) \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}^h_{o,i}} \|[z_i]\|^2_{0,e} \right)^{\frac{1}{2}} \qquad (3.41) \\
&\le C \left(\sum_{K\in\mathcal{T}_i^h} \Big(\|\nabla c_i\|^2_{0,K} + h^{-2} \|c_i\|^2_{0,K} \Big) \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_i^h} h^{-1} \|[z_i]\|^2_{0,e} \right)^{\frac{1}{2}} \\
&\le C \Big(\|c_i\|^2_{1,\mathcal{T}_i^h} + h^{-2} \|c_i\|^2_{0,\Omega_i} \Big)^{\frac{1}{2}} J_i^{\frac{1}{2}}(z_i, z_i).
\end{aligned}$$

Similarly, we have

$$J_\gamma\big(c_\gamma, z_\gamma\big) \le C \Big(\|c_\gamma\|^2_{1,\mathcal{T}_\gamma^h} + h^{-2} \|c_\gamma\|^2_{0,\gamma} \Big)^{\frac{1}{2}} J_\gamma^{\frac{1}{2}}\big(z_\gamma, z_\gamma\big). \qquad (3.42)$$

We find that (3.41) and (3.42) yield

$$O_7 \leq C\left(\|c\|_{1,\mathcal{T}^h}^2 + h^{-2}\|c\|_{0,\Omega}^2\right)^{\frac{1}{2}} J^{\frac{1}{2}}(z,z). \tag{3.43}$$

O_8 can be estimated as,

$$\begin{aligned} O_8 &= \sum_{i=1,2}\int_\gamma \frac{2\eta_\gamma}{2\xi-1}\left(\xi\left(c_i-c_\gamma\right)+(1-\xi)\left(c_{i+1}-c_\gamma\right)\right)\left(z_i-z_\gamma\right) \\ &\leq C\sum_{i=1,2}\left(\|c_i-c_\gamma\|_{0,\gamma}^2+\|c_{i+1}-c_\gamma\|_{0,\gamma}^2\right)^{\frac{1}{2}}\|z_i-z_\gamma\|_{0,\gamma}. \end{aligned} \tag{3.44}$$

Taking the estimates for O_1–O_8 back into (3.34), the desired result holds.

Lemma 5. *For any $c, \hat{c}, z \in H^1\left(\mathcal{T}^h\right)$, a constant $C > 0$ independent of h exists such that*

$$|B(c,z) - B(\hat{c},z)| \leq C\left(\|c-\hat{c}\|_{0,\Omega}^2 + h^2\|c-\hat{c}\|_{1,\mathcal{T}^h}^2\right)^{\frac{1}{2}} |||z|||, \tag{3.45}$$

and, especially for any $c^h, \hat{c}^h, z^h \in D_k\left(\mathcal{T}^h\right)$

$$\left|B\left(c^h,z^h\right) - B\left(\hat{c}^h,z^h\right)\right| \leq C\|c^h-\hat{c}^h\|_{0,\Omega}|||z^h|||. \tag{3.46}$$

Proof. In view of the definition of $B(\cdot,\cdot)$, we have the following for $c, \hat{c}, z \in H^1\left(\mathcal{T}^h\right)$:

$$\begin{aligned} B(c,z) - B(\hat{c},z) = &-\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\int_K \mathbf{u}_i(c_i-\hat{c}_i)\cdot\nabla z_i + \sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}\int_e \{\mathbf{u}_i(c_i-\hat{c}_i)\cdot\mathbf{n}_e\}[z_i] \\ &-\sum_{K\in\mathcal{T}_\gamma^h}\int_K \mathbf{u}_\gamma\left(c_\gamma-\hat{c}_\gamma\right)\cdot\nabla_\tau z_\gamma + \sum_{e\in\mathcal{E}_\gamma^h}\int_e\{\mathbf{u}_\gamma\left(c_\gamma-\hat{c}_\gamma\right)\cdot\mathbf{n}_e\}[z_\gamma] \\ :=\ & T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{3.47}$$

Using the assumption of $\mathbf{u}_i$ and the Cauchy–Schwarz inequality, we get

$$|T_1| \leq C\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\left(\int_K |c_i-\hat{c}_i|^2\right)^{\frac{1}{2}}\left(\int_K |\nabla z_i|^2\right)^{\frac{1}{2}} \leq C\sum_{i=1,2}\|c_i-\hat{c}_i\|_{0,\Omega_i}\|z_i\|_{1,\mathcal{T}_i^h}. \tag{3.48}$$

Similarly, we have

$$|T_3| \leq C\|c_\gamma-\hat{c}_\gamma\|_{0,\gamma}\|z_\gamma\|_{1,\mathcal{T}_\gamma^h}. \tag{3.49}$$

Using the Cauchy–Schwarz inequality and Lemma 2, it follows that

$$
\begin{aligned}
|T_2| &= \sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{\partial,i}}\int_e \mathbf{u}_i\left(c_i-\hat{c}_i\right)\cdot\mathbf{n}_e[z_i] + \sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{o,i}}\int_e \frac{1}{2}\left(\mathbf{u}_i\left(c_i-\hat{c}_i\right)\cdot\mathbf{n}_e|_{K_e^1} + \mathbf{u}_i\left(c_i-\hat{c}_i\right)\cdot\mathbf{n}_e|_{K_e^2}\right)[z_i] \\
&\le C\sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{\partial,i}}\left(\int_e |\left(c_i-\hat{c}_i\right)|_{K_e}|^2\right)^{\frac{1}{2}}\left(\int_e |[z_i]|^2\right)^{\frac{1}{2}} \\
&\quad + C\sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{o,i}}\left(\int_e\left(|\left(c_i-\hat{c}_i\right)|_{K_e^1}|^2 + |\left(c_i-\hat{c}_i\right)|_{K_e^2}|^2\right)\right)^{\frac{1}{2}}\left(\int_e |[z_i]|^2\right)^{\frac{1}{2}} \\
&\le C\sum_{i=1,2}\left(\sum_{e\in\mathcal{E}^h_{\partial,i}}\left(h^{-1}\|c_i-\hat{c}_i\|^2_{0,K_e} + h\|\nabla\left(c_i-\hat{c}_i\right)\|^2_{0,K_e}\right)\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}^h_{\partial,i}}\|[z_i]\|^2_{0,e}\right)^{\frac{1}{2}} \\
&\quad + C\sum_{i=1,2}\left(\sum_{e\in\mathcal{E}^h_{o,i}}\left(h^{-1}\|c_i-\hat{c}_i\|^2_{0,K_e^1} + h^{-1}\|c_i-\hat{c}_i\|^2_{0,K_e^2} + h\|\nabla\left(c_i-\hat{c}_i\right)\|^2_{0,K_e^1} + h\|\nabla\left(c_i-\hat{c}_i\right)\|^2_{0,K_e^2}\right)\right)^{\frac{1}{2}} \\
&\quad \times\left(\sum_{e\in\mathcal{E}^h_{o,i}}\|[z_i]\|^2_{0,e}\right)^{\frac{1}{2}} \\
&\le C\left(\sum_{i=1,2}\left(\|c_i-\hat{c}_i\|^2_{0,\Omega_i} + h^2\|c_i-\hat{c}_i\|^2_{1,\mathcal{T}_i^h}\right)\right)^{\frac{1}{2}}\left(\sum_{i=1,2}\sum_{e\in\mathcal{E}_i^h}h^{-1}\|[z_i]\|^2_{0,e}\right)^{\frac{1}{2}} \\
&\le C\left(\sum_{i=1,2}\left(\|c_i-\hat{c}_i\|^2_{0,\Omega_i} + h^2\|c_i-\hat{c}_i\|^2_{1,\mathcal{T}_i^h}\right)\right)^{\frac{1}{2}}\sum_{i=1,2}J_i^{\frac{1}{2}}\left(z_i,z_i\right).
\end{aligned}
\tag{3.50}
$$

Similarly, we have the following estimate of T_4:

$$
|T_4| \le C\left(\|c_\gamma-\hat{c}_\gamma\|^2_{0,\gamma} + h^2\|c_\gamma-\hat{c}_\gamma\|^2_{1,\mathcal{T}_\gamma^h}\right)^{\frac{1}{2}} J_\gamma^{\frac{1}{2}}\left(z_\gamma,z_\gamma\right). \tag{3.51}
$$

Taking the bounds for $T_1 - T_4$ back into (3.47) yields

$$
\begin{aligned}
|B\left(c,z\right) - B\left(\hat{c},z\right)| &\le C\left(\sum_{i=1,2}\left(\|c_i-\hat{c}_i\|^2_{0,\Omega_i} + h^2\|c_i-\hat{c}_i\|^2_{1,\mathcal{T}_i^h}\right) + \|c_\gamma-\hat{c}_\gamma\|^2_{0,\gamma} + h^2\|c_\gamma-\hat{c}_\gamma\|^2_{1,\mathcal{T}_\gamma^h}\right)^{\frac{1}{2}} \\
&\quad \times\left(\sum_{i=1,2}J_i\left(z_i,z_i\right) + J_\gamma\left(z_\gamma,z_\gamma\right) + \sum_{i=1,2}\|z_i\|^2_{1,\mathcal{T}_i^h} + \|z_\gamma\|^2_{1,\mathcal{T}_\gamma^h}\right)^{\frac{1}{2}} \\
&\le C\left(\|c-\hat{c}\|^2_{0,\Omega} + h^2\|c-\hat{c}\|^2_{1,\mathcal{T}^h}\right)^{\frac{1}{2}}|||z|||.
\end{aligned}
\tag{3.52}
$$

Further, if $c^h, \hat{c}^h, z^h \in \mathcal{D}_k\left(\mathcal{T}^h\right)$, using Lemma 1, we have the following estimates for T_2 and T_4:

$$
\begin{aligned}
|T_2| \le\, & C \sum_{i=1,2} \left(\sum_{e\in\mathcal{E}^h_{\partial,i}} h^{-1} \|c_i^h - \hat{c}_i^h\|^2_{0,k_e} \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}^h_{\partial,i}} \|[z_i^h]\|^2_{0,e} \right)^{\frac{1}{2}} \\
& + C \sum_{i=1,2} \left(\sum_{e\in\mathcal{E}^h_{o,i}} h^{-1} \left(\|c_i^h - \hat{c}_i^h\|^2_{0,k_e^1} + \|c_i^h - \hat{c}_i^h\|^2_{0,k_e^2} \right) \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}^h_{o,i}} \|[z_i^h]\|^2_{0,e} \right)^{\frac{1}{2}} \\
\le\, & C \sum_{i=1,2} \|c_i^h - \hat{c}_i^h\|_{0,\Omega_i} \left(\sum_{e\in\mathcal{E}^h_{i}} h^{-1} \|[z_i^h]\|^2_{0,e} \right)^{\frac{1}{2}} \\
\le\, & C \sum_{i=1,2} \|c_i^h - \hat{c}_i^h\|_{0,\Omega_i} J_i^{\frac{1}{2}}\left(z_i^h, z_i^h\right), \\
|T_4| \le\, & C \|c_\gamma^h - \hat{c}_\gamma^h\|_{0,\gamma} J_\gamma^{\frac{1}{2}}\left(z_\gamma^h, z_\gamma^h\right).
\end{aligned}
\tag{3.53}
$$

Hence, we have

$$
\left| B\left(c^h, z^h\right) - B\left(\hat{c}^h, z^h\right) \right| \le C \|c^h - \hat{c}^h\|_{0,\Omega} |||z^h|||. \tag{3.54}
$$

Let $\bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_\gamma) \in \mathcal{D}_k(\mathcal{T}^h)$ be an interpolant of the exact solution $c = (c_1, c_2, c_\gamma)$ with the following optimal approximation properties:

$$
\begin{aligned}
|c_i - \bar{c}_i|_{s,K} \le C h_K^{k+1-s} |c_i|_{k+1,K}, \quad & \forall K \in \mathcal{T}_i^h, \quad \forall 0 \le s \le k+1, \\
|c_\gamma - \bar{c}_\gamma|_{s,K} \le C h_K^{k+1-s} |c_\gamma|_{k+1,K}, \quad & \forall K \in \mathcal{T}_\gamma^h, \quad \forall 0 \le s \le k+1, \\
\int_0^T \left\| \frac{\partial(c_i - \bar{c}_i)}{\partial t} \right\|_{0,K} dt \le C h^k \int_0^T \left\| \frac{\partial c_i}{\partial t} \right\|_{k,K} dt, \quad & \forall K \in \mathcal{T}_i^h, \\
\int_0^T \left\| \frac{\partial\left(c_\gamma - \bar{c}_\gamma\right)}{\partial t} \right\|_{0,K} dt \le C h^k \int_0^T \left\| \frac{\partial c_\gamma}{\partial t} \right\|_{k,K} dt, \quad & \forall K \in \mathcal{T}_\gamma^h.
\end{aligned}
\tag{3.55}
$$

