Electronic
Research Archive

https://www.aimspress.com/journal/era

ERA, 34(6): 4131–4157.
DOI: 10.3934/era.2026185
Received: 25 March 2026
Revised: 30 April 2026
Accepted: 07 May 2026
Published: 18 May 2026

Research article

Normalized solutions to p-Laplacian equations with a potential term and mass-supercritical nonlinearities

Yanfeng Li, Xuejiao Liu*and Haicheng Liu

College of Science, Heilongjiang Institute of Technology, Harbin 150001, China

* **Correspondence:** Email: liuxuejiao@hljit.edu.cn.

Abstract: In this paper, we investigate the existence of normalized ground state solutions for a class of quasilinear elliptic equations. More precisely, we consider the p-Laplace equation with a prescribed L^p-norm constraint. The problem involves a potential function and a nonlinear term with mass supercritical growth. By employing variational methods, we aim to find solutions with a given mass, where the parameter λ appears as a Lagrange multiplier. Under appropriate assumptions of potential and nonlinearity, we establish the existence of such solutions for any given positive mass.

Keywords: normalized solutions; general nonlinearities; mass supercritical; non-trapping potential

1. Introduction

The study of normalized solutions to nonlinear partial differential equations (PDEs) has attracted significant attention due to their physical implications. This paper investigates the existence of ground state normalized solutions for a class of p-Laplacian equations with potential and mass-supercritical nonlinearity.

The p-Laplacian operator arises in various nonlinear problems [1–4]. For instance, it is used in image restoration (when $1 \leq p \leq 2$), in non-Newtonian fluid dynamics (when $\vec{v} = a|\nabla u|^{p-2}\nabla u$ relates shear stress to the velocity gradient), and in nonlinear elasticity and reaction–diffusion processes. A detailed physical background can be found in [2].

The primary objective of this work is to analyze the following p-Laplace equation with an L^p-normalized constraint:

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u + \lambda|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c, \end{cases} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is p-Laplace operator, $1 < p < N$, $f : \mathbb{R} \to \mathbb{R}$, and $\lambda \in \mathbb{R}$. The solution

(u_c, λ_c) is referred to as a normalized solution. The energy functional associated with this problem is

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} F(u)dx$$

on

$$S(c) = \left\{u \in E : \int_{\mathbb{R}^N} |u|^p dx = c\right\},$$

where $E = \left\{u \in W^{1,p}(\mathbb{R}^N) : \left|\int_{\mathbb{R}^N} V(x)|u|^p dx\right| < +\infty\right\}$ and $F(t) = \int_0^t f(s)ds$, and the norm of E is defined by

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p + V(x)|u|^p dx\right)^{\frac{1}{p}},$$

which is equivalent to the Sobolev norm $\|u\|_{W^{1,p}(\mathbb{R}^N)}$ under the assumption (V_1) in this paper. Then, we study the critical point of the functional I on the restricted manifold $S(c)$. Problem (1.1) with the prescribed L^p-norm arises naturally when seeking standing wave solutions of the time-dependent p-Laplacian equation $i\partial_t\Psi + \Delta_p\Psi = f(\Psi)$. Substituting $\Psi(x,t) = e^{-i\lambda t}u(x)$ yields the stationary Eq (1.1), and the constraint $\int_{\mathbb{R}^N} |u|^p dx = c$ corresponds to the conservation of mass (e.g., the number of particles in Bose–Einstein condensates). Unlike fixed-frequency problems where λ is given, here, λ is an unknown Lagrange multiplier determined by the prescribed mass c.

The normalized solution comes from the study of the standing wave of nonlinear Schrödinger equation. In the case $p = 2$ and $V(x) = 0$, Problem (1.1) comes from the research of solitary waves for the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\Psi + \Delta\Psi = f(\Psi) \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.2}$$

where $\Psi = \Psi(x,t)$ denotes the wave function. Equation (1.2) has important applications in problems such as Bose–Einstein condensates (see [5–7]). In Bose–Einstein condensates, $|\Psi(x,t)|^2$ represents the probability density of a single particle at time t appearing at position x in space, and $\int_{\mathbb{R}^N} |\Psi(x,t)|^2 dx$ is the total number of atoms in whole space. Recalling that a solution of the form $\Psi(x,t) = e^{-\mathrm{i}\lambda t}u(x)$ is called a standing wave solution, by substituting the expression of the standing wave solution into Eq (1.2), it can be obtained that

$$-\Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

and the conservation of mass condition is

$$\int_{\mathbb{R}^N} |u|^2 dx = c > 0.$$

In this case, $\lambda = \lambda_c$ is unknown, regarded as a Lagrange multiplier, and the solution (λ_c, u_c) is a couple of normalized solutions of Eq (1.3). From the viewpoint of physics, it is more meaningful to study the normalized solution.

The problem of normalized solutions has attracted the attention of many scholars in recent years, and many wonderful results have been obtained in the study of normalized solution. In 2015, Jeanjean et al. [8] proved the existence of a mountain pass solution and a minimum either local or global solution having a prescribled L^2-norm for quasilinear Schrödinger equations. Under different growth conditions, Soave N. [9,10] studied the existence and properties of ground states for nonlinear Schrödinger equations with combined nonlinearities when $f(u) = \mu|u|^{q-2}u + |u|^{p-2}u$ in Eq (1.3). The biggest difference between this situation and the previous one is that the structure of the functional I becomes more complex. We define the ground state solution of (1.1) as follows:

Definition 1.1. *We write that u_c is a ground state of (1.1) on S_c if it is a solution to (1.1) having minimal energy among all the solutions which belong to S_c :*

$$dI|_{S(c)}(u_c) = 0 \quad and \quad I(u_c) = \inf\left\{I(u) : dI|_{S(c)}(u) = 0 \; and \; u \in S(c)\right\}.$$

In the case $N > p \neq 2$ and $V(x) \equiv 0$, important contributions to normalized solution to the p-Laplacian equations

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + \mu|u|^{p-2}u + f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x = a^p \end{cases} \tag{1.4}$$

can be found in [11]. Zhang and Zhang assumed that f is odd and L^p-supercritical, and they discussed the existence of a positive radial ground state under different growth conditions of disturbance terms by Schwarz rearrangement and the Ekeland variational principle. Via a fountain theorem-type argument, they obtained infinitely many solutions radial and nonradial. In [12], Wang et al studied the following p-Laplacian equation with a L^2-norm constraint:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda u + |u|^{s-2}u \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = \rho. \end{cases} \tag{1.5}$$

Other literature on the normalized solution of a p-Laplacian operator can be found in [13–15], where in [14], Wang and Sun introduced a special potential functional into the p-Laplacian equation

$$-\Delta_p u + |x|^k|u|^{p-2}u = \lambda|u|^{r-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

The existence of a ground state under an L^p-norm constraint and an L^2-norm constraint are obtained respectively by a compact embedding lemma in [16].

In fact, due to diverse physical backgrounds, many equations inevitably involve potential functions in place. The fixed mass problem with potential is also studied under suitable assumptions on the potential function by many scholars. In [17], Ikoma and Miyamoto used the concentration compactness principle to obtain stable standing waves under potential conditions $V(x) \in C\left(\mathbb{R}^N\right), 0 \not\equiv V(x) \leq 0$, and $\lim_{|x|\to\infty} V(x) = 0$. Soon afterward, the case of non-negative potential and vanishing at infinity was considered in [18]. Under the circumstances, a new variational structure Pohozaev identity was used. By constructing a bounded Palais–Smale sequence, Thomas with his partners obtained the existence of normalized solutions with a high Morse index. In the case of nonpositive potential $V(x) \leq 0$ and vanishing at infinity, Ding and Zhong [19] proved the existence of $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^{\mathbb{N}})$ to the following Schrödinger equation:

$$\begin{cases} -\Delta u(x) + V(x)u(x) + \lambda u(x) = g(u(x)) & \text{in } \mathbb{R}^N, \\ 0 \leq u(x) \in H^1\left(\mathbb{R}^N\right), N \geq 3, \end{cases}$$

satisfying the L^2-norm under some explicit smallness assumptions on V. He et al. [20] studied the existence of the normalized solution of the Kirchhoff equation under similar assumptions of potential function.

There are also many novel results in the study of normalized solutions of other equations with nonlinear terms. For example, the study of nonlinear Schrödinger equations with a mixed fractional Laplacian can be found in [21, 22]. For (p, q)-Laplacian equations with mass supercritical growth, normalized solutions have been extensively studied in various contexts [23–25]. Related variational methods and energy estimates for general nonlinear systems can also be found in [26–28].

In the present paper, we investigate the existence of a ground state for Problem (1.1). Drawing on the assumptions of the nonlinear terms f in [11], we present some assumptions on f as follows.

(F_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and odd.

(F_2) There exist $\bar{p} < \alpha \leq \beta < p^*$ such that

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t), \ \forall t \in \mathbb{R} \setminus \{0\}$$

where $F(t) = \int_0^t f(s)ds$, $\bar{p} = p + \frac{p^2}{N}$, $p^* = \begin{cases} \dfrac{Np}{N-p}, & p < N, \\ +\infty, & p \geq N. \end{cases}$

(F_3) The functional defined by $\bar{F}(t) = f(t)t - pF(t)$ is of class C^1, and $F'(t)t \geq \alpha \bar{F}(t), \forall t \in \mathbb{R}$, where α is mentioned in (F_2).

In the case of $p = 2$ and $V(x) \equiv 0$, under the assumptions (F_1) and (F_2), Jeanjean [29] proved that the energy functional satisfies the mountain pass geometry and that a bounded Palais–Smale sequence exists. By the embedding $H_r^1 \hookrightarrow L^p(\mathbb{R}^N)$ being compact for $p \in (2, 2^*)$, Jeanjean obtained a radial normalized solution when $N \geq 2$.

For the potential function in this article, we focus on the non-trapping potential, that is, $\lim\limits_{|x| \to +\infty} V(x) = 0$. If $\limsup\limits_{|x| \to +\infty} V(x) =: V_\infty < +\infty$ exists, we can make a replacement that does not affect the existence of a normalized solution. Inspired by [19, 20], we make the following assumptions about the potential function $V(x)$ in this paper:

(V_1) $\lim\limits_{|x| \to +\infty} V(x) = \sup\limits_{x \in \mathbb{R}^N} V(x) = 0$, and there exists $\sigma_1 \in \left[0, 1 - \dfrac{p^2}{N(\alpha - p)}\right)$ such that

$$\left| \int_{\mathbb{R}^N} V(x)|u|^p dx \right| \leq \sigma_1 \|\nabla u\|_p^p, \ \forall u \in E.$$

(V_2) $\nabla V(x)$ exists *a.e.* in $\mathbb{R}^N$. Putting $W(x) = 1/p \langle \nabla V(x), x \rangle$, there exists $0 < \sigma_2 < \min\left\{N(1-\sigma_1)(\alpha - p)/p^2 - 1, [\beta p - (\beta - p)N]/\beta p\right\}$ such that

$$\left| \int_{\mathbb{R}^N} W(x)|u|^p dx \right| \leq \sigma_2 \|\nabla u\|_p^p, \ \forall u \in E.$$

(V_3) $\nabla W(x)$ exists *a.e.* in $\mathbb{R}$. Let $Y(x) = \langle \nabla W(x), x \rangle + (N\alpha/p - N) W(x)$, there exists $\sigma_3 \in \left[0, \frac{N}{p}\alpha - p - N\right)$ such that

$$\int_{\mathbb{R}^N} Y_+(x)|u|^p dx \leq \sigma_3 \|\nabla u\|_p^p, \ \forall u \in E,$$

where $Y_+(x) = \max\{0, Y(x)\}$.

Remark: We emphasize that such potential functions do exist, and the conditions for these potential functions are sufficient but not necessary.

Examples of potentials satisfying (V_1)–(V_3)**.**

Radial smooth decay. Let

$$V(x) = -\varepsilon e^{-|x|^2}, \qquad \varepsilon > 0 \text{ small},$$

where $x \in \mathbb{R}^N$, $N \geq 3$, and $1 < p < N$. By the weighted Sobolev inequality, we can verify that $V(X)$ satisfies (V_1)–(V_3).

In addition, $V(x) = -\varepsilon \sin^2(|x|/(1 + |x|^{N+1})$, and $\varepsilon > 0$ also satisfies (V_1)–(V_3).

Sufficiency. The assumptions (V_1)–(V_3) are sufficient for our main result (Theorem 1.2), but they are not necessary. Weaker conditions (e.g., different smallness thresholds or weighted norms) could be explored in future work. The present formulation is chosen to balance generality with technical feasibility.

Although recent years have witnessed significant progress in the study of normalized solutions for p-Laplacian equations with mass constraints, existing works, such as [11, 14, 19], are largely confined to settings without potential, with trapping potentials, or with specific power-type nonlinearities. In contrast, this paper addresses the existence of ground state normalized solutions in a more general and technically challenging framework involving non-trapping potentials and mass supercritical nonlinearities of general type. The main novelties and contributions of this work can be summarized as follows:

- Non-trapping potentials: Unlike previous works that focus on trapping potentials or zero potential, we introduce hypotheses (V_1)–(V_3) that allow for non-trapping potentials. By constructing weighted terms $W(x)$ and $Y(x)$ from the potential, we establish precise control over the energy functional, ensuring the stability of the Pohozaev manifold.
- A new condition for general nonlinearities: Although we adopt the general mass supercritical assumptions (F_1)–(F_2) from [11], we replace the monotonicity condition (H_3) in [11] with a new inequality (F_3). This modification is better suited to handle the presence of a non-trapping potential and facilitates the proof that the Pohozaev manifold is a natural constraint. For $t > 0$, set

  $$f(t) = |t|^{q-2}t\Big(1 + \varepsilon \cos(\log|t|)\Big), \qquad t \neq 0, \quad f(0) = 0,$$

  where $q \in (p, p^*)$, and $0 < \varepsilon \ll 1$. Then, we verify:

  - (F_1) holds (oddness and continuity).
  - (F_2) holds by choosing $\alpha = q - \delta$, $\beta = q + \delta$ with sufficiently small $\delta > 0$.
  - (F_3) holds for sufficiently small ε because $\bar{F}'(t)t/\bar{F}(t)$ is uniformly close to $q > p$.
  - The monotonicity condition $f(t)/t^{p-1}$ being nondecreasing fails because its derivative changes sign (e.g., $\log t = \pi/2$ gives a negative derivative, and $\log t = 0$ gives a positive derivative).