We define the following notations,

$$
\begin{aligned}
\varrho^j = \left(\varrho_1^j, \varrho_2^j, \varrho_\gamma^j\right), \quad & \varrho_i^j = c_i^j - \bar{c}_i^j, i = 1,2, \quad \varrho_\gamma^j = c_\gamma^j - \bar{c}_\gamma^j, \\
\varsigma^j = \left(\varsigma_1^j, \varsigma_2^j, \varsigma_\gamma^j\right), \quad & \varsigma_i^j = \bar{c}_i^j - c_i^{h,j}, i = 1,2, \quad \varsigma_\gamma^j = \bar{c}_\gamma^j - c_\gamma^{h,j}.
\end{aligned}
\tag{3.56}
$$

We first give the estimate between c and $\bar{c}$ in the $|||\cdot|||$-norm.

$$
\begin{aligned}
|||\varrho^j|||^2 = & \sum_{i=1,2} \|\varrho_i^j\|^2_{1,\mathcal{T}_i^h} + \|\varrho_\gamma^j\|^2_{1,\mathcal{T}_\gamma^h} + \sum_{i=1,2} J_i(\varrho_i^j, \varrho_i^j) + J_\gamma(\varrho_\gamma^j, \varrho_\gamma^j) + \sum_{i=1,2} \|\varrho_i^j - \varrho_\gamma^j\|^2_{0,\gamma} \\
:= & M_1 + M_2 + M_3 + M_4 + M_5.
\end{aligned}
\tag{3.57}
$$

By using (3.55), the following results are obtained for M_1 and M_2:

$$
M_1 \le C \sum_{i=1,2} h^{2k} |c_i|^2_{k+1,\Omega_i}, \tag{3.58}
$$

and

$$M_2 \le Ch^{2k}|c_\gamma|^2_{k+1,\gamma}. \tag{3.59}$$

Using Lemma 2 and (3.55), we obtain

$$\begin{aligned}
M_3 &= \sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{\partial,i}} |e|^{-1}\|\varrho_i^j|_{K_e}\|^2_{0,e} + \sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{o,i}} |e|^{-1}\|\varrho_i^j|_{K_e^1} - \varrho_i^j|_{K_e^2}\|^2_{0,e} \\
&\le C\sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{\partial,i}} h^{-1}\left(h^{-1}\|\varrho_i^j\|^2_{0,K_e} + h\|\nabla\varrho_i^j\|^2_{0,K_e}\right) \\
&\quad + C\sum_{i=1,2}\sum_{e\in\mathcal{E}^h_{o,i}} h^{-1}\left(h^{-1}\|\varrho_i^j\|^2_{0,K_e^1} + h\|\nabla\varrho_i^j\|^2_{0,K_e^1} + h^{-1}\|\varrho_i^j\|^2_{0,K_e^2} + h\|\nabla\varrho_i^j\|^2_{0,K_e^2}\right) \\
&\le C\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\left(h^{-2}\|\varrho_i^j\|^2_{0,K} + \|\nabla\varrho_i^j\|^2_{0,K}\right) \\
&\le C\sum_{i=1,2} h^{2k}|c_i|^2_{k+1,\Omega_i}.
\end{aligned} \tag{3.60}$$

Similarly,

$$M_4 \le Ch^{2k}|c_\gamma|^2_{k+1,\gamma}. \tag{3.61}$$

For M_5, we also use Lemma 2 to obtain

$$\begin{aligned}
M_5 &\le \sum_{i=1,2}\sum_{e\subset\gamma}\|\varrho_i^j\|^2_{0,e} + 2\|\varrho_\gamma^j\|^2_{0,\gamma} \\
&\le C\sum_{i=1,2}\sum_{e\subset\gamma}\left(h^{-1}\|\varrho_i^j\|^2_{0,K_e} + h\|\nabla\varrho_i^j\|^2_{0,K_e}\right) + C\|\varrho_\gamma^j\|^2_{0,\gamma} \\
&\le C\sum_{i=1,2}\sum_{K\in\mathcal{T}_i^h}\left(h^{-1}\|\varrho_i^j\|^2_{0,K} + h\|\nabla\varrho_i^j\|^2_{0,K}\right) + C\|\varrho_\gamma^j\|^2_{0,\gamma} \\
&\le C\sum_{i=1,2}\left(h^{-1}\|\varrho_i^j\|^2_{0,\Omega_i} + h\|\varrho_i^j\|^2_{1,\mathcal{T}_i^h}\right) + C\|\varrho_\gamma^j\|^2_{0,\gamma} \\
&\le Ch^{2k}|c|^2_{k+1,\Omega}.
\end{aligned} \tag{3.62}$$

Finally, by (3.58)–(3.62), we have

$$|||\varrho^j||| \le Ch^k|c|_{k+1,\Omega}. \tag{3.63}$$

Theorem 1. *Assume that* $c^j = \left(c_1^j, c_2^j, c_\gamma^j\right)$ *with* $c_i^j \in H^{k+1}(\Omega_i)$ *and* $c_\gamma^j \in H^{k+1}(\gamma)$ *is the exact solution to the problem in* (2.1)–(2.4)*, and* $c^{h,j} = \left(c_1^{h,j}, c_2^{h,j}, c_\gamma^{h,j}\right) \in \mathcal{D}_k(\mathcal{T}^h)$ *is the numerical solution to Algorithm 1. Then, if* σ *is large enough and* h *is sufficiently small, a constant* C *exists such that*

$$\Delta t\sum_{j=1}^{N} |||c^j - c^{h,j}|||^2 \le C\left(\|c\|_{L^\infty([0,T],H^{k+1}(\Omega))}, \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2((0,T),L^2(\Omega))}\right)\left(h^{2k} + \Delta t^2\right). \tag{3.64}$$

Proof. Subtracting (3.8) from (3.7), we have the following error equation:

$$\begin{aligned}
&\sum_{i=1,2}\int_{\Omega_i}\phi_i\frac{\partial c_i^j}{\partial t}z_i + \int_\gamma d\phi_\gamma\frac{\partial c_\gamma^j}{\partial t}z_\gamma - \sum_{i=1,2}\int_{\Omega_i}\phi_i\frac{c_i^{h,j} - c_i^{h,j-1}}{\Delta t}z_i - \int_\gamma d\phi_\gamma\frac{c_\gamma^{h,j} - c_\gamma^{h,j-1}}{\Delta t}z_\gamma \\
&+ A\left(c^j, z\right) - A\left(c^{h,j}, z\right) + B\left(c^j, z\right) - B\left(c^{h,j}, z\right) = 0.
\end{aligned} \tag{3.65}$$

Choosing $z = \varsigma^j = (\varsigma_1^j, \varsigma_2^j, \varsigma_\gamma^j)$, the equation above can be rewritten as

$$\begin{aligned}
&\sum_{i=1,2} \int_{\Omega_i} \phi_i \frac{\varsigma_i^j - \varsigma_i^{j-1}}{\Delta t} \varsigma_i^j + \int_\gamma d\phi_\gamma \frac{\varsigma_\gamma^j - \varsigma_\gamma^{j-1}}{\Delta t} \varsigma_\gamma^j + A\left(\varsigma^j, \varsigma^j\right) \\
&= - A\left(\varrho^j, \varsigma^j\right) - B\left(\varrho^j, \varsigma^j\right) - B\left(\varsigma^j, \varsigma^j\right) \\
&\quad - \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(\frac{\partial c_i^j}{\partial t} - \frac{\overline{c}_i^j - \overline{c}_i^{j-1}}{\Delta t} \right) \varsigma_i^j - \int_\gamma d\phi_\gamma \left(\frac{\partial c_\gamma^j}{\partial t} - \frac{\overline{c}_\gamma^j - \overline{c}_\gamma^{j-1}}{\Delta t} \right) \varsigma_\gamma^j.
\end{aligned} \tag{3.66}$$

The first two terms on the left-hand of the equation above can be bounded as

$$\begin{aligned}
&\sum_{i=1,2} \int_{\Omega_i} \phi_i \frac{\varsigma_i^j - \varsigma_i^{j-1}}{\Delta t} \varsigma_i^j + \int_\gamma d\phi_\gamma \frac{\varsigma_\gamma^j - \varsigma_\gamma^{j-1}}{\Delta t} \varsigma_\gamma^j \\
&\geq \frac{1}{2\Delta t} \left(\left(\sum_{i=1,2} \|\phi_i^{\frac{1}{2}} \varsigma_i^j\|_{0,\Omega_i}^2 + \|d^{\frac{1}{2}} \phi_\gamma^{\frac{1}{2}} \varsigma_\gamma^j\|_{0,\gamma}^2 \right) - \left(\sum_{i=1,2} \|\phi_i^{\frac{1}{2}} \varsigma_i^{j-1}\|_{0,\Omega_i}^2 + \|d^{\frac{1}{2}} \phi_\gamma^{\frac{1}{2}} \varsigma_\gamma^{j-1}\|_{0,\gamma}^2 \right) \right).
\end{aligned} \tag{3.67}$$

By using Lemma 3, the third term in the left-hand side of (3.66) holds the bound

$$A\left(\varsigma^j, \varsigma^j\right) \geq \alpha |||\varsigma^j|||^2. \tag{3.68}$$

We then are concerned with the right-hand side terms in (3.66). The last two terms can be rewritten as

$$\begin{aligned}
&- \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(\frac{\partial c_i^j}{\partial t} - \frac{\overline{c}_i^j - \overline{c}_i^{j-1}}{\Delta t} \right) \varsigma_i^j - \int_\gamma d\phi_\gamma \left(\frac{\partial c_\gamma^j}{\partial t} - \frac{\overline{c}_\gamma^j - \overline{c}_\gamma^{j-1}}{\Delta t} \right) \varsigma_\gamma^j \\
&= - \sum_{i=1,2} \int_{\Omega_i} \phi_i \left(\left(\frac{\partial c_i^j}{\partial t} - \frac{c_i^j - c_i^{j-1}}{\Delta t} \right) + \frac{\varrho_i^j - \varrho_i^{j-1}}{\Delta t} \right) \varsigma_i^j \\
&\quad - \int_\gamma d\phi_\gamma \left(\left(\frac{\partial c_\gamma^j}{\partial t} - \frac{c_\gamma^j - c_\gamma^{j-1}}{\Delta t} \right) + \frac{\varrho_\gamma^j - \varrho_\gamma^{j-1}}{\Delta t} \right) \varsigma_\gamma^j.
\end{aligned} \tag{3.69}$$

By using the Taylor expansion with an integral remainder, we have

$$\begin{aligned}
&\left| \frac{\partial c_i^j}{\partial t} - \frac{c_i^j - c_i^{j-1}}{\Delta t} \right| \leq \frac{1}{2\Delta t} \left(\int_{t^{j-1}}^{t^j} \left| \frac{\partial^2 c_i}{\partial t^2} \right|^2 dt \right)^{\frac{1}{2}} \left(\int_{t^{j-1}}^{t^j} \left(t - t^{j-1} \right)^2 dt \right)^{\frac{1}{2}}, \\
&\left\| \frac{\partial c_i^j}{\partial t} - \frac{c_i^j - c_i^{j-1}}{\Delta t} \right\|_{0,\Omega_i}^2 \leq C\Delta t \int_{t^{j-1}}^{t^j} \left\| \frac{\partial^2 c_i}{\partial t^2} \right\|_{0,\Omega_i}^2 dt.
\end{aligned} \tag{3.70}$$

Then by using the Hölder inequality, we have

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i \left(\frac{\partial c_i}{\partial t} - \frac{c_i^j - c_i^{j-1}}{\Delta t} \right) \varsigma_i^j \leq C \sum_{i=1,2} \|\varsigma_i^j\|_{0,\Omega_i}^2 + C\Delta t \sum_{i=1,2} \int_{t^{j-1}}^{t^j} \left\| \frac{\partial^2 c_i}{\partial t^2} \right\|_{0,\Omega_i}^2 dt. \tag{3.71}$$

The Taylor expansion is applied to the second term in (3.69) as well, yielding

$$\sum_{i=1,2}\int_{\Omega_i}\phi_i\frac{\varrho_i^j-\varrho_i^{j-1}}{\Delta t}\varsigma_i^j \le C\sum_{i=1,2}\|\varsigma_i^j\|_{0,\Omega_i}^2+\frac{1}{\Delta t}\sum_{i=1,2}\int_{t^{j-1}}^{t^j}\left\|\frac{\partial\varrho_i}{\partial t}\right\|_{0,\Omega_i}^2 dt. \tag{3.72}$$