  This example clearly shows that (F_3) is strictly weaker than the monotonicity condition and therefore extends the applicability of the theory to a wider class of nonlinearities, including those with oscillatory growth rates.
- Unified variational framework: Inspired by [30, 31], we construct an auxiliary functional $J(u) = \Psi_u(s)|_{S(c)}$ that simplifies the construction of Palais–Smale sequences and provides a flexible framework extendable to other nonlocal operators.

The combination of a general nonlinearity satisfying (F_1)–(F_3) with a non-trapping potential satisfying the smallness conditions (V1)–(V3) represents a comprehensive and technically challenging framework that extends previous results in [11, 29].

After showing the assumptions of the nonlinear term f and the potential function $V(x)$, our main result are shown as follows.

Theorem 1.2. *Assume that $N \geq 3, 1 < p < N$, f satisfies the assumptions (F_1)–(F_3), and $V(x) \not\equiv 0$ satisfies the hypotheses (V_1)–(V_3). Then, for any $c > 0$, there exists a couple $(\lambda_c, u_c) \in \mathbb{R}^+ \times E$ that solves Eq (1.1) satisfying $\int_{\mathbb{R}^N} |u_c|^p dx = c$. Moreover, u_c is a normalized ground state solution.*

The remaining parts of this paper are arranged as follows. In the next section, we will show some preliminary results and the so-called Pohozaev identity. The rest of this paper is devoted to the proof of **Theorem 1.2**. Throughout the paper, the notation $\|u\|_p$ denotes the L^p-norm, and the notation $\rightharpoonup$ stands for weak convergence. Capital latter C and C_i denote positive constants whose precise value may change from line to line.

2. Preliminaries

In this section, we show some important lemmas that will be used in the subsequent proof, and we construct a submanifold $\mathcal{P}_c$ on S_c based on the Pohozaev identity, which can be found in [9,11,12,14,19], and so forth.

To begin, we give the Gagliardo–Nirenberg inequality [11,32] and introduce some properties of F shown in [11] under the assumptions (F_1) and (F_2).

Lemma 2.1. *[(1.7) in 11]For $\forall s \in (p, p^*)$, there exists a positive constant $C_{N,s}$ depending on N and s such that*

$$\|u\|_s \leq C_{N,s} \|\nabla u\|_p^{\gamma_s} \|u\|_p^{1-\gamma_s}, \quad \forall u \in W^{1,p}(\mathbb{R}^N),$$

where $\gamma_s := N(s-p)/sp$.

Lemma 2.2. *[[11], Lemma 2.1] Assume that (F_1) and (F_2) hold, then for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$, we have*

$$\begin{cases} |s|^\beta F(t) \leqslant F(ts) \leqslant |s|^\alpha F(t), & \text{if } |s| \leqslant 1, \\ |s|^\alpha F(t) \leqslant F(ts) \leqslant |s|^\beta F(t), & \text{if } |s| \geqslant 1. \end{cases} \tag{2.1}$$

Lemma 2.3. *[[11], Lemma 2.2] Assume that (F_1) and (F_2) hold; then, with $t \in \mathbb{R}$, we have*

$$\begin{cases} \frac{1}{\beta-p}\bar{F}(t) \leqslant F(t) \leqslant \frac{1}{\alpha-p}\bar{F}(t), \\ \frac{\beta}{\beta-p}\bar{F}(t) \leqslant f(t)t \leqslant \frac{\alpha}{\alpha-p}\bar{F}(t). \end{cases} \tag{2.2}$$

Next, we introduce the so-called Pohozaev identity for the p-Laplacian equation. The Pohozaev identity for the p-Laplacian operator was first established in [33] for the Laplacian case and was extended to the p-Laplacian in [34]. For a modern exposition, see also [18, Section 2].

Lemma 2.4. *Let $N \geq 3$, $p \in (1, N)$, and $\lambda \in \mathbb{R}$. If $u \in E$ is a weak solution of Eq (1.1), then the Pohozaev identity*

$$\begin{aligned} P(u) &= \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u|^p dx - \frac{N}{p}\int_{\mathbb{R}^N} f(u)u - pF(u) dx \\ &= \|\nabla u\|_p^p - \int_{\mathbb{R}^N} W(x)|u|^p dx - \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u) dx = 0 \end{aligned} \tag{2.3}$$

holds.

Proof. First we suppose $u \in C^2(\mathbb{R}^N)$. Multiplying by $(x \cdot \nabla u)$ and integrating by parts in $\mathbb{R}^N$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} (\Delta_p u)(x \cdot \nabla u)dx &= \sum_{j=1}^{N} \int_{\mathbb{R}^N} \mathrm{div}(|\nabla u|^{p-2}\nabla u) x_j u_{x_j} dx \\
&= \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} u_{x_i}\right)_{x_i} x_j u_{x_j} dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} |\nabla u|^{p-2} u_{x_i} u_{x_j} dx - \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} |\nabla u|^{p-2} u_{x_i} x_j u_{x_i x_j} dx \\
&= -\int_{\mathbb{R}^N} |\nabla u|^p dx - \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^p}{p}\right)_{x_j} x_j dx = \frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx
\end{aligned}$$

and

$$\int_{\mathbb{R}^N} |u|^{p-2} u (x \cdot \nabla u) dx = \sum_{j=1}^{N} \int_{\mathbb{R}^N} \left(\frac{|u|^p}{p}\right)_{x_j} x_j dx = -\frac{N}{p} \int_{\mathbb{R}^N} |u|^p dx.$$

In the same way, we can get

$$\begin{aligned}
\int_{\mathbb{R}^N} f(u)(x \cdot \nabla u) dx &= \sum_{i=1}^{N} \int_{\mathbb{R}^N} f(u) x_i u_{x_i} dx = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{\partial F(u)}{\partial x_i} dx \\
&= -\sum_{i=1}^{N} \int_{\mathbb{R}^N} F(u) dx = -N \int_{\mathbb{R}^N} F(u) dx,
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(V(x)|u|^{p-2}u\right)(x \cdot \nabla u) dx &= \sum_{j=1}^{N} \int_{\mathbb{R}^N} V(x)|u|^{p-2} u x_j u_{x_j} dx \\
&= \sum_{j=1}^{N} \int_{\mathbb{R}^N} \left(\frac{|u|^p}{p}\right)_{x_j} x_j V(x) dx \\
&= -\sum_{j=1}^{N} \frac{|u|^p}{p} \left(V(x) + x_j V(x)_{x_j}\right) dx \\
&= -\frac{N}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |u|^p dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{p-N}{N} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{N}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |u|^p dx \\
- \frac{N}{p} \lambda \int_{\mathbb{R}^N} |u|^p dx = -N \int_{\mathbb{R}^N} F(u) dx.
\end{aligned} \tag{2.4}$$

We multiply (1.1) by u and integrate on $\mathbb{R}^N$, obtaining

$$\int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} V(x)|u|^p dx + \lambda \int_{\mathbb{R}^N} |u|^p dx = \int_{\mathbb{R}^N} f(u) u dx. \tag{2.5}$$

Combining (2.4) with (2.5), we have the Pohozaev identity $P(u) = 0$.

The above derivation assumes $u \in C^2(\mathbb{R}^N)$. For weak solutions $u \in E$ of Eq (1.1), we note that $C_c^\infty(\mathbb{R}^N)$ is dense in E. Hence, there exists a sequence $u_n \subset C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in E. For each u_n, the Pohozaev identity $P(u_n) = 0$ holds by the computation above. By the growth conditions (F_1)–(F_2) and the Sobolev embedding theorem, each term in $P(u)$ is continuous with respect to the E-norm. Therefore, passing to the limit $n \to \infty$ yields $P(u) = 0$. Consequently, the Pohozaev identity is valid for all weak solutions $u \in E$ of (1.1). □

According to the Pohozaev identity, the Pohozaev manifold is defined by

$$\mathcal{P}_c = \{u \in S_c : P(u) = 0\}.$$

It is well-known that any critical point of $I|_{S_c}$ stays in $\mathcal{P}_c$. Then we shall see that $\mathcal{P}_c$ is also a natural constraint (see Lemma 3.2). To find the normalized ground state of (1.1) for a given $c > 0$, we define the ground state energy

$$E_c = \inf_{u \in \mathcal{P}_c} I(u).$$

To study the behavior of $I(u)$ on the sphere $S(c)$, we introduce a scaling transformation. For any $u \in S(c)$ and $s \in \mathbb{R}$, define $(s * u)(x) = e^{Ns/p} u(e^s x)$. A direct computation shows that $\|s * u\|_p^p = \|u\|_p^p = c$, and $\|\nabla(s * u)\|_p^p = e^{ps}\|\nabla u\|_p^p$. Substituting this scaled function into $I(u)$ yields the fibering map

$$\Psi_u(s) = I(s * u) = \frac{1}{p} e^{ps} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V\left(\frac{x}{e^s}\right) |u|^p dx - e^{-Ns} \int_{\mathbb{R}^N} F\left(e^{\frac{Ns}{p}} u(x)\right) dx. \tag{2.6}$$

By a direct calculation, we find that $d\Psi_u(s)/ds = P(s * u)$, and thus, we have another way to define the Pohozaev manifold:

$$\mathcal{P}_c = \left\{u \in S_c : \Psi_u'(0) = 0\right\}.$$

In this direction, we consider the decomposition of $\mathcal{P}_c$ into the disjoint union $\mathcal{P}_c = \mathcal{P}_c^+ \cup \mathcal{P}_c^0 \cap \mathcal{P}_c^-$, in which

$$\mathcal{P}_c^+ = \{u \in \mathcal{P}_c : \Psi_u''(0) > 0\}; \mathcal{P}_c^0 = \{u \in \mathcal{P}_c : \Psi_u''(0) = 0\}; \mathcal{P}_c^- = \{u \in \mathcal{P}_c : \Psi_u''(0) < 0\}.$$

3. Proof of Theorem 1.2

In this section, we will present the the proof of Theorem 1.2. Several lemmas that play an important role in the proof process are established.

Lemma 3.1. *Assume that $V(x)$ and f satisfy $(V_1) - (V_3)$ and $(F_1) - (F_3)$, respectively. For any $u \in E \setminus \{0\}$, we have*

(i) $\Psi_u(s) \to 0^+$ as $s \to -\infty$.

(ii) $\Psi_u(s) \to -\infty$ as $s \to +\infty$.

Proof. By (2.1), we have

$$F(1) \min\left\{|t|^\alpha, |t|^\beta\right\} \leqslant F(t) \leqslant F(1)\left(|t|^\alpha + |t|^\beta\right), \quad \forall t \in \mathbb{R}. \tag{3.1}$$

Thus, for any fixed $u \in E\backslash\{0\}$, the quantities $\|u\|_p$ and $\|\nabla u\|_p$ are finite positive constants. Then, by (3.1) and the Gagliardo–Nirenberg inequality,

$$\begin{aligned}\Psi_u(s) &\geq \frac{1}{p}e^{ps}\|\nabla u\|_p^p + \frac{1}{p}\int_{\mathbb{R}^N} V\left(\frac{x}{e^s}\right)|u|^p dx - e^{-Ns}F(1)\int_{\mathbb{R}^N} \left|e^{\frac{Ns}{p}}u\right|^\alpha + \left|e^{\frac{Ns}{p}}u\right|^\beta dx \\ &= \frac{1}{p}e^{ps}\|\nabla u\|_p^p + \frac{1}{p}\int_{\mathbb{R}^N} V\left(\frac{x}{e^s}\right)|u|^p dx - F(1)e^{\alpha\gamma_\alpha s}\|u\|_\alpha^\alpha - F(1)e^{\beta\gamma_\beta s}\|u\|_\beta^\beta \\ &\geq \frac{1}{p}e^{ps}\|\nabla u\|_p^p + \frac{1}{p}\int_{\mathbb{R}^N} V\left(\frac{x}{e^s}\right)|u|^p dx - C_1 e^{\alpha\gamma_\alpha s}\|\nabla u\|_p^{\alpha\gamma_\alpha} - C_2 e^{\beta\gamma_\beta s}\|\nabla u\|_p^{\beta\gamma_\beta},\end{aligned}$$

where $C_1 = F(1)C_{N,\alpha}^\alpha\|u\|_p^{\alpha(1-\gamma_\alpha)}$, $C_2 = F(1)C_{N,\beta}^\beta\|u\|_p^{\beta(1-\gamma_\beta)}$.
Further,

$$\begin{aligned}\Psi_u(s) &\leq \frac{1}{p}e^{ps}\|\nabla u\|_p^p + \frac{1}{p}\int_{\mathbb{R}^N} V\left(\frac{x}{e^s}\right)|u|^p dx - e^{-Ns}F(1)\int_{\mathbb{R}^N} \min\left\{e^{\frac{Ns}{p}\alpha}|u|^\alpha, e^{\frac{Ns}{p}\beta}|u|^\beta\right\} \\ &\leq \frac{1}{p}e^{ps}\|\nabla u\|_p^p + \frac{1}{p}\int_{\mathbb{R}^N} V\left(\frac{x}{e^s}\right)|u|^p dx - F(1)\min\left\{e^{\alpha\gamma_\alpha s}, e^{\beta\gamma_\beta s}\right\} \times \min\left\{\|u\|_\alpha^\alpha, \|u\|_\beta^\beta\right\}.\end{aligned}$$

Because $0 < p < \alpha\gamma_\alpha \leq \beta\gamma_\beta$, we get $\Psi_u(s) \to 0^+$ as $s \to -\infty, \Psi_u(s) \to -\infty$ as $s \to +\infty$. □

Lemma 3.2. *Assume that the assumptions (F_1)–(F_3) and (V_3) hold; then, $\mathcal{P}_c^- = \mathcal{P}_c$ is closed in E, and it is a natural constraint of $I|_{S_c}$.*

Proof. We divide the proof into three steps.