Similarly, the following result holds in γ:

$$\begin{aligned}&\int_\gamma d\phi_\gamma\left(\left(\frac{\partial c_\gamma^j}{\partial t}-\frac{c_\gamma^j-c_\gamma^{j-1}}{\Delta t}\right)-\frac{\varrho_\gamma^j-\varrho_\gamma^{j-1}}{\Delta t}\right)\varsigma_\gamma^j\\&\le C\|\varsigma_\gamma^j\|_{0,\gamma}^2+C\left(\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left\|\frac{\partial\varrho_\gamma}{\partial t}\right\|_{0,\gamma}^2 dt+\Delta t\int_{t^{j-1}}^{t^j}\left\|\frac{\partial^2 c_\gamma}{\partial t^2}\right\|_{0,\gamma}^2 dt\right).\end{aligned} \tag{3.73}$$

Hence, we have the following bound for the last two terms in (3.66) from (3.70)–(3.73):

$$\begin{aligned}&\left|\sum_{i=1,2}\int_{\Omega_i}\phi_i\left(\frac{\partial c_i^j}{\partial t}-\frac{\bar{c}_i^j-\bar{c}_i^{j-1}}{\Delta t}\right)\varsigma_i^j-\int_\gamma d\phi_\gamma\left(\frac{\partial c_\gamma^j}{\partial t}-\frac{\bar{c}_\gamma^j-\bar{c}_\gamma^{j-1}}{\Delta t}\right)\varsigma_\gamma^j\right|\\&\le C\|\varsigma^j\|_{0,\Omega}^2+C\left(\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left\|\frac{\partial\varrho}{\partial t}\right\|_{0,\Omega}^2 dt+\Delta t\int_{t^{j-1}}^{t^j}\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{0,\Omega}^2 dt\right),\end{aligned} \tag{3.74}$$

with $\frac{\partial\varrho}{\partial t}=\left(\frac{\partial\varrho_1}{\partial t},\frac{\partial\varrho_2}{\partial t},\frac{\partial\varrho_\gamma}{\partial t}\right)$ and $\frac{\partial^2 c}{\partial t^2}=\left(\frac{\partial^2 c_1}{\partial t^2},\frac{\partial^2 c_2}{\partial t^2},\frac{\partial^2 c_\gamma}{\partial t^2}\right)$.

We then estimate the first three terms on the right-hand side of Eq (3.66). By using Lemma 4 and the Young's inequality, we have

$$\begin{aligned}-A\left(\varrho^j,\varsigma^j\right)&\le C\left(|||\varrho^j|||+h\|\varrho^j\|_{2,\mathcal{T}^h}+h^{-1}\|\varrho^j\|_{0,\Omega}\right)|||\varsigma^j|||\\&\le C\left(|||\varrho^j|||^2+h^2\|\varrho^j\|_{2,\mathcal{T}^h}^2+h^{-2}\|\varrho^j\|_{0,\Omega}^2\right)+\varepsilon|||\varsigma^j|||^2.\end{aligned} \tag{3.75}$$

Similarly, using Lemma 5 and Young's inequality, we have

$$\begin{aligned}-B\left(\varsigma^j,\varsigma^j\right)&\le C\|\varsigma^j\|_{0,\Omega}|||\varsigma^j|||\le C\|\varsigma^j\|_{0,\Omega}^2+\varepsilon|||\varsigma^j|||^2,\\-B\left(\varrho^j,\varsigma^j\right)&\le C\left(\|\varrho^j\|_{0,\Omega}^2+h^2\|\varrho^j\|_{1,\mathcal{T}^h}^2\right)^{\frac{1}{2}}|||\varsigma^j|||\\&\le C\left(\|\varrho^j\|_{0,\Omega}^2+h^2\|\varrho^j\|_{1,\mathcal{T}^h}^2\right)+\varepsilon|||\varsigma^j|||^2.\end{aligned} \tag{3.76}$$

Taking (3.67), (3.68), and (3.74)–(3.76) back to (3.66), it follows that

$$\begin{aligned}&(\alpha-3\varepsilon)\,|||\varsigma^j|||^2+\frac{1}{2\Delta t}\left(\|\phi^{\frac{1}{2}}\varsigma^j\|_{0,\Omega}^2-\|\phi^{\frac{1}{2}}\varsigma^{j-1}\|_{0,\Omega}^2\right)\\&\le C\|\varsigma^j\|_{0,\Omega}^2+C\left(\Delta t\int_{t^{j-1}}^{t^j}\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{0,\Omega}^2 dt+\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left\|\frac{\partial\varrho}{\partial t}\right\|_{0,\Omega}^2 dt\right)\\&\quad+C\left(|||\varrho^j|||^2+h^2\|\varrho^j\|_{2,\mathcal{T}^h}^2+h^{-2}\|\varrho^j\|_{0,\Omega}^2\right)+C\left(\|\varrho^j\|_{0,\Omega}^2+h^2\|\varrho^j\|_{1,\mathcal{T}^h}^2\right),\end{aligned} \tag{3.77}$$

with $\phi^{\frac{1}{2}}\varsigma^j=(\phi_1^{\frac{1}{2}}\varsigma_1^j,\phi_2^{\frac{1}{2}}\varsigma_2^j,d^{\frac{1}{2}}\phi_\gamma^{\frac{1}{2}}\varsigma_\gamma^j)$. Multiplying the equation above by $2\Delta t$, summing on $j=1,\cdots,N$ and choosing $\varepsilon=\frac{\alpha}{6}$, $\|\varsigma^0\|_{0,\Omega}=0$, the equation above can be rewritten as

$$\Delta t\sum_{j=1}^N|||\varsigma^j|||^2+\|\varsigma^N\|_{0,\Omega}^2\le C\Delta t\sum_{j=1}^N\|\varsigma^j\|_{0,\Omega}^2+C\left(\|c\|_{L^\infty([0,T],H^{k+1}(\Omega))},\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2((0,T),L^2(\Omega))}\right)\left(h^{2k}+\Delta t^2\right). \tag{3.78}$$

The following results hold from the Gronwall's inequality:

$$\Delta t \sum_{j=1}^{N} |||\varsigma^j|||^2 \leq C\left(\|c\|_{L^\infty([0,T],H^{k+1}(\Omega))}, \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2((0,T),L^2(\Omega))}\right)\left(h^{2k} + \Delta t^2\right). \tag{3.79}$$

By using the triangle inequality and (3.63), we finish the proof.

4. Model with intersecting fractures

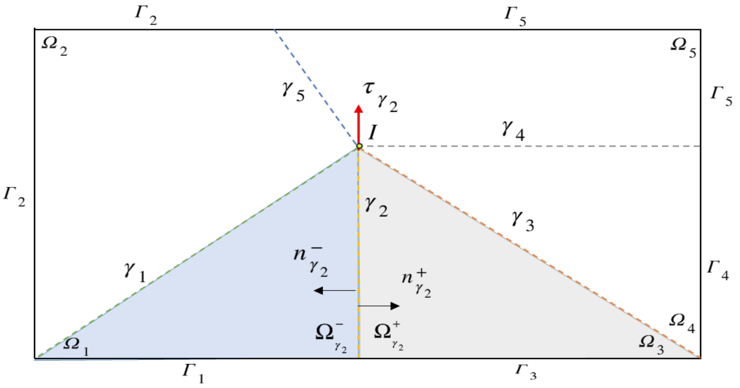


Figure 2. The domain with intersecting fractures.

In this section, we extend the hybrid-dimensional fracture model from one single fracture to a network with intersecting fractures. The fractured network γ now comprises n_γ discrete fractures, expressed as $\gamma = \cup_{\ell=1}^{n_\gamma} \gamma_\ell$, where each fracture γ_ℓ constitutes an open, bounded, planar $(n-1)$-dimensional orientable manifold. The fractured network exhibits topological connectivity, which requires all fractures to intersect at a single point (for $n = 2$) or line segment (for $n = 3$). For each fracture, the intersection point/line corresponds to one of its endpoints if $n = 2$ or to part of its boundary edges if $n = 3$. Let I be the intersection point/line, i.e., $I = \cap_{\ell=1}^{n_\gamma} \overline{\gamma}_\ell$. We impose geometric constraints that ensure minimum intersection angles between fractures and between fractures and the domain boundary $\partial\Omega$, where contact occurs. The boundary of the fractured network (excluding I) is entirely contained within $\partial\Omega$, which eliminates the possibility of fully immersed fractures. The fractured network partitions the global domain Ω into Lipschitz-continuous subdomains Ω_i $(i = 1, \cdots, n_\Omega)$, with the matrix domain defined as $\Omega_m = \cup_{i=1}^{n_\Omega} \Omega_i$. The two adjacent subdomains $\Omega^+_{\gamma_\ell}$ and $\Omega^-_{\gamma_\ell}$ are precisely separated by each fracture γ_ℓ, which satisfies $\gamma_\ell = \partial\Omega^+_{\gamma_\ell} \cap \partial\Omega^-_{\gamma_\ell}$. In fracture γ_ℓ, let $\mathbf{n}^+_{\gamma_\ell}$ and $\mathbf{n}^-_{\gamma_\ell}$ represent the outward unit normal vectors associated with the subdomains $\Omega^+_{\gamma_\ell}$ and $\Omega^-_{\gamma_\ell}$ respectively, and $\mathbf{n}_{\gamma_\ell} = \mathbf{n}^+_{\gamma_\ell} = -\mathbf{n}^-_{\gamma_\ell}$. Moreover, we use $\mathbf{n}_\gamma$ to denote the normal vector to the whole fractured network, meaning that $\mathbf{n}_\gamma = \mathbf{n}_{\gamma_\ell}$ in γ_ℓ, similar to $\mathbf{n}^+_\gamma$ and $\mathbf{n}^-_\gamma$. Figure 2 presents a clear discription of the intersecting fractured network and the corresponding notations.

For the functions q_{Ω_i} defined in subdomains Ω_i, we construct the composite matrix field through the product space notation, $q_m = \left(q_{\Omega_1}, q_{\Omega_2}, \ldots, q_{\Omega_{n_\Omega}}\right)$. Similarly, for the functions q_{γ_ℓ} defined in the fractures γ_ℓ, we define the composite fracture field, $q_\gamma = \left(q_{\gamma_1}, q_{\gamma_2}, \ldots, q_{\gamma_{n_\gamma}}\right)$. Vector-valued fields follow analogous constructions, $\mathbf{v}_m = \left(\mathbf{v}_{\Omega_1}, \mathbf{v}_{\Omega_2}, \ldots, \mathbf{v}_{\Omega_{n_\Omega}}\right)$ and $\mathbf{v}_\gamma = \left(\mathbf{v}_{\gamma_1}, \mathbf{v}_{\gamma_2}, \ldots, \mathbf{v}_{\gamma_{n_\gamma}}\right)$. Moreover, $d = (d_1, d_2, \cdots, d_{n_\gamma})$ is the width of the fracture system. In addition, in order to describe the model with intersecting fractures, we also need to introduce the trace-restricted functions $q_m^{\gamma+} = \left(q_{\Omega_{\gamma_1}^+}|_{\gamma_1}, q_{\Omega_{\gamma_2}^+}|_{\gamma_2}, \dots, q_{\Omega_{\gamma_{n_\gamma}}^+}|_{\gamma_{n_\gamma}}\right)$ and $q_m^{\gamma-} = \left(q_{\Omega_{\gamma_1}^-}|_{\gamma_1}, q_{\Omega_{\gamma_2}^-}|_{\gamma_2}, \dots, q_{\Omega_{\gamma_{n_\gamma}}^-}|_{\gamma_{n_\gamma}}\right)$, where $q_{\Omega_{\gamma_\ell}^\pm}|_{\gamma_\ell}$ denotes the trace of the matrix field restricted to the fracture interface. Vector trace fields follow a similar construction: $\mathbf{v}_m^{\gamma+} = \left(\mathbf{v}_{\Omega_{\gamma_1}^+}|_{\gamma_1}, \dots, \mathbf{v}_{\Omega_{\gamma_{n_\gamma}}^+}|_{\gamma_{n_\gamma}}\right)$ and $\mathbf{v}_m^{\gamma-} = \left(\mathbf{v}_{\Omega_{\gamma_1}^-}|_{\gamma_1}, \dots, \mathbf{v}_{\Omega_{\gamma_{n_\gamma}}^-}|_{\gamma_{n_\gamma}}\right)$.

We define the boundaries for the matrix and fractures network as $\Gamma_m = \cup_{i=1}^{n\Omega}\Gamma_i$, with $\Gamma_i = \partial\Omega_i \cap \partial\Omega$, and $\partial\gamma = \left(\cup_{\ell=1}^{n_\gamma}\partial\gamma_\ell\right) \backslash I$. The hybrid-dimensional fracture model with intersecting fractures is described as follows, for $t \in [0, T]$.