Step 1 : $\mathcal{P}_c^- = \mathcal{P}_c$.
For any $u \in \mathcal{P}_c$, we have

$$\|\nabla u\|_p^p - \int_{\mathbb{R}^N} W(x)|u|^p dx = \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u)dx. \tag{3.2}$$

A direct computation of $\Psi_u''(0)$ yields

$$\Psi_u''(0) = p\|\nabla u\|_p^p + \int_{\mathbb{R}^N} \langle\nabla W(x), x\rangle\, |u|^p dx - \frac{N^2}{p^2}\int_{\mathbb{R}^N} \bar{F}'(u)udx + \frac{N^2}{p}\int_{\mathbb{R}^N} \bar{F}(u)dx.$$

Applying (F_3), we obtain

$$\Psi_u''(0) \leq p\|\nabla u\|_p^p + \int_{\mathbb{R}^N} \langle\nabla W(x), x\rangle\, |u|^p dx + \frac{N}{p}\left(N - \frac{N}{p}\alpha\right)\int_{\mathbb{R}^N} \bar{F}(u)dx.$$

Furthermore, by combining the identity (3.2) with the definition of $Y(x)$ and (V_3), we obtain

$$\begin{aligned}\Psi_u''(0) &\leq \left(N + p - \frac{N}{p}\alpha\right)\|\nabla u\|_p^p + \int_{\mathbb{R}^N}\left(\langle\nabla W(x), x\rangle + \left(\frac{N}{p}\alpha - N\right)W(x)\right)|u|^p dx \\ &= \left(N + p - \frac{N}{p}\alpha\right)\|\nabla u\|_p^p + \int_{\mathbb{R}^N} Y(x)|u|^p dx \\ &\leq -\left(\frac{N}{p}\alpha - N - p - \sigma_3\right)\|\nabla u\|_p^p < 0.\end{aligned}$$

Therefore, $\mathcal{P}_c^+ = \mathcal{P}_c^0 = \emptyset$, which implies that $\mathcal{P}_c^- = \mathcal{P}_c$. Moreover, $\mathcal{P}_c$ is closed in E as the preimage of $\{0\}$ under the continuous map P.

Step 2 : $\mathcal{P}_c$ is a C^1 manifold.

To show that $\mathcal{P}_c$ is a C^1 manifold, it suffices to prove that $P'(u) \neq 0$ for all $u \in \mathcal{P}_c$. A direct computation gives

$$P'(u)u = p\left[\int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} W(x)|u|^p dx\right] - \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}'(u)u dx.$$

By (3.2) and the hypotheses (V_2) and (F_3), we obtain

$$\begin{aligned} P'(u)u &\leq (p-\alpha)\left(\|\nabla u\|_p^p - \int_{\mathbb{R}^N} W(x)|u|^p dx\right) \\ &\leq (p-\alpha)(1+\sigma_2)\|\nabla u\|_p^p < 0, \end{aligned}$$

where we use (V_2) to estimate $\int_{\mathbb{R}^N} W(x)|u|^p dx \leq \sigma_2\|\nabla u\|_p^p$. Hence, $P'(u)u \neq 0$ for all $u \in \mathcal{P}_c$, and by the implicit function theorem, the manifold $\mathcal{P}_c$ is class of C^1 for all $c > 0$.

Step 3 : $\mathcal{P}_c$ is a natural constraint.

Let u be a critical point of $I(u)$ as a constraint on $\mathcal{P}_c$; then, we construct the Lagrangian function

$$\Phi(u) = I(u) + \frac{1}{p}\lambda\|u\|_p^p + \mu P(u),$$

in which λ and μ are Lagrange multipliers. If $u \in \mathcal{P}_c$ is a critical point of $I|_{\mathcal{P}_c}$, then there exist $\lambda,\ \mu \in \mathbb{R}$ such that

$$I'(u) + \lambda|u|^{p-2}u + \mu P'(u) = 0. \tag{3.3}$$

Evaluating $d\Phi(s \star u)/ds$ at $s = 0$ and using $P(u) = 0$ and $\Psi_u''(0) < 0$ (from Step 1), we obtain $\mu\Psi_u''(0) = 0$, which forces $\mu = 0$. Hence, u is also a critical point of $I|_{S(c)}$, proving that $\mathcal{P}_c$ is a natural constraint.

□

Lemma 3.3. *Under the assumptions* (V_1)–(V_3) *and* (F_1)–(F_3), *for any* $u \in E \setminus \{0\}$, *the following results hold:*

(i) There exists a unique $s_u \in \mathbb{R}$ *such that* $P(s * u) = 0$.

(ii) $\Psi_u(s_u) > \Psi_u(s)$ *for any* $s \neq s_u$ *and* $\Psi_u(s_u) > 0$.

(iii) The mapping $u \mapsto s_u$ *is continuous for* $u \in E \setminus \{0\}$.

Proof. (i) From (3.1), we know

$$\lim_{s\to-\infty}\Psi_u(s) = 0^+,\ \lim_{s\to+\infty}\Psi_u(s) = -\infty;$$

then, $\Psi_u(s)$ reaches the global maximum at some $s_u \in \mathbb{R}$. According to the continuity of $\Psi_u(s)$ and Fermat's theorem,

$$\Psi_u'(s_u) = P(s_u * u) = 0.$$

To prove the uniqueness of $\Psi_u(s)$, we use proof by contradiction. Suppose that there exist $s_1, s_2 \in \mathbb{R}$ such that $P(s_1 * u) = P(s_2 * u) = 0$. Without loss of generality, we can assume that $s_1 < s_2$. Then, by

Lemma 3.2, s_1 and s_2 are local maximum points of $\Psi_u(s)$. Therefore, there exists a point $s_3 \in (s_1, s_2)$ such that s_3 is a local minimum point of $\Psi_u(s)$, and then $\Psi'_u(s_3) = 0, \Psi''_u(s_3) \geq 0$, which implies $s_3 * u \in \mathcal{P}_c^0 \cup \mathcal{P}_c^+$ and contradicts with Lemma 3.2.

(ii) On the basis of (i) and Lemma 3.1, we find that $\Psi_u(s)$ reaches the global maximum point at s_u, and s_u is unique; thus, $\Psi_u(s_u) > \Psi_u(s)$ for any $s \neq s_u$ and $\Psi_u(s_u) > 0$.

(iii) Because of (i), the mapping $u \mapsto s_u$ is well-defined. (iii) is equivalent to prove the following: Let $u \in E \setminus \{0\}$ and $\{u_n\} \subset E \setminus \{0\}$ be any sequence such that $u_n \to u$ in E, then up to a subsequence $s_{u_n} \to s_u$ as $n \to \infty$.

First, we claim that $\{s_{u_n}\}$ is bounded in $\mathbb{R}$. Suppose $\{s_{u_n}\}$ is unbounded; then, one of $\lim\limits_{n\to\infty} s_{(u_n)} = +\infty$ and $\lim\limits_{n\to\infty} s_{(u_n)} = -\infty$ is true.

Case 1: $\lim\limits_{n\to\infty} s_{(u_n)} = +\infty$.
We define a function $g_\nu : \mathbb{R} \to \mathbb{R}$ as follows:

$$g_\nu(t) = \begin{cases} \dfrac{F(t)}{|t|^{\frac{\alpha+\bar{p}}{2}}} + \nu, & \text{for } t \neq 0, \\ \nu, & \text{for } t = 0. \end{cases}$$

Then we obtain $F(t) = g_\nu(t)|t|^{(\alpha+\bar{p})/2} - \nu|t|^{(\alpha+\bar{p})/2}$ for all $t \in \mathbb{R}$. The purpose of introducing g_nu is to extract the dominant growth term $|t|^{(\alpha+\beta)/2}$ while ensuring the remaining factor is non-negative, which allows us to apply Fatou's lemma or direct comparison to obtain a contradiction as $s_{u_n} \to +\infty$. Based on (F_2) and (3.1), we know that $g_\nu(t)$ is continuous, and

$$g_\nu(t) \to +\infty \quad \text{as } t \to +\infty.$$

So, we can take $\nu > 0$ large enough such that $g_\nu(t) \geq 0$ for all $t \in \mathbb{R}$. Thereby,

$$\lim_{s\to+\infty} \int_{\mathbb{R}^N} g_\nu\left(e^{\frac{Ns}{p}} u\right)|u|^{\frac{\alpha+\bar{p}}{2}} dx \geq 0.$$

On account of Lemma 3.3(ii) and (V_1), we have

$$\begin{aligned} 0 \leq \Psi_{u_n}(s_{u_n}) &= \frac{1}{p} e^{p s_{u_n}} \|\nabla u_n\|_p^p + \frac{1}{p} \int_{\mathbb{R}^N} V\left(\frac{x}{e^{s_{u_n}}}\right)|u_n|^p dx \\ &\quad - e^{-N s_{u_n}} \int_{\mathbb{R}^N} F\left(e^{\frac{N s_{u_n}}{p}} u_n\right) dx \\ &\leq \frac{1}{p} e^{p s_{u_n}} \|\nabla u_n\|_p^p - e^{-N s_{u_n}} \int_{\mathbb{R}^N} F\left(e^{\frac{N s_{u_n}}{p}} u_n\right) dx \\ &= \frac{1}{p} e^{p s_{u_n}} \|\nabla u_n\|_p^p - e^{-N s_{u_n}} \int_{\mathbb{R}^N} g_0\left(e^{\frac{N s_{u_n}}{p}} u_n\right)\left|e^{\frac{N s_{u_n}}{p}} u_n\right|^{\frac{\alpha+\bar{p}}{2}} dx \\ &= \frac{1}{p} e^{p s_{u_n}} \|\nabla u_n\|_p^p - e^{\left(\frac{N(\alpha+\bar{p})}{2p} - n\right) s_{u_n}} \int_{\mathbb{R}^N} g_0\left(e^{\frac{N s_{u_n}}{p}} u_n\right)|u_n|^{\frac{\alpha+\bar{p}}{2}} dx \\ &\to -\infty \quad \text{as } s_{u_n} \to +\infty, \end{aligned}$$

which is a contradiction.

Case 2: $\lim_{n\to\infty} s_{(u_n)} = -\infty$.
It follows from (ii) and $u_n \to u$ in E that $\Psi_{u_n}(s_{u_n}) \geq \Psi_{u_n}(s_u)$, and $s_{u_n} * u_n \to s_u * u$ in E. Accordingly, $\Psi_{u_n}(s_{u_n}) = \Psi_u(s_u) + o_n(1)$, and by Fatou's lemma, we have

$$\limsup_{n\to\infty} \Psi_{(u_n)}(s_{u_n}) \geq \Psi_u(s_u) > 0,$$

which is a contradiction with $\lim_{n\to\infty} s_{(u_n)} = -\infty$ due to $\lim_{s\to-\infty} \Psi_u(s) = 0^+$ (see Lemma 3.1(i)).

In conclusion, we obtain that $\{s_{u_n}\}$ is bounded $\in \mathbb{R}$; then, there exist an s_1, and a subsequence still denoted by $\{s_{u_n}\}$ such that $s_{u_n} \to s_1$, and $s_{u_n} * u_n \to s_1 * u$ in E. On the other hand, $P(s_{u_n} * u_n) = 0$ for any $n \in \mathbb{N}$, and it follows that $P(s_1 * u) = 0$. Based on the uniqueness mentioned in (i), we have $s_1 = s_u$, and (iii) is completed. □

Lemma 3.4. *Assume that the hypotheses* (V_1)–(V_2) *and* (F_1)–(F_3) *hold. Then:*

(i) $\mathcal{P} \neq \emptyset$.

(ii) $I(u)$ *is coercive on* $\mathcal{P}_c$.

(iii) $\inf_{u\in\mathcal{P}_c} \|\nabla u\|_p^p > 0$.

(iv) $\inf_{u\in\mathcal{P}_c} I(u) > 0$.

Proof. (i) On the basis of the definition of $\mathcal{P}_c$ and Lemma 3.3(i), we know that $\mathcal{P} \neq \emptyset$.

(ii) For any $u \in \mathcal{P}_c$, by (2.2) and (2.3), under the assumption of (V_1) and (V_2), we obtain

$$\begin{aligned}(1+\sigma_2)\|\nabla u\|_p^p \geq \|\nabla u\|_p^p - \int_{\mathbb{R}^N} W(x)|u|^p dx &= \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u)dx \\ &\geq \frac{N}{p}(\alpha - p)\int_{\mathbb{R}^N} F(u)dx\end{aligned}$$

and

$$I(u) \geq \frac{1}{p}(1-\sigma_1)\|\nabla u\|_p^p - \int_{\mathbb{R}^N} F(u)dx \geq \left[\frac{1}{p}(1-\sigma_1) - \frac{p(1+\sigma_2)}{N(\alpha-p)}\right]\|\nabla u\|_p^p. \tag{3.4}$$

Thus, $I(u)$ is coercive on $\mathcal{P}_c$.

(iii) Based on (F_1), we know $\bar{F} \geq 0$ *a.e.* in $\mathbb{R}^N$. If there exists $\{u_n\} \subset \mathcal{P}_c$ such that $\|\nabla u_n\|_p \to 0$, then $P(u_n) = 0$. On the other hand,

$$P(u) = \|\nabla u\|_p^p - \int_{\mathbb{R}^N} W(x)|u|^p dx - \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u)dx > (1-\sigma_2)\|\nabla u\|_p^p > 0.$$

Combining with the above two aspects, we get a contradiction, and $\inf_{u\in\mathcal{P}_c} \|\nabla u\|_p > 0$.

(iv) At the beginning, we claim that for any $c > 0$, there exists $\delta > 0$ small enough such that $I(u) \geq (1-\sigma_1)\|\nabla u\|_p^p/(2p)$ for $u \in S_c$ and $\|\nabla u\|_p < \delta$.

For any $u \in S_c$, it follows from (V_1), (2.1), and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
I(u) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} F(u)dx \\
&\geq \frac{1}{p}(1-\sigma_1)\,\|\nabla u\|_p^p - F(1)\int_{\mathbb{R}^N} |u|^\alpha + |u|^\beta dx \\
&\geq \left[\frac{1}{p}(1-\sigma_1) - F(1)C_{N,\alpha}^{\alpha} c^{\frac{\alpha(1-\gamma_\alpha)}{p}}\,\|\nabla u\|_p^{\alpha\gamma_\alpha - p} - F(1)C_{N,\beta}^{\beta} c^{\frac{\beta(1-\gamma_\beta)}{p}}\,\|\nabla u\|_p^{\beta\gamma_\beta - p}\right]\|\nabla u\|_p^p .
\end{aligned}$$

So, we can take $\delta > 0$ small enough such that

$$I(u) \geq \frac{1-\sigma_1}{2p}\,\|\nabla(u)\|_p^p .$$

For $u \in \mathcal{P}_c$, based on (2.6) and Lemma 3.3(i), we know that $I(u) = \Psi_u(0) \geq \Psi_u(s)$ for all $s \in \mathbb{R}$. Let $\delta > 0$ small enough and $\bar{s} = ln\left(\delta/\,\|\nabla u\|_p\right)$; then, $\|\nabla(\bar{s} * u)\|_p = \delta$.

$$I(u) \geq \Psi_u(\bar{s}) = I(\bar{s} * u) \geq \frac{1-\sigma_1}{2p}\,\|\nabla(\bar{s} * u)\|_p^p > 0.$$

Therefore, the proof of (iv) is completed. □

Lemma 3.5. *Let* $\{u_n\}$ *be a bounded sequence in* E *and* $\{a_n\}$ *be a sequence satisfying* $\lim\limits_{n\to\infty} a_n = 1$. *Then,* $\lim\limits_{n\to\infty} |I(a_n u_n) - I(u_n)| = 0$.