In the matrix Ω_m, we have

$$\begin{aligned}
\phi_m \frac{\partial c_m}{\partial t} + \nabla\cdot\boldsymbol{\chi}_m &= q_m && \text{in } \Omega_m,\\
\boldsymbol{\chi}_m &= -\mathbf{K}_m \nabla c_m + \mathbf{u}_m c_m && \text{in } \Omega_m,\\
c_m &= c_{\partial,m} && \text{on } \Gamma_m.
\end{aligned} \tag{4.1}$$

In the fracture γ, we have

$$\begin{aligned}
d\phi_\gamma \frac{\partial c_\gamma}{\partial t} + \nabla_\tau\cdot\boldsymbol{\chi}_\gamma &= q_\gamma + \boldsymbol{\chi}_m^{\gamma+}\cdot\mathbf{n}_\gamma^+ + \boldsymbol{\chi}_m^{\gamma-}\cdot\mathbf{n}_\gamma^- && \text{in } \gamma,\\
\boldsymbol{\chi}_\gamma &= -\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma + d\mathbf{u}_\gamma c_\gamma && \text{in } \gamma,\\
c_\gamma &= c_{\partial,\gamma} && \text{on } \partial\gamma.
\end{aligned} \tag{4.2}$$

For the transmission conditions in γ, we have

$$\boldsymbol{\chi}_m^{\gamma *}\cdot\mathbf{n}_\gamma^* = \frac{2\eta_\gamma}{2\xi - 1}\left(\xi c_m^{\gamma *} + (1-\xi)c_m^{\gamma(-*)} - c_\gamma\right), \qquad \text{in } \gamma,\ * = +, -. \tag{4.3}$$

At the intersection I, we have

$$\begin{aligned}
\sum_{\ell=1}^{n_\gamma} \boldsymbol{\chi}_{\gamma_\ell}\cdot\tau_{\gamma_\ell} &= 0,\\
c_{\gamma_1} = c_{\gamma_2} &= \cdots = c_{\gamma_{n_\gamma}}.
\end{aligned} \tag{4.4}$$

At the initial time $t = 0$, we have

$$c_m = c_m^0 \quad \text{in } \Omega_m, \qquad c_\gamma = c_\gamma^0 \quad \text{in } \gamma, \tag{4.5}$$

where $*$ and $(-*)$ mean the signs are opposite, and τ_{γ_ℓ} is defined on each fracture γ_ℓ as the vector in its tangent plane, pointing outward from the intersection point/line I. We use the homogeneous Dirichlet boundary conditions $c_{\partial,m} = 0$ and $c_{\partial,\gamma} = 0$. The symbols defined in Section 3 continue to be used, such as the partitions $\mathcal{T}_m^h$ and $\mathcal{T}_\gamma^h$ with $\mathcal{T}_m^h = \cup_{i=1}^{n\Omega}\mathcal{T}_{\Omega_i}^h$ and $\mathcal{T}_\gamma^h = \cup_{\ell=1}^{n_\gamma}\mathcal{T}_{\gamma_\ell}^h$, the sets of interior edges or faces in the matrix $\mathcal{E}_{o,m}^h = \bigcup_{i=1}^{n\Omega}\mathcal{E}_{o,\Omega_i}^h$, the sets of interior points or edges in the fracture $\mathcal{E}_{o,\gamma}^h = \bigcup_{\ell=1}^{n_\gamma}\mathcal{E}_{o,\gamma_\ell}^h$, the sets of exterior edges or faces $\mathcal{E}_{\partial,m}^h = \bigcup_{i=1}^{n\Omega}\mathcal{E}_{\partial,\Omega_i}^h$, the sets of exterior points or edges in the fracture $\mathcal{E}_{\partial,\gamma}^h = \bigcup_{\ell=1}^{n_\gamma}\mathcal{E}_{\partial,\gamma_\ell}^h$ (not including points or edges on I), and $\mathcal{E}_m^h = \mathcal{E}_{o,m}^h \cup \mathcal{E}_{\partial,m}^h$, $\mathcal{E}_\gamma^h = \mathcal{E}_{o,\gamma}^h \cup \mathcal{E}_{\partial,\gamma}^h$. In addition, we also need to introduce some notations for the intersection. Let $\mathcal{E}_h^I$ be the set of points/edges on the intersection of fractures. Note that, in this paper, since we consider a network with one single intersection, when $n = 2$, this set consists only of one single point. Similar to [33], the jump and average operators on the intersection are defined.

Definition 1. We define the jump and average operators for z_γ and $\mathbf{v}_\gamma$ on I as follows:

$$
\begin{aligned}
\{z_\gamma\}_I &= \frac{1}{n_\gamma}\left(z_{\gamma_1} + z_{\gamma_2} + \cdots + z_{\gamma_{n_\gamma}}\right), \\
\left[z_\gamma\right]_I &= (z_{\gamma_j} - z_{\gamma_\ell})_{1\le j<\ell\le n_\gamma}, \\
\{\mathbf{v}_\gamma\}_I &= \frac{1}{n_\gamma}\left(\mathbf{v}_{\gamma_j}\cdot\tau_{\gamma_j} - \mathbf{v}_{\gamma_\ell}\cdot\tau_{\gamma_\ell}\right)_{1\le j<\ell\le n_\gamma}, \\
\left[\mathbf{v}_\gamma\right]_I &= \mathbf{v}_{\gamma_1}\cdot\tau_{\gamma_1} + \mathbf{v}_{\gamma_2}\cdot\tau_{\gamma_2} + \cdots + \mathbf{v}_{\gamma_{n_\gamma}}\cdot\tau_{\gamma_{n_\gamma}}.
\end{aligned}
\tag{4.6}
$$

The following result for the jump and average can be seen in [33], as well as the corresponding proof.

Proposition 1. The following identity holds:

$$
[z_\gamma\mathbf{v}_\gamma]_I = [\mathbf{v}_\gamma]_I\{z_\gamma\}_I + \{\mathbf{v}_\gamma\}_I\cdot[z_\gamma]_I, \tag{4.7}
$$

where the vector-valued function $z_\gamma\mathbf{v}_\gamma$ is defined as $z_\gamma\mathbf{v}_\gamma = (z_{\gamma_1}\mathbf{v}_{\gamma_1}, z_{\gamma_2}\mathbf{v}_{\gamma_2}, \cdots, z_{\gamma_{n_\gamma}}\mathbf{v}_{\gamma_{n_\gamma}})$.

In the matrix Ω_m, similar to (3.4), for any $z_m \in H^1(\mathcal{T}_m^h)$, we can derive

$$
\begin{aligned}
&\int_{\Omega_m}\phi_m\frac{\partial c_m}{\partial t}z_m + \sum_{K\in\mathcal{T}_m^h}\int_K \mathbf{K}_m\nabla c_m\cdot\nabla z_m - \sum_{K\in\mathcal{T}_m^h}\int_K \mathbf{u}_m c_m\cdot\nabla z_m - \sum_{e\in\mathcal{E}_m^h}\int_e\{\mathbf{K}_m\nabla c_m\cdot\mathbf{n}_e\}[z_m] \\
&+\sum_{e\in\mathcal{E}_m^h}\int_e\{\mathbf{u}_m c_m\cdot\mathbf{n}_e\}[z_m] + \sum_{*=+,-}\int_\gamma\frac{2\eta_\gamma}{2\xi-1}\left(\xi c_m^{\gamma*} + (1-\xi)c_m^{\gamma(-*)} - c_\gamma\right)z_m^{\gamma*} = \int_{\Omega_m} q_m z_m.
\end{aligned}
\tag{4.8}
$$

In each fracture γ_ℓ, multiplying the first equation in (4.2) by a test function $z_{\gamma_\ell} \in H^1\left(\mathcal{T}_{\gamma_\ell}^h\right)$, integrating within each element $K \in \mathcal{T}_{\gamma_\ell}^h$, and using Green's formulation, we obtain

$$
\begin{aligned}
&\int_K d_\ell\phi_{\gamma_\ell}\frac{\partial c_{\gamma_\ell}}{\partial t}z_{\gamma_\ell} - \int_K \chi_{\gamma_\ell}\cdot\nabla_\tau z_{\gamma_\ell} + \int_{\partial K\setminus I}\chi_{\gamma_\ell}\cdot\mathbf{n}_{\partial K}z_{\gamma_\ell} + \int_{\partial K\cap I}\chi_{\gamma_\ell}\cdot\tau_{\gamma_\ell}z_{\gamma_\ell} \\
&= \int_K q_{\gamma_\ell}z_{\gamma_\ell} + \int_K\left(\chi_{\Omega_{\gamma_\ell}^+}|_{\gamma_\ell}\cdot\mathbf{n}_{\gamma_\ell}^+ + \chi_{\Omega_{\gamma_\ell}^-}|_{\gamma_\ell}\cdot\mathbf{n}_{\gamma_\ell}^-\right)z_{\gamma_\ell}.
\end{aligned}
\tag{4.9}
$$

Summing over all elements $K \in \mathcal{T}_{\gamma_\ell}^h$, we obtain

$$
\begin{aligned}
&\int_{\gamma_\ell} d_\ell\phi_{\gamma_\ell}\frac{\partial c_{\gamma_\ell}}{\partial t}z_{\gamma_\ell} + \sum_{K\in\mathcal{T}_{\gamma_\ell}^h}\int_K \mathbf{K}_{\gamma_\ell,\tau}d_\ell\nabla_\tau c_{\gamma_\ell}\cdot\nabla_\tau z_{\gamma_\ell} - \sum_{e\in\mathcal{E}_{\gamma_\ell}^h}\int_e\{\mathbf{K}_{\gamma_\ell,\tau}d_\ell\nabla_\tau c_{\gamma_\ell}\cdot\mathbf{n}_e\}[z_{\gamma_\ell}] \\
&- \sum_{K\in\mathcal{T}_{\gamma_\ell}^h}\int_K d_\ell\mathbf{u}_{\gamma_\ell}c_{\gamma_\ell}\cdot\nabla_\tau z_{\gamma_\ell} + \sum_{e\in\mathcal{E}_{\gamma_\ell}^h}\int_e\{d_\ell\mathbf{u}_{\gamma_\ell}c_{\gamma_\ell}\cdot\mathbf{n}_e\}[z_{\gamma_\ell}] + \int_I \chi_{\gamma_\ell}\cdot\tau_{\gamma_\ell}z_{\gamma_\ell} \\
&- \int_{\gamma_\ell}\left(\chi_{\Omega_{\gamma_\ell}^+}|_{\gamma_\ell}\cdot\mathbf{n}_{\gamma_\ell}^+ + \chi_{\Omega_{\gamma_\ell}^-}|_{\gamma_\ell}\cdot\mathbf{n}_{\gamma_\ell}^-\right)z_{\gamma_\ell} = \int_{\gamma_\ell} q_{\gamma_\ell}z_{\gamma_\ell}.
\end{aligned}
\tag{4.10}
$$

Summing over all the fractures $\ell = 1, 2, \cdots, n_\gamma$, we have

$$\begin{aligned}
&\int_\gamma d\phi_\gamma \frac{\partial c_\gamma}{\partial t} z_\gamma + \sum_{K\in\mathcal{T}_\gamma^h} \int_K \mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \nabla_\tau z_\gamma - \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] - \sum_{K\in\mathcal{T}_\gamma^h} \int_K d\mathbf{u}_\gamma c_\gamma \cdot \nabla_\tau z_\gamma \\
&+ \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{d\mathbf{u}_\gamma c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] + \sum_{\ell=1}^{n_\gamma} \int_I \boldsymbol{\chi}_{\gamma_\ell} \cdot \tau_{\gamma_\ell} z_{\gamma_\ell} - \int_\gamma \left(\boldsymbol{\chi}_m^{\gamma+} \cdot \mathbf{n}_\gamma^+ + \boldsymbol{\chi}_m^{\gamma-} \cdot \mathbf{n}_\gamma^-\right) z_\gamma = \int_\gamma q_\gamma z_\gamma.
\end{aligned} \tag{4.11}$$

Based on the definition of the jump operator in Definition 1, Proposition 1, and the first condition in (4.4), the sixth term in the equation above can be rewritten as

$$\sum_{\ell=1}^{n_\gamma} \boldsymbol{\chi}_{\gamma_\ell} \cdot \tau_{\gamma_\ell} z_{\gamma_\ell} = [z_\gamma \boldsymbol{\chi}_\gamma]_I = [\boldsymbol{\chi}_\gamma]_I \{z_\gamma\}_I + \{\boldsymbol{\chi}_\gamma\}_I \cdot [z_\gamma]_I = \{\boldsymbol{\chi}_\gamma\}_I \cdot [z_\gamma]_I. \tag{4.12}$$