Proof. Based on a direct calculation, we obtain

$$\begin{aligned}
I(a_n u_n) - I(u_n) = &\frac{1}{p}(a_n^p - 1)\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p}(a_n^p - 1)\int_{\mathbb{R}^N} V(x)|u|^p dx \\
&- \int_{\mathbb{R}^N} F(a_n u_n) - F(u_n)dx.
\end{aligned}$$

By a change of form,

$$\begin{aligned}
\int_{\mathbb{R}^N} F(a_n u_n) - F(u_n)dx &= \int_{\mathbb{R}^N}\left(\int_0^1 f\left(u_n + (a_n-1)\theta u_n\right)(a_n - 1)u_n d\theta\right)dx \\
&= (a_n - 1)\int_{\mathbb{R}^N}\left(\int_0^1 f\left(u_n + (a_n-1)\theta u_n\right)u_n d\theta\right)dx.
\end{aligned}$$

Hence, we only need to prove that $\int_{\mathbb{R}^N}\left(\int_0^1 f\left(u_n + (a_n-1)\theta u_n\right)u_n d\theta\right)dx$ is bounded.

On the other hand, combining $0 \leq |u_n + (a_n-1)\theta u_n| < (a_n+2)|u_n|$ with $|f(t)t| \leq \beta F(1)\left(|t|^\alpha + |t|^\beta\right)$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_0^1 f\left(u_n + (a_n-1)\theta u_n\right)u_n d\theta\right)dx \\
&\leq \beta F(1)\int_{\mathbb{R}^N}\left(\int_0^1 \left(|u_n + (a_n-1)\theta u_n|^{\alpha-1}u_n + |u_n + (a_n-1)\theta u_n|^{\beta-1}u_n\right)d\theta\right)dx \\
&\leq \beta F(1)\int_{\mathbb{R}^N}\left(\int_0^1 (a_n+2)^{\alpha-1}|u_n|^\alpha + (a_n+2)^{\beta-1}|u_n|^\beta\right) \\
&\leq C\left(\|\nabla u_n\|_p^{\alpha\gamma_\alpha}\,\|u\|_p^{\alpha(1-\gamma_\alpha)} + \|\nabla u_n\|_p^{\beta\gamma_\beta}\,\|u\|_p^{\beta(1-\gamma_\beta)}\right)
\end{aligned}$$

is bounded because $\{u_n\}$ is bounded in E. $\square$

Lemma 3.6. *For any* $u \in S_c$, e^{s_u} *either satisfies* $e^{s_u[(\alpha-p)N/p-p]} \leq C\|\nabla u\|_p^p/\|u\|_\alpha^\alpha$ *or* $e^{s_u[(\beta-p)N/p-p]} \leq C\|\nabla u\|_p^p/\|u\|_\beta^\beta$*, where* $C = C(\sigma_2, p, N, \alpha)$ *is a constant.*

Proof of Lemma 3.6. By Lemma 3.3(i), we have $s_u * u \in \mathcal{P}_c$, so the Pohozaev identity gives

$$\int_{\mathbb{R}^N} |\nabla(s_u * u)|^p dx - \int_{\mathbb{R}^N} W(x)|s_u * u|^p dx = \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(s_u * u)dx.$$

Using (V_2) and Lemma 2.3, we obtain

$$(1+\sigma_2)\|\nabla(s_u * u)\|_p^p \geq \frac{N}{p}(\alpha - p)\int_{\mathbb{R}^N} F(s_u * u)dx.$$

By (F_2) and (3.1), we have $F(t) \geq F(1)\min\{|t|^\alpha, |t|^\beta\}$ for all $t \in \mathbb{R}$. Hence,

$$\int_{\mathbb{R}^N} F(s_u * u)dx \geq F(1)\int_{\mathbb{R}^N} \min\{|s_u * u|^\alpha, |s_u * u|^\beta\}dx.$$

Note that $|s_u * u|^\alpha = e^{N\alpha s_u/p}|u|^\alpha$ and $|s_u * u|^\beta = e^{N\beta s_u/p}|u|^\beta$. Therefore,

$$\min\{|s_u * u|^\alpha, |s_u * u|^\beta\} \geq \min\{e^{N\alpha s_u/p}, e^{N\beta s_u/p}\} \cdot \min\{|u|^\alpha, |u|^\beta\}.$$

Integrating and using the fact that $\|\nabla(s_u * u)\|_p^p = e^{ps_u}\|\nabla u\|_p^p$, we obtain

$$(1+\sigma_2)e^{ps_u}\|\nabla u\|_p^p \geq \frac{N}{p}(\alpha - p)F(1)\min\{e^{\frac{N\alpha s_u}{p}}, e^{\frac{N\beta s_u}{p}}\}\min\{\|u\|_\alpha^\alpha, \|u\|_\beta^\beta\}.$$

Rearranging the terms, we get

$$\min\left\{e^{s_u(\frac{N\alpha}{p}-p)}\|u\|_\alpha^\alpha,\ e^{s_u(\frac{N\beta}{p}-p)}\|u\|_\beta^\beta\right\} \leq \frac{p(1+\sigma_2)}{N(\alpha-p)F(1)}\|\nabla u\|_p^p.$$

Because $N\alpha/p - p = N(\alpha - p)/p$, and $N\beta/p - p = N(\beta - p)/p$, we conclude that either

$$e^{s_u(\frac{\alpha-p}{p}N-p)} \leq C\frac{\|\nabla u\|_p^p}{\|u\|_\alpha^\alpha} \quad \text{or} \quad e^{s_u(\frac{\beta-p}{p}N-p)} \leq C\frac{\|\nabla u\|_p^p}{\|u\|_\beta^\beta},$$

where $C = p(1+\sigma_2)/[N(\alpha - p)F(1)]$. This completes the proof. $\square$

Lemma 3.7. *Assume that* $1 < p < N$ *and* F *satisfy* (F_1)*–*(F_2)*. Then, the function* $c \mapsto E_c$ *is continuous and nonincreasing on (0,+∞).*

Proof. For any $0 < c_2 < c_1$, we shall prove that $E_{c_1} \leq E_{c_2}$. Based on the definition of E_c, we first have that, for any $\epsilon > 0$, there exists $u \in \mathcal{P}_{c_2}$ such that

$$I(u) \leq E_{c_2} + \frac{\epsilon}{2}. \tag{3.5}$$

Let $\gamma \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a cut-off function that satisfies

$$\gamma(x) = \begin{cases} 1, & |x| \leq 1, \\ \in [0,1], & |x| \in (1,2), \\ 0, & |x| \geq 2. \end{cases}$$