We then rewrite (4.11) as follows:

$$\begin{aligned}
&\int_\gamma d\phi_\gamma \frac{\partial c_\gamma}{\partial t} z_\gamma + \sum_{K\in\mathcal{T}_\gamma^h} \int_K \mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \nabla_\tau z_\gamma - \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] \\
&- \sum_{K\in\mathcal{T}_\gamma^h} \int_K d\mathbf{u}_\gamma c_\gamma \cdot \nabla_\tau z_\gamma + \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{d\mathbf{u}_\gamma c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] - \sum_{e\in\mathcal{E}_I^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma\}_I \cdot [z_\gamma]_I \\
&+ \sum_{e\in\mathcal{E}_I^h} \int_e \{d\mathbf{u}_\gamma c_\gamma\}_I \cdot [z_\gamma]_I - \int_\gamma \frac{2\eta_\gamma}{2\xi - 1} \left(c_m^{\gamma+} + c_m^{\gamma-} - 2c_\gamma\right) z_\gamma = \int_\gamma q_\gamma z_\gamma, \quad \forall z_\gamma \in H^1(\mathcal{T}_\gamma^h).
\end{aligned} \tag{4.13}$$

The definitions of the variation forms $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ continue to be used.

$$\begin{aligned}
A\left(c, z\right) = &\sum_{K\in\mathcal{T}_m^h} \int_K \mathbf{K}_m \nabla c_m \cdot \nabla z_m - \sum_{e\in\mathcal{E}_m^h} \int_e \{\mathbf{K}_m \nabla c_m \cdot \mathbf{n}_e\}[z_m] + \nu \sum_{e\in\mathcal{E}_m^h} \int_e \{\mathbf{K}_m \nabla z_m \cdot \mathbf{n}_e\}[c_m] \\
&+ \sum_{K\in\mathcal{T}_\gamma^h} \int_K \mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \nabla_\tau z_\gamma - \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma \cdot \mathbf{n}_e\}[z_\gamma] + \nu \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau z_\gamma \cdot \mathbf{n}_e\}[c_\gamma] \\
&+ \sigma J(c, z) + \sum_{*=+,-} \int_\gamma \frac{2\eta_\gamma}{2\xi - 1} \left(\xi c_m^{\gamma*} + (1 - \xi) c_m^{\gamma(-*)} - c_\gamma\right)\left(z_m^{\gamma*} - z_\gamma\right),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
B\left(c, z\right) = &- \sum_{K\in\mathcal{T}_m^h} \int_K \mathbf{u}_m c_m \cdot \nabla z_m + \sum_{e\in\mathcal{E}_m^h} \int_e \{\mathbf{u}_m c_m \cdot \mathbf{n}_e\}[z_m] - \sum_{K\in\mathcal{T}_\gamma^h} \int_K d\mathbf{u}_\gamma c_\gamma \cdot \nabla z_\gamma \\
&+ \sum_{e\in\mathcal{E}_\gamma^h} \int_e \{d\mathbf{u}_\gamma c_\gamma \cdot \mathbf{n}_e\}[z_\gamma].
\end{aligned} \tag{4.15}$$

Moreover, we also the introduce variation forms defined on the intersection I

$$A_I\left(c_\gamma, z_\gamma\right) = - \sum_{e\in\mathcal{E}_I^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau c_\gamma\}_I \cdot [z_\gamma]_I + \nu \sum_{e\in\mathcal{E}_I^h} \int_e \{\mathbf{K}_{\gamma,\tau} d\nabla_\tau z_\gamma\}_I \cdot [c_\gamma]_I + \sigma J_I(c_\gamma, z_\gamma), \tag{4.16}$$

with

$$J_I(c_\gamma, z_\gamma) = \begin{cases} \displaystyle\sum_{e\in\mathcal{E}_I^h} \frac{1}{|e|}\int_e [c_\gamma]_I \cdot [z_\gamma]_I, \text{two-dimensional fracture,} \\ \displaystyle\sum_{e\in\mathcal{E}_I^h} \max_{K_e\in\{\mathcal{E}_{\gamma_1}^h,\, \mathcal{E}_{\gamma_2}^h,\, \cdots,\, \mathcal{E}_{\gamma_{n_\gamma}}^h\}} \frac{1}{|K_e|}[c_\gamma]_I \cdot [z_\gamma]_I, \text{one-dimensional fracture,} \end{cases} \tag{4.17}$$

and

$$B_I\left(c_\gamma, z_\gamma\right) = \sum_{e\in\mathcal{E}_I^h} \int_e \{d\mathbf{u}_\gamma c_\gamma\}_I \cdot [z_\gamma]_I. \tag{4.18}$$

The exact solution $c = (c_m, c_\gamma)$ for the hybrid-dimensional fracture model of advection–diffusion equation with intersecting fractures satisfies the following for any $z = (z_m, z_\gamma) \in H^1(\mathcal{T}^h)$:

$$\int_{\Omega_m} \phi_m \frac{\partial c_m}{\partial t} z_m + \int_\gamma d\phi_\gamma \frac{\partial c_\gamma}{\partial t} z_\gamma + A(c,z) + B(c,z) + A_I(c,z) + B_I(c,z) = \int_{\Omega_m} q_m z_m + \int_\gamma q_\gamma z_\gamma. \tag{4.19}$$

Algorithm 2 (DG-backward Euler method for the model with intersecting fractures)**.** For $j = 1, 2, 3, ..., N$, given $c^{h,j-1} = \left(c_m^{h,j-1}, c_\gamma^{h,j-1}\right)$, find $c^{h,j} = \left(c_m^{h,j}, c_\gamma^{h,j}\right) \in \mathcal{D}_k(\mathcal{T}^h)$ such that for any $z = (z_m, z_\gamma) \in \mathcal{D}_k(\mathcal{T}^h)$, we have

$$\begin{aligned} &\int_{\Omega_m} \phi_m \frac{c_m^{h,j} - c_m^{h,j-1}}{\Delta t} z_m + \int_\gamma d\phi_\gamma \frac{c_\gamma^{h,j} - c_\gamma^{h,j-1}}{\Delta t} z_\gamma + A(c^{h,j}, z) + B(c^{h,j}, z) \\ &+ A_I(c_\gamma^{h,j}, z_\gamma) + B_I(c_\gamma^{h,j}, z_\gamma) = \int_{\Omega_m} q_m z_m + \int_\gamma q_\gamma z_\gamma, \\ &\int_{\Omega_m} c_m^{h,0} z_m + \int_\gamma c_\gamma^{h,0} z_\gamma = \int_{\Omega_m} c_m^0 z_m + \int_\gamma c_\gamma^0 z_\gamma. \end{aligned} \tag{4.20}$$

Theorem 2. *Assume that $c = \left(c_m, c_\gamma\right)$ with $c_{\Omega_i} \in H^{k+1}(\Omega_i)$ and $c_{\gamma_\ell} \in H^{k+1}(\gamma_\ell)$ is the solution to the problem* (4.1)–(4.5)*, and $c^h = (c_m^h, c_\gamma^h) \in \mathcal{D}_k(\mathcal{T}^h)$ is the numerical solution to Algorithm 2, respectively. Then, if σ is large enough and h is sufficiently small, a constant C exists such that*

$$\Delta t \sum_{j=1}^N |||c^j - c^{h,j}|||^2 \le C\left(\|c\|_{L^\infty([0,T],H^{k+1}(\Omega))}, \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2((0,T),L^2(\Omega))}\right)\left(h^{2k} + \Delta t^2\right). \tag{4.21}$$

Proof. Subtracting (4.20) from (4.19) and taking $z = \varsigma^j = (\varsigma_m^j, \varsigma_\gamma^j) \in \mathcal{D}_k\left(\mathcal{T}^h\right)$, we have

$$\begin{aligned} &\int_{\Omega_m} \phi_m \frac{\partial c_m^j}{\partial t} \varsigma_m^j + \int_\gamma d\phi_\gamma \frac{\partial c_\gamma^j}{\partial t} \varsigma_\gamma^j - \int_{\Omega_m} \phi_m \frac{c_m^{h,j} - c_m^{h,j-1}}{\Delta t} \varsigma_m^j - \int_\gamma d\phi_\gamma \frac{c_\gamma^{h,j} - c_\gamma^{h,j-1}}{\Delta t} \varsigma_\gamma^j + A\left(c^j, \varsigma^j\right) \\ &- A\left(c^{h,j}, \varsigma^j\right) + B\left(c^j, \varsigma^j\right) - B\left(c^{h,j}, \varsigma^j\right) + A_I(c_\gamma^j, \varsigma_\gamma^j) - A_I(c_\gamma^{h,j}, \varsigma_\gamma^j) + B_I(c_\gamma^j, \varsigma_\gamma^j) - B_I(c_\gamma^{h,j}, \varsigma_\gamma^j) = 0. \end{aligned} \tag{4.22}$$

This means

$$\begin{aligned} &\int_{\Omega_m} \phi_m \frac{\varsigma_m^j - \varsigma_m^{j-1}}{\Delta t} \varsigma_m^j + \int_\gamma d\phi_\gamma \frac{\varsigma_\gamma^j - \varsigma_\gamma^{j-1}}{\Delta t} \varsigma_\gamma^j + A\left(\varsigma^j, \varsigma^j\right) + A_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right) \\ =& - A\left(\varrho^j, \varsigma^j\right) - B\left(\varrho^j, \varsigma^j\right) - B\left(\varsigma^j, \varsigma^j\right) - A_I\left(\varrho_\gamma^j, \varsigma_\gamma^j\right) - B_I(\varrho_\gamma^j, \varsigma_\gamma^j) - B_I(\varsigma_\gamma^j, \varsigma_\gamma^j) \\ &- \int_{\Omega_m} \phi_m \left(\frac{\partial c_m^j}{\partial t} - \frac{\bar{c}_m^j - \bar{c}_m^{j-1}}{\Delta t}\right) \varsigma_m^j - \int_\gamma d\phi_\gamma \left(\frac{\partial c_\gamma^j}{\partial t} - \frac{\bar{c}_\gamma^j - \bar{c}_\gamma^{j-1}}{\Delta t}\right) \varsigma_\gamma^j. \end{aligned} \tag{4.23}$$

For the first two terms on the left-hand side of the equation above, we have the same estimate as (3.67)

$$
\begin{aligned}
&\int_{\Omega_m} \phi_m \frac{\varsigma_m^j - \varsigma_m^{j-1}}{\Delta t} \varsigma_m^j + \int_\gamma d\phi_\gamma \frac{\varsigma_\gamma^j - \varsigma_\gamma^{j-1}}{\Delta t} \varsigma_\gamma^j \\
&\geq \frac{1}{2\Delta t}\left(\left(\|\phi_m^{\frac{1}{2}}\varsigma_m^j\|_{0,\Omega_m}^2 + \|d^{\frac{1}{2}}\phi_\gamma^{\frac{1}{2}}\varsigma_\gamma^j\|_{0,\gamma}^2\right) - \left(\|\phi_m^{\frac{1}{2}}\varsigma_m^{j-1}\|_{0,\Omega_m}^2 + \|d^{\frac{1}{2}}\phi_\gamma^{\frac{1}{2}}\varsigma_\gamma^{j-1}\|_{0,\gamma}^2\right)\right).
\end{aligned}
\tag{4.24}
$$

According to Lemma 3, we have

$$
A\left(\varsigma^j, \varsigma^j\right) \geq \alpha |||\varsigma^j|||^2. \tag{4.25}
$$

By applying the definition of A_I to the left-hand side of (4.23), the last term can be bounded as

$$
\begin{aligned}
A_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right) =\ & (\nu - 1) \sum_{e \in \mathcal{E}_I^h} \int_e \{\mathbf{K}_{\gamma,\tau} d \nabla_\tau \varsigma_\gamma^j\}_I \cdot [\varsigma_\gamma^j]_I + \sigma J_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right) \\
=\ & \frac{\nu - 1}{n_\gamma} \sum_{e \in \mathcal{E}_I^h} \int_e \left(\mathbf{K}_{\gamma_p,\tau} d \nabla_\tau \varsigma_{\gamma_p}^j \cdot \tau_{\gamma_p} - \mathbf{K}_{\gamma_\ell,\tau} d \nabla_\tau \varsigma_{\gamma_\ell}^j \cdot \tau_{\gamma_\ell}\right)_{1 \leq p < \ell \leq n_\gamma} \cdot [\varsigma_\gamma^j]_I + \sigma J_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right) \\
\geq\ & - C \sum_{\ell=1}^{n_\gamma} \sum_{e \in \mathcal{E}_I^h} \|\nabla_\tau \varsigma_{\gamma_\ell}^j\|_{0,e} \|[\varsigma_\gamma^j]_I\|_{0,e} + \sigma J_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right) \\
\geq\ & - C h^{-\frac{1}{2}} \sum_{\ell=1}^{n_\gamma} \sum_{e \in \mathcal{E}_I^h} \|\nabla_\tau \varsigma_{\gamma_\ell}^j\|_{0,K_e} \|[\varsigma_\gamma^j]_I\|_{0,e} + \sigma J_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right) \\
\geq\ & - \varepsilon \|\varsigma_\gamma^j\|_{1,\mathcal{T}_\gamma^h}^2 + (\sigma - C) J_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right).
\end{aligned}
\tag{4.26}
$$

From (4.24)–(4.26), the left-hand side of (4.23) can be estimated as

$$
\mathcal{L} \geq \frac{1}{2\Delta t}\left(\|\phi^{\frac{1}{2}}\varsigma^j\|_{0,\Omega}^2 - \|\phi^{\frac{1}{2}}\varsigma^{j-1}\|_{0,\Omega}^2\right) + (\alpha - \varepsilon) |||\varsigma^j|||^2 + (\sigma - C) J_I\left(\varsigma_\gamma^j, \varsigma_\gamma^j\right). \tag{4.27}
$$

According to Lemmas 4 and 5, the first three terms on the right-hand side of (4.23) can be bounded as

$$
\begin{aligned}
-A\left(\varrho^j, \varsigma^j\right) &\leq C\left(|||\varrho^j|||^2 + h^2\|\varrho^j\|_{2,\mathcal{T}^h}^2 + h^{-2}\|\varrho^j\|_{0,\Omega}\right) + \varepsilon |||\varsigma^j|||^2, \\
-B\left(\varsigma^j, \varsigma^j\right) &\leq C\|\varsigma^j\|_{0,\Omega}^2 + \varepsilon |||\varsigma^j|||^2, \\
-B\left(\varrho^j, \varsigma^j\right) &\leq C\left(\|\varrho^j\|_{0,\Omega}^2 + h^2\|\varrho^j\|_{1,\mathcal{T}^h}^2\right) + \varepsilon |||\varsigma^j|||^2.
\end{aligned}
\tag{4.28}
$$

For the fourth term on the right-hand side of (4.23), it reads as follows:

$$
\begin{aligned}
-A_I\left(\varrho_\gamma^j, \varsigma_\gamma^j\right) &= \sum_{e \in \mathcal{E}_m^h} \int_e \{\mathbf{K}_{\gamma,\tau} d \nabla_\tau \varrho_\gamma^j\}_I \cdot [\varsigma_\gamma^j]_I - \nu \sum_{e \in \mathcal{E}_m^h} \int_e \{\mathbf{K}_{\gamma,\tau} d \nabla_\tau \varsigma_\gamma^j\}_I \cdot [\varrho_\gamma^j]_I - \sigma J_I\left(\varrho_\gamma^j, \varsigma_\gamma^j\right) \\
&:= R_1 + R_2 + R_3.
\end{aligned}
\tag{4.29}
$$

Similar to (3.38), we obtain

$$\begin{aligned} R_1 &= \frac{1}{n_\gamma}\sum_{e\in\mathcal{E}_I^h}\int_e \left(\mathbf{K}_{\gamma_p,\tau} d_p \nabla_\tau \varrho^j_{\gamma_p}\cdot\tau_{\gamma_p} - \mathbf{K}_{\gamma_\ell,\tau} d_\ell \nabla_\tau \varrho^j_{\gamma_\ell}\cdot\tau_{\gamma_\ell}\right)_{1\le p<\ell\le n_\gamma}\cdot[\varsigma^j_\gamma]_I \\ &\le C\sum_{\ell=1}^{n_\gamma}\sum_{e\in\mathcal{E}_I^h}\|\nabla_\tau\varrho^j_{\gamma_\ell}\|_{0,e}\|[\varsigma^j_\gamma]_I\|_{0,e} \\ &\le C\left(h^2\|\varrho^j_\gamma\|^2_{2,\mathcal{T}^h_\gamma}+\|\varrho^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}\right)^{\frac{1}{2}} J_I^{\frac{1}{2}}\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right). \end{aligned} \tag{4.30}$$

Similar to (3.40) and (4.30), it is found that

$$R_2 \le C\|\varsigma^j_\gamma\|_{1,\mathcal{T}^h_\gamma}\left(\|\varrho^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}+h^{-2}\|\varrho^j_\gamma\|^2_{0,\gamma}\right)^{\frac{1}{2}}. \tag{4.31}$$

Similar to (3.42), we show

$$R_3 \le C\left(\|\varrho^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}+h^{-2}\|\varrho^j_\gamma\|^2_{0,\gamma}\right)^{\frac{1}{2}} J_I^{\frac{1}{2}}\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right). \tag{4.32}$$

Introducing the results of (4.30)–(4.32) to (4.29) shows that

$$\begin{aligned} -A_I\left(\varrho^j_\gamma,\varsigma^j_\gamma\right) &\le C\left(h^2\|\varrho^j_\gamma\|^2_{2,\mathcal{T}^h_\gamma}+\|\varrho^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}+h^{-2}\|\varrho^j_\gamma\|^2_{0,\gamma}\right)^{\frac{1}{2}}\left(\|\varsigma^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}+J_I^{\frac{1}{2}}\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right)\right) \\ &\le C\left(h^2\|\varrho^j_\gamma\|^2_{2,\mathcal{T}^h_\gamma}+\|\varrho^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}+h^{-2}\|\varrho^j_\gamma\|^2_{0,\gamma}\right)+\varepsilon|||\varsigma^j_\gamma|||^2+\varepsilon J_I\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right). \end{aligned} \tag{4.33}$$

For the fifth term on the right-hand side of (4.23), it holds the bound

$$\begin{aligned} B_I(\varrho^j_\gamma,\varsigma^j_\gamma) &= \frac{1}{n_\gamma}\sum_{e\in\mathcal{E}_I^h}\int_e\left(d_p\mathbf{u}_{\gamma_p}\varrho^j_{\gamma_p}\cdot\tau_{\gamma_p}-d_\ell\mathbf{u}_{\gamma_\ell}\varrho^j_{\gamma_\ell}\cdot\tau_{\gamma_\ell}\right)_{1\le p<\ell\le n_\gamma}\cdot[\varsigma^j_\gamma]_I \\ &\le C\sum_{\ell=1}^{n_\gamma}\sum_{e\in\mathcal{E}_I^h}\|\varrho^j_{\gamma_\ell}\|_{0,e}\|[\varsigma^j_\gamma]_I\|_{0,e} \\ &\le C\sum_{\ell=1}^{n_\gamma}\sum_{e\in\mathcal{E}_I^h}\left(h^{-\frac{1}{2}}\|\varrho^j_{\gamma_\ell}\|_{0,K_e}+h^{\frac{1}{2}}\|\nabla_\tau\varrho^j_{\gamma_\ell}\|_{0,K_e}\right)\|[\varsigma^j_\gamma]_I\|_{0,e} \\ &\le C\left(\sum_{\ell=1}^{n_\gamma}\left(\|\varrho^j_{\gamma_\ell}\|_{0,\gamma_\ell}+h\|\varrho^j_{\gamma_\ell}\|_{1,\mathcal{T}^h_{\gamma_\ell}}\right)\right)J_I^{\frac{1}{2}}\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right) \\ &\le C\left(\|\varrho^j_\gamma\|^2_{0,\gamma}+h^2\|\varrho^j_\gamma\|^2_{1,\mathcal{T}^h_\gamma}\right)+\varepsilon J_I\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right). \end{aligned} \tag{4.34}$$

Similarly, we have

$$B_I(\varsigma^j_\gamma,\varsigma^j_\gamma) \le C\|\varsigma^j_\gamma\|^2_{0,\gamma}+\varepsilon J_I\left(\varsigma^j_\gamma,\varsigma^j_\gamma\right). \tag{4.35}$$

Similar to (3.74), we have the following estimate for the last two terms on the right-hand side of (4.23):

$$\begin{aligned}&\left|\int_{\Omega_m}\phi_m\left(\frac{\partial c_m^j}{\partial t}-\frac{\overline{c}_m^j-\overline{c}_m^{j-1}}{\Delta t}\right)\varsigma_m^j-\int_\gamma d\phi_\gamma\left(\frac{\partial c_\gamma^j}{\partial t}-\frac{\overline{c}_\gamma^j-\overline{c}_\gamma^{j-1}}{\Delta t}\right)\varsigma_\gamma^j\right|\\ &\le C\|\varsigma^j\|_{0,\Omega}^2+C\left(\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left\|\frac{\partial\varrho}{\partial t}\right\|_{0,\Omega}^2 dt+\Delta t\int_{t^{j-1}}^{t^j}\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{0,\Omega}^2 dt\right),\end{aligned}\tag{4.36}$$

with $\frac{\partial\varrho}{\partial t}=\left(\frac{\partial\varrho_m}{\partial t},\frac{\partial\varrho_\gamma}{\partial t}\right)$and $\frac{\partial^2 c}{\partial t^2}=\left(\frac{\partial^2 c_m}{\partial t^2},\frac{\partial^2 c_\gamma}{\partial t^2}\right)$.
It shows that the right-hand side of (4.23) has the following estimate from (4.28) and (4.33)–(4.36):

$$\begin{aligned}\mathcal{R}\le{}& C\|\varsigma^j\|_{0,\Omega}^2+C\left(|||\varrho^j|||^2+h^2\|\varrho^j\|_{2,\mathcal{T}^h}^2+h^{-2}\|\varrho^j\|_{0,\Omega}^2\right)+C\left(\|\varrho^j\|_{0,\Omega}^2+h^2\|\varrho^j\|_{1,\mathcal{T}^h}^2\right)\\ &+C\left(\frac{1}{\Delta t}\int_{t^j-1}^{t^j}\left\|\frac{\partial\varrho}{\partial t}\right\|_{0,\Omega}^2 dt+\Delta t\int_{t^{j-1}}^{t^j}\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{0,\Omega}^2 dt\right)+4\varepsilon|||\varsigma^j|||^2+3\varepsilon J_I\left(\varsigma_\gamma^j,\varsigma_\gamma^j\right).\end{aligned}\tag{4.37}$$

Taking (4.27) and (4.37) back to (4.23), it follows that

$$\begin{aligned}&(\alpha-5\varepsilon)\,|||\varsigma^j|||^2+\frac{1}{2\Delta t}\left(\|\phi^{\frac{1}{2}}\varsigma^j\|_{0,\Omega}^2-\|\phi^{\frac{1}{2}}\varsigma^{j-1}\|_{0,\Omega}^2\right)+(\sigma-C-3\varepsilon)\,J_I\left(\varsigma_\gamma^j,\varsigma_\gamma^j\right)\\ &\le C\|\varsigma^j\|_{0,\Omega}^2+C\left(\Delta t\int_{t^{j-1}}^{t^j}\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{0,\Omega}^2 dt+\frac{1}{\Delta t}\int_{t^{j-1}}^{t^j}\left\|\frac{\partial\varrho}{\partial t}\right\|_{0,\Omega}^2 dt\right)\\ &\quad+C\left(|||\varrho^j|||^2+h^2\|\varrho^j\|_{2,\mathcal{T}^h}^2+h^{-2}\|\varrho^j\|_{0,\Omega}^2\right).\end{aligned}\tag{4.38}$$

Similar to (3.79), we have the following estimate:

$$\Delta t\sum_{j=1}^N|||\varsigma^j|||^2\le C\left(\|c\|_{L^\infty([0,T],H^{k+1}(\Omega))},\left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2((0,T),L^2(\Omega))}\right)\left(h^{2k}+\Delta t^2\right).\tag{4.39}$$

By using the triangle inequality and (3.63), we finish the proof.

Remark 2. In addition to the assumptions in Remark 1, under the condition that $[d\mathbf{u}]_I=0$, similar L^2 stability holds for the model with intersecting fractures. Similar to (4.25) and (4.26), we have

$$A\left(c^{h,j},c^{h,j}\right)+A_I\left(c_\gamma^{h,j},c_\gamma^{h,j}\right)\ge\alpha|||c^{h,j}|||^2+(\sigma-C)\,J_I\left(c_\gamma^{h,j},c_\gamma^{h,j}\right).\tag{4.40}$$

In the term $B(c^{h,j},c^{h,j})$, due to the intersecting fractures, we need to analyze the terms related to fractures again.

$$
\begin{aligned}
&-\sum_{K\in\mathcal{T}_\gamma^h}\int_K d\mathbf{u}_\gamma c_\gamma^{h,j}\cdot\nabla c_\gamma^{h,j}\\
&= -\frac{1}{2}\sum_{\ell=1}^{n_\gamma}\sum_{K\in\mathcal{T}_{\gamma_\ell}^h}\int_K d_\ell \mathbf{u}_{\gamma_\ell}\cdot\nabla\left(c_{\gamma_\ell}^{h,j}\right)^2\\
&= \frac{1}{2}\sum_{\ell=1}^{n_\gamma}\sum_{K\in\mathcal{T}_{\gamma_\ell}^h}\int_K d_\ell\nabla\cdot\mathbf{u}_{\gamma_\ell}\left(c_{\gamma_\ell}^{h,j}\right)^2 - \frac{1}{2}\sum_{\ell=1}^{n_\gamma}\sum_{K\in\mathcal{T}_{\gamma_\ell}^h}\int_{\partial K} d_\ell\mathbf{u}_{\gamma_\ell}\cdot\mathbf{n}_{\partial K}\left(c_{\gamma_\ell}^{h,j}\right)^2\\
&= -\frac{1}{2}\sum_{e\in\mathcal{E}_\gamma^h}\int_e d\mathbf{u}_\gamma\cdot\mathbf{n}_e\left[\left(c_\gamma^{h,j}\right)^2\right] - \frac{1}{2}\sum_{\ell=1}^{n_\gamma}\sum_{K\in\mathcal{T}_{\gamma_\ell}^h}\int_{\partial K\cap I} d_\ell\mathbf{u}_{\gamma_\ell}\cdot\mathbf{n}_{\partial K}\left(c_{\gamma_\ell}^{h,j}\right)^2 \qquad (4.41)\\
&= -\frac{1}{2}\sum_{e\in\mathcal{E}_\gamma^h}\int_e d\mathbf{u}_\gamma\cdot\mathbf{n}_e\left[\left(c_\gamma^{h,j}\right)^2\right] - \frac{1}{2}\sum_{e\in\mathcal{E}_I^h}\int_e\left[d\mathbf{u}_\gamma\left(c_\gamma^{h,j}\right)^2\right]\\
&= -\frac{1}{2}\sum_{e\in\mathcal{E}_\gamma^h}\int_e d\mathbf{u}_\gamma\cdot\mathbf{n}_e\left[\left(c_\gamma^{h,j}\right)^2\right] - \frac{1}{2}\sum_{e\in\mathcal{E}_I^h}\int_e\left[d\mathbf{u}_\gamma\right]\left\{\left(c_\gamma^{h,j}\right)^2\right\} - \frac{1}{2}\sum_{e\in\mathcal{E}_I^h}\int_e\left\{d\mathbf{u}_\gamma\right\}\cdot\left[\left(c_\gamma^{h,j}\right)^2\right]\\
&= -\frac{1}{2}\sum_{e\in\mathcal{E}_\gamma^h}\int_e d\mathbf{u}_\gamma\cdot\mathbf{n}_e\left[\left(c_\gamma^{h,j}\right)^2\right].
\end{aligned}
$$

Then by using the same steps in (3.31) and (3.32), the bilinear form of the diffusion part vanishes identically, i.e., $B(c^{h,j}, c^{h,j}) = 0$. According to the second condition in (4.4), we have $B_I(c_\gamma^{h,j}, c_\gamma^{h,j}) = 0$. Similar to (3.31) and (3.32), the L^2 stability is proved.

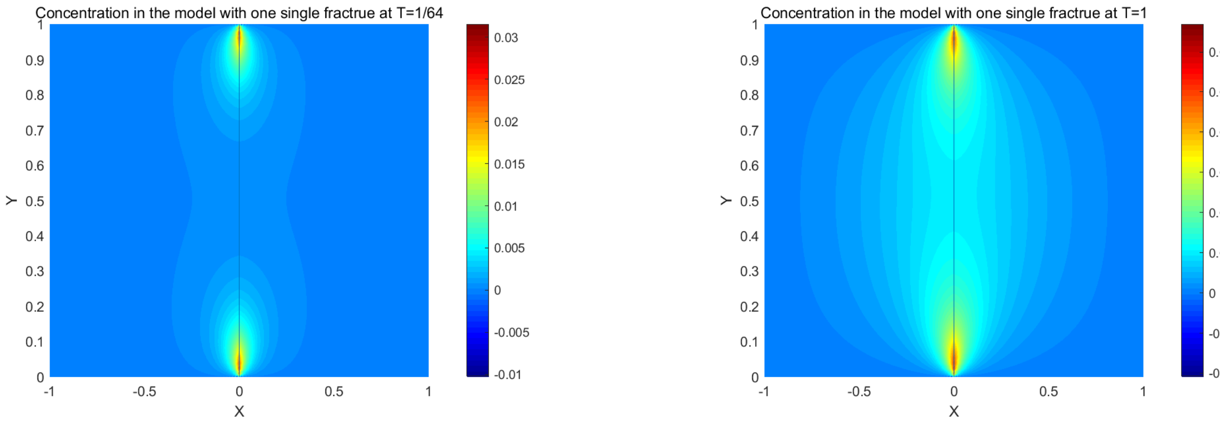
In this section, although we only consider a single intersection, the definitions of A_I and B_I, the algorithm design, the theoretical analysis, and the numerical examples can all be extended to multiple intersections.

5. Numerical experiment

In this section, numerical experiments are conducted to verify the accuracy and efficiency of the DG–backward Euler method for solving the hybrid–dimensional fracture models of the advection–diffusion equation with both one single fracture and intersecting fractures. In terms of spatial discretization, piecewise linear polynomial spaces are used, and the mesh partitioning is performed as follows: The matrix domain Ω_i is partitioned into square cells of side length h_{Ω_i}, each of which is further subdivided into two right-angled triangles, and concurrently, each fracture γ_ℓ is discretized into segments of length h_{γ_ℓ}. Within this discrete framework, we compute the errors in the discrete H^1-norm for the concentration in both the matrix and fracture domains, along with the corresponding convergence rates. To clearly demonstrate the convergence behavior with respect to the mesh size h_{Ω_i}, the numerical solution on a refined mesh (with size $h_{\Omega_i}/2$) is used as the reference solution. The errors and convergence rates for each coarser mesh are then obtained by evaluating their solutions against this reference. The coefficients in the proposed scheme are chosen as follows: $\sigma = 100$ and $\nu = -1$.

Example 1. Experiment with a single fracture

The first example refers to the geometry made of two unitary squares separated by a single vertical fracture, i.e., $\Omega_1 = (-1, 0) \times (0, 1)$, $\Omega_2 = (0, 1) \times (0, 1)$ and $\gamma = \{x = 0\} \times (0, 1)$. The permeability parameters are set as $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{E}$, where $\mathbf{E}$ is the 2×2 identity matrix, and $\mathbf{K}_{\gamma,\tau} = 0.1$ and $\mathbf{K}_{\gamma,n} = 0.001$. The fracture width is $d = 0.01$, and $\xi = 4/5$. We consider the source terms $q_i = 0$ and $q_\gamma = 0$, the Dirichlet boundary conditions $c_{\partial,i} = 0$ and $c_{\partial,\gamma} = 0.5$, the parameters $\phi_i = \phi_\gamma = 0.5$ and $\mathbf{u}_i = (3,1)^{\mathbf{T}}, u_\gamma = 1$; and the concentration $c_i^0 = c_\gamma^0 = 0$ at the initial time. The total duration is $T = 1$, and the time step Δt equals the spatial step size h_{Ω_i}. The errors and convergence rates are shown in Table 1, and Figure 3 presents the concentration distributions at times $T = 1/64$ and $T = 1$.


Concentration in the model with one single fractrue at T=1/64

Concentration in the model with one single fractrue at T=1

(a) **(b)**

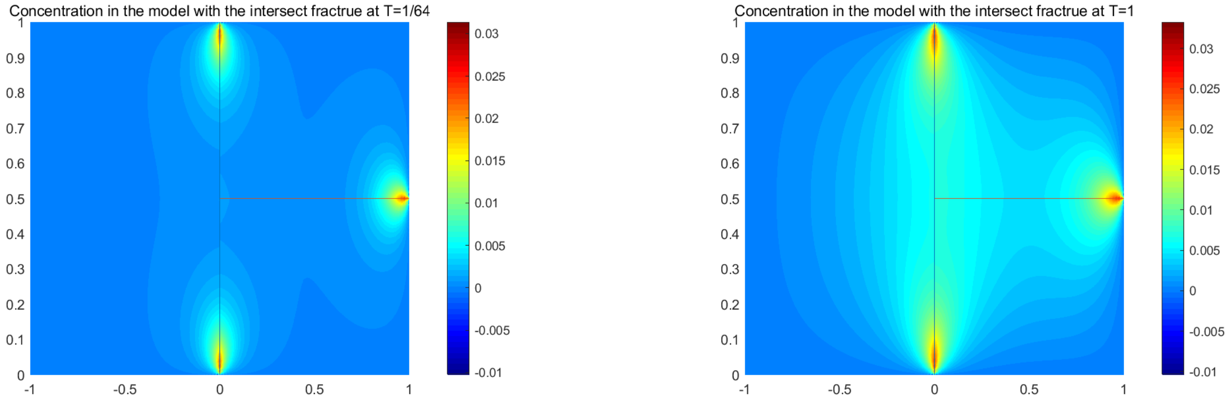
Figure 3. Concentration distributions in the model with one single fracture. (a) T = 1/64. (b) T = 1.

Example 2. Experiment with intersecting fractures

We next consider an intersecting fractured network, in which Ω is subdivided into three different areas $\Omega_1 = (-1,0)\times(0,1)$, $\Omega_2 = (0,1)\times(0,0.5)$, and $\Omega_3 = (0,1)\times(0.5,1)$ by intersecting the interfaces $\gamma_1 = \partial\Omega_1 \cap \partial\Omega_2, \gamma_2 = \partial\Omega_1 \cap \partial\Omega_3$, and $\gamma_3 = \partial\Omega_2 \cap \partial\Omega_3$. The Dirichlet condition for the concentrations $c_{\partial,m} = 0$ and $c_{\partial,\gamma} = 0.5$ are used, the source terms $q_{\Omega_i} = 0$ and $q_{\gamma_\ell} = 0$ are added in the matrix and fractures, and the concentration $c_m^0 = c_\gamma^0 = 0$ at the initial time. The tensors of permeability in the matrix and fractures are set as $\mathbf{K}_m = \mathbf{E}$, $\mathbf{K}_{\gamma,\tau} = 1$, and $\mathbf{K}_{\gamma,n} = 0.001$. The width of fracture is $d = 0.01$, and $\xi = 4/5$, and the parameters are $\phi_m = \phi_\gamma = 0.5$, $\mathbf{u}_m = (3,1)^{\mathbf{T}}, u_\gamma = 1$. Similarly, the time step Δt equals the spatial step size h_{Ω_i}, and the total duration is $T = 1$. The errors and convergence rates are shown in Table 2, and Figure 4 presents the concentration distributions at times $T = 1/64$ and $T = 1$.

Table 1. Errors and convergence rates for Example 1.

h_{Ω_i}	$\|\|c_1 - c_1^h\|\|_{1,\mathcal{T}_1^h}$	Rate	$\|\|c_2 - c_2^h\|\|_{1,\mathcal{T}_2^h}$	Rate	$\|\|c_\gamma - c_\gamma^h\|\|_{1,\mathcal{T}_\gamma^h}$	Rate
2^{-2}	1.32E-2	-	1.32E-2	-	3.66E-1	-
2^{-3}	7.86E-3	0.75	7.86E-3	0.75	1.89E-1	0.95
2^{-4}	4.67E-3	0.75	4.67E-3	0.75	9.55E-2	0.99
2^{-5}	2.71E-3	0.78	2.71E-3	0.78	4.80E-2	0.99
2^{-6}	1.53E-3	0.82	1.53E-3	0.82	2.42E-2	0.99
2^{-7}	8.50E-4	0.85	8.50E-4	0.85	1.24E-2	0.98



(a) **(b)**

Figure 4. Concentration distributions in the model with the intersecting fractures. (a) T = 1/64. (b) T = 1.

Table 2. Errors and convergence rates for Example 2.

h_{Ω_i}	$\|\|c_1 - c_1^h\|\|_{1,\mathcal{T}_1^h}$	Rate	$\|\|c_2 - c_2^h\|\|_{1,\mathcal{T}_2^h}$	Rate	$\|\|c_3 - c_3^h\|\|_{1,\mathcal{T}_3^h}$	Rate	$\|\|c_\gamma - c_\gamma^h\|\|_{1,\mathcal{T}_\gamma^h}$	Rate
2^{-2}	3.60E-1	-	2.41E-1	-	2.46E-1	-	2.71E-01	-
2^{-3}	2.24E-1	0.68	1.48E-1	0.70	1.51E-1	0.70	1.30E-01	1.06
2^{-4}	1.30E-1	0.78	8.62E-2	0.78	8.76E-2	0.79	6.33E-02	1.03
2^{-5}	7.29E-2	0.84	4.84E-2	0.83	4.90E-2	0.84	3.13E-01	1.02
2^{-6}	3.98E-2	0.87	2.66E-2	0.86	2.68E-2	0.87	1.56E-02	1.01
2^{-7}	2.14E-2	0.89	1.44E-3	0.89	1.45E-2	0.89	7.75E-03	1.00

Example 3. Experiment with the exact solution

Next, we investigate an experiment with an exact solution, where the errors and convergence rates are evaluated against the exact solution. The tensor of permeability in the matrix and fracture are set as $\mathbf{K}_1 = \mathbf{K}_2 = 10\,\mathbf{E}$ and $\mathbf{K}_{f,\tau} = 15, \mathbf{K}_{f,n} = 10^{-3}$. The width of the fracture is $d = 0.001$ and $\xi = 56/73$, and the parameters are $\phi_i = \phi_\gamma = 0.5$, $\mathbf{u}_i = (5/13, 2/13)^{\mathbf{T}}$, and $u_\gamma = 0.5$. The errors and convergence rates are listed in Table 3.

$$\begin{cases} c_1 = \dfrac{(x+9)t\sin(\pi y)}{9} \\ c_2 = \dfrac{(-2x+4)t\sin(\pi y)}{9} \\ c_\gamma = t\sin(\pi y) \end{cases}$$

Tables 1–3 respectively record the numerical errors and convergence rates of the discrete H^1-norm for the concentration which is solved by using the DG-backward Euler method. From the tables, it can be observed that for the hybrid-dimensional fracture models involving both a single fracture and intersecting fractures, the DG-backward Euler method achieves the expected $O(h + \Delta t)$ convergence rate for the concentration in the discrete H^1-norm in both the matrix and fractures, which is consistent with the theoretical analysis provided in Theorems 1 and 2.

Table 3. Errors and convergence rates for Example 3.

h_{Ω_i}	$\|c_1 - c_1^h\|_{1,\mathcal{T}_1^h}$	Rate	$\|c_2 - c_2^h\|_{1,\mathcal{T}_2^h}$	Rate	$\|c_\gamma - c_\gamma^h\|_{1,\mathcal{T}_\gamma^h}$	Rate
2^{-4}	1.24E-00	-	4.75E-02	-	9.85E-02	-
2^{-5}	6.18E-01	1.00	2.37E-02	1.00	4.91E-02	1.00
2^{-6}	3.09E-01	1.00	1.19E-02	1.00	2.46E-02	1.00
2^{-7}	1.55E-01	1.00	5.93E-03	1.00	1.23E-02	1.00
2^{-8}	7.73E-02	1.00	2.96E-03	1.00	6.14E-03	1.00

6. Conclusions

In this paper, the DG-backward Euler method is firstly used for a kind of hybrid–dimensional fracture models of the advection–diffusion equation with one single fracture and with intersecting fractures. We verify the accuracy of the method from both the theoretical and practical aspects. The desired optimal error bounds are achieved in the discrete H^1-norm for the concentration. Numerical examples with both one fracture and intersecting fractures are carried out, and the numerical results are consistent with the conclusions of Theorems 1 and 2.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

The work was supported by the National Natural Science Foundation of China under Grant 12101025. The material in this paper was not presented at any conference.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. B. Lepillier, A. Daniilidis, N. D. Gholizadeh, P. O. Bruna, J. Kummerow, D. Bruhn, A fracture flow permeability and stress dependency simulation applied to multi-reservoirs, multi-production scenarios analysis, *Geotherm. Energy*, **7** (2019), 24. https://doi.org/10.1186/s40517-019-0141-8
2. S. Salimzadeh, H. Nick, A coupled model for reactive flow through deformable fractures in enhanced geothermal systems, *Geothermics*, **81** (2019), 88–100. https://doi.org/10.1016/j.geothermics.2019.04.010
3. R. March, F. Doster, S. Geiger, Assessment of CO_2 storage potential in naturally fractured reservoirs with dual-porosity models, *Water Resour. Res.*, **54** (2018), 1650–1668. https://doi.org/10.1002/2017WR022159

4. F. Ren, G. Ma, Y. Wang, L. Fan, H. Zhu, Two-phase flow pipe network method for simulation of CO_2 sequestration in fractured saline aquifers, *Int. J. Rock Mech. Min. Sci.*, **98** (2017), 39–53. https://doi.org/10.1016/j.ijrmms.2017.07.010

5. A. R. Khoei, N. Hosseini, T. Mohammadnejad, Numerical modeling of two-phase fluid flow in deformable fractured porous media using the extended finite element method and an equivalent continuum model, *Adv. Water Resour.*, **94** (2016), 510–528. https://doi.org/10.1016/j.advwatres.2016.02.017

6. J. Jiang, J. Yang, Coupled fluid flow and geomechanics modeling of stress-sensitive production behavior in fractured shale gas reservoirs, *Int. J. Rock Mech. Min. Sci.*, **101** (2018), 1–12. https://doi.org/10.1016/j.ijrmms.2017.11.003

7. T. Ma, H. Xu, C. Guo, X. Fu, W. Liu, R. Yang, A discrete fracture modeling approach for analysis of coalbed methane and water flow in a fractured coal reservoir, *Geofluids*, **1** (2020), 8845348. https://doi.org/10.1155/2020/8845348

8. J. E. Saiers, E. Barth, Potential contaminant pathways from hydraulically fractured shale aquifers, *Groundwater*, **50** (2012), 826–828. https://doi.org/10.1111/j.1745-6584.2012.00930.x

9. C. Alboin, J. Jaffré, J. E. Roberts, X. Wang, C. Serres, Domain decomposition for some transmission problems in flow in porous media, *Lect. Notes Phys.*, **552** (2000), 22–34. https://doi.org/10.1007/3-540-45467-5_2

10. V. Martin, J. Jaffré, J. E. Roberts, Modeling fractures and barriers as interfaces for flow in porous media, *SIAM J. Sci. Comput.*, **26** (2005), 1667–1691. https://doi.org/10.1137/S1064827503429363

11. M. Lesinigo, C. D'Angelo, A. Quarteroni, A multiscale Darcy–Brinkman model for fluid flow in fractured porous media, *Numer. Math.*, **117** (2011), 717–752. https://doi.org/10.1007/s00211-010-0343-2

12. S. Chen, Q. Huang, F. Xu, A two-grid decoupled algorithm for a multi-dimensional Darcy–Brinkman fracture model, *J. Sci. Comput.*, **90** (2022), 88. https://doi.org/10.1007/s10915-021-01738-y

13. N. Frih, J. E. Roberts, A. Saada, Modeling fractures as interfaces: A model for Forchheimer fractures, *Comput. Geosci.*, **12** (2008), 91–104. https://doi.org/10.1007/s10596-007-9062-x

14. J. Aghili, K. Brenner, J. Hennicker, R. Masson, L. Trenty, Two-phase discrete fracture matrix models with linear and nonlinear transmission conditions, *GEM Int. J. Geomath.*, **10** (2019), 1. https://doi.org/10.1007/s13137-019-0118-6

15. E. Ahmed, J. Jaffré, J. E. Roberts, A reduced fracture model for two-phase flow with different rock types, *Math. Comput. Simul.*, **137** (2017), 49–70. http://dx.doi.org/10.1016/j.matcom.2016.10.005

16. S. Chen, Discontinuous Galerkin method for hybrid-dimensional fracture models of two-phase flow, *J. Comput. Phys.*, **488** (2023), 112244. https://doi.org/10.1016/j.jcp.2023.112244

17. H. Diao, X. Fei, H. Liu, Geometrical characterizations of radiating and non-radiating elastic sources and mediums with applications, preprint, arXiv:2508.05401.

18. H. Diao, H. Liu, Q. Meng, Dislocations with corners in an elastic body with applications to fault detection, *SIAM J. Appl. Math.*, **85** (2025), 2399–2424.

19. H. Diao, H. Liu, Q. Meng, Dislocations in a multi-layered elastic solid with applications to fault and interface identifications, preprint, arXiv:2505.20763.
20. N. Frih, V. Martin, J. E. Roberts, A. Saâda, Modeling fractures as interfaces with nonmatching grids, *Comput. Geosci.*, **16** (2012), 1043–1060. https://doi.org/10.1007/s10596-012-9302-6
21. W. M. Boon, J. M. Nordbotten, I. Yotov, Robust discretization of flow in fractured porous media, *SIAM J. Numer. Anal.*, **56** (2018), 2203–2233. https://doi.org/10.1137/17M1139102
22. A. Fumagalli, E. Keilegavlen, S. Scialò, Conforming, non-conforming and non-matching discretization couplings in discrete fracture network simulations, *J. Comput. Phys.*, **376** (2019), 694–712. https://doi.org/10.1016/j.jcp.2018.09.048
23. B. Giovanardi, L. Formaggia, A. Scotti, P. Zunino, Unfitted FEM for modelling the interaction of multiple fractures in a poroelastic medium, *Lect. Notes Comput. Sci. Eng.*, **121** (2018), 331–352. https://doi.org/10.1007/978-3-319-71431-8_11
24. S. Chen, X. Li, H. Rui, The finite volume method based on the Crouzeix–Raviart element for a fracture model, *Numer. Methods Partial Differ. Equations*, **35** (2019), 1904–1927. https://doi.org/10.1002/num.22384
25. P. F. Antonietti, L. Formaggia, A. Scotti, M. Verani, N. Verzotti, Mimetic finite difference approximation of flows in fractured porous media, *ESAIM Math. Model. Numer. Anal.*, **50** (2016), 809–832. https://doi.org/10.1051/m2an/2015087
26. A. Fumagalli, A. Scotti, A reduced model for flow and transport in fractured porous media with non-matching grids, in *Numerical Mathematics and Advanced Applications 2011: Proceedings of ENUMATH 2011, the 9th European Conference on Numerical Mathematics and Advanced Applications, Leicester, September 2011*, Springer, Berlin, Heidelberg, (2012), 499–507. https://doi.org/10.1007/978-3-642-33134-3_53
27. B. Cockburn, G. E. Karniadakis, C. W. Shu, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, Springer Science & Business Media, 2012.
28. H. Egger, J. Schöberl, A hybrid mixed discontinuous Galerkin finite-element method for convection–diffusion problems, *IMA J. Numer. Anal.*, **30** (2010), 1206–1234. https://doi.org/10.1093/imanum/drn083
29. S. Sun, M. F. Wheeler, Symmetric and nonsymmetric discontinuous Galerkin methods for reactive transport in porous media, *SIAM J. Numer. Anal.*, **43** (2005), 195–219. https://doi.org/10.1137/S003614290241708X
30. S. Bartels, M. Jensen, R. Müller, Discontinuous Galerkin finite element convergence for incompressible miscible displacement problems of low regularity, *SIAM J. Numer. Anal.*, **47** (2009), 3720–3743. https://doi.org/10.1137/070712079
31. H. Leng, H. Chen, Adaptive interior penalty hybridized discontinuous Galerkin methods for Darcy flow in fractured porous media, *IMA J. Numer. Anal.*, **44** (2024), 2165–2197. https://doi.org/10.1093/imanum/drad054
32. P. F. Antonietti, C. Facciolà, A. Russo, M. Verani, Discontinuous Galerkin approximation of flows in fractured porous media on polytopic grids, *SIAM J. Sci. Comput.*, **41** (2019), A109–A138. https://doi.org/10.1137/17M1138194

33. P. F. Antonietti, C. Facciolà, M. Verani, Polytopic Discontinuous Galerkin methods for the numerical modelling of flow in porous media with networks of intersecting fractures, *Comput. Math. Appl.*, **116** (2022), 116–139. https://doi.org/10.1016/j.camwa.2021.08.015
34. P. F. Antonietti, C. Facciolà, M. Verani, Unified analysis of discontinuous Galerkin approximations of flows in fractured porous media on polygonal and polyhedral grids, *Math. Eng.*, **2** (2020), 340–385. https://doi.org/10.3934/mine.2020017
35. L. Zhao, E. Chung, An adaptive discontinuous Galerkin method for the Darcy system in fractured porous media, *Comput. Geosci.*, **26** (2022), 1581–1596. https://doi.org/10.1007/s10596-022-10171-5
36. H. Leng, H. Chen, An adaptive hybridizable discontinuous Galerkin method for Darcy-Forchheimer flow in fractured porous media, *Math. Models Methods Appl. Sci.*, **34** (2024), 2497–2536. https://doi.org/10.1142/S0218202524500532
37. H. Hoteit, A. Firoozabadi, Multicomponent fluid flow by discontinuous Galerkin and mixed methods in unfractured and fractured media, *Water Resour. Res.*, **41** (2005), 389–404. https://doi.org/10.1029/2005WR004339
38. J. Moortgat, M. A. Amooie, M. R. Soltanian, Implicit finite volume and discontinuous Galerkin methods for multicomponent flow in unstructured 3D fractured porous media, *Adv. Water Resour.*, **96** (2016), 389–404. https://doi.org/10.1016/j.advwatres.2016.08.007
39. X. Yan, Z. Huang, J. Yao, Y. Li, D. Fan, An efficient embedded discrete fracture model based on mimetic finite difference method, *J. Pet. Sci. Eng.*, **145** (2016), 11–21. https://doi.org/10.1016/j.petrol.2016.03.013
40. B. Rivière, M. F. Wheeler, V. Girault, A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems, *SIAM J. Numer. Anal.*, **39** (2001), 902–931. https://doi.org/10.1137/S003614290037174X
41. S. Agmon, *Lectures on Elliptic Boundary Value Problems*, American Mathematical Society, **369** (2010). https://doi.org/10.1090/chel/369