For $\delta > 0$ small enough, we define $u_\delta(x) = u(x) \cdot \gamma(\delta x) \in E \setminus \{0\}$. It is easy to check $u_\delta \to u$ in E as $\delta \to 0^+$. Because $u \mapsto s_u$ is continuous, we then get $\lim\limits_{\delta \to 0^+} s_{u_\delta} = s_u = 0$ and $s_{u_\delta} * u_\delta \to s_u * u = u$ in E as $\delta \to 0+$. Therefore, there exists a constant $\delta > 0$ small enough such that

$$I(s_{u_\delta} * u_\delta) \leq I(s_u * u) + \frac{\epsilon}{4} = I(u) + \frac{\epsilon}{4}. \tag{3.6}$$

Take $v \in C_0^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}(v) \subset B(0, 1 + 4/\delta) \setminus B(0, 4/\delta)$, and set

$$\bar{v} = \left(\frac{c_1 - \|u_\delta\|_p^p}{\|v\|_p^p} \right)^{\frac{1}{p}} v.$$

For any $\mu < 0$, we define $\omega_\mu = u_\delta + \mu * \bar{v} = u_\delta + e^{\frac{N}{p}\mu} \bar{v}(e^\mu x)$. Note that $\omega_\mu \in S_{c_1}$, and $\mathrm{supp}(u_\delta) \cap \mathrm{supp}(\mu * \bar{v}) = \emptyset$. Next, we claim that s_{ω_μ} is bounded from above as $\mu \to -\infty$. If $s_{\omega_\mu} \to +\infty$ as $\mu \to -\infty$, we observe that $I(s_{\omega_\mu} * \omega_\mu) > 0$ by Lemma (3.4)(iv). On the flip side, we obtain

$$\begin{aligned} & I(s_{\omega_\mu} * \omega_\mu) = \frac{1}{p} e^{p s_{\omega_\mu}} \left\| \nabla \omega_\mu \right\|_p^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V\left(\frac{x}{e^{s_{\omega_\mu}}} \right) |u|^p dx \\ & - e^{-N s_{\omega_\mu}} \int_{\mathbb{R}^N} F\left(e^{\frac{N s_{\omega_\mu}}{p}} \omega_\mu \right) dx \\ & \leq \frac{1}{p} e^{p s_{\omega_\mu}} \left\| \nabla \omega_\mu \right\|_p^p dx - F(1) \min\left\{ e^{\frac{N s_{\omega_\mu}}{p}\alpha - N s_{\omega_\mu}}, e^{\frac{N s_{\omega_\mu}}{p}\beta - N s_{\omega_\mu}} \right\} \int_{\mathbb{R}^N} \min\left\{ |u|^\alpha, |u|^\beta \right\} dx \\ & \to -\infty \quad \text{as } s_{\omega_\mu} \to +\infty, \end{aligned}$$

which is a contradiction, so the claim is true.

Next, because

$$s_{\omega_\mu} + \mu \to -\infty \quad \text{as } \mu \to -\infty,$$

we deduce that

$$\begin{aligned} & \left\| \nabla [(s_{\omega_\mu} + \mu) * \bar{v}] \right\|_p \to 0, \\ & \left| \int_{\mathbb{R}^N} V(x) |(s_{\omega_\mu} + \mu) * \bar{v}|^p dx \right| \leq \sigma_1 \left\| \nabla [(s_{\omega_\mu} + \mu) * \bar{v}] \right\|_p \to 0, \\ & (s_{\omega_\mu} + \mu) * \bar{v} \to 0 \quad \text{in} \quad L^{\bar{p}}(\mathbb{R}^N). \end{aligned}$$

By [[11], Lemma 2.3], we obtain $F((s_{\omega_\mu} + \mu) * \bar{v}) \to F(0)$ in $L^1(\mathbb{R}^N)$. Then, we deduce that for any $X > 0$ large enough, when $\mu < -X$,

$$I((s_{\omega_\mu} + \mu) * \bar{v}) \leq \frac{\epsilon}{4}. \tag{3.7}$$

In the end, in view of Lemmas 3.3(ii) and (3.5)–(3.7), we obtain

$$\begin{aligned} E_{c_1} \leq I(s_{\omega_\mu} * \omega_\mu) &= I(s_{\omega_\mu} * u_\delta) + I(s_{\omega_\mu} * (\mu * \bar{v})) \\ &\leq I(s_{u_\delta} * u_\delta) + I((s_{\omega_\mu} + \mu) * \bar{v}) \\ &\leq I(u) + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq E_{c_2} + \epsilon. \end{aligned}$$

Hence, the function $c \mapsto E_c$ is nonincreasing on $(0, +\infty)$.

Next, we prove the continuity of the function $c \mapsto E_c$. For any fixed $c > 0$, we know that $E_{c-h} \geq E_c \geq E_{c+h}$ for $h > 0$. Consequently, $\lim\limits_{h\to 0^+} E_{c-h} \geq E_c \geq \lim\limits_{h\to 0^+} E_{c+h}$. So, it is sufficient to prove that $\lim\limits_{h\to 0^+} E_{c-h} \leq E_c \leq \lim\limits_{h\to 0^+} E_{c+h}$. In the first, we claim $\lim\limits_{h\to 0^+} E_{c-h} \leq E_c$. Let $u \in \mathcal{P}_c, c > h > 0$, and then $P(u) = 0$. Setting $u_h(x) = (1 - h/c)^{\frac{1}{p}} u(x)$, it is easy to check $u_h \in S_{c-h}$; thus, there exists $s_h \in \mathbb{R}$ such that $P(s_h * u_h) = 0$. It follows from Lemma 3.6 that

$$e^{s_h(\frac{\alpha-p}{p}N-p)} \leq C\frac{\|\nabla u_h\|_p^p}{\|u_h\|_\alpha^\alpha} = C\frac{\left(1-\frac{h}{c}\right)}{\left(1-\frac{h}{c}^{\frac{\alpha}{p}}\right)}\frac{\|\nabla u\|_p^p}{\|u\|_\alpha^\alpha}$$

or

$$e^{s_h(\frac{\beta-p}{p}N-p)} \leq C\frac{\|\nabla u_h\|_p^p}{\|u_h\|_\beta^\beta} = C\frac{\left(1-\frac{h}{c}\right)}{\left(1-\frac{h}{c}^{\frac{\beta}{p}}\right)}\frac{\|\nabla u\|_p^p}{\|u\|_\beta^\beta}.$$

In either case, it follows that s_h is bounded as $h \to 0^+$. Hence, $s_h * u$ is bounded in E. Based on Lemma 3.5, we deduce

$$E_{c-h} \leq I(s_h * u_h) \leq I(s_h * u) + o_n(h) \leq I(u) + o_n(h).$$

By the arbitrariness of $u \in \mathcal{P}_c$, the claim is true.

For $\lim\limits_{h\to 0^+} E_{c+h} \leq E_c$, we take $u \in \mathcal{P}_c$ and for any $h > 0$. Let $v_h(x) = (1 + h/c)^{\frac{1}{p}} u(x)$. The rest is similar, so we omit it.

So far, we have completed the proof that under the assumptions (F_1) and (F_2), the function $c \mapsto E_c$ is continuous and nonincreasing on (0,+∞). □

Lemma 3.8. *Assume that $V(x)$ satisfies (V_1)–(V_3), and f satisfies (F_1)–(F_3). If there exist $u \in S_c$ with $I(u) = E_c$ and $\lambda \in \mathbb{R}$ such that*

$$-\Delta_p u + V(x)|u|^{p-2}u + \lambda|u|^{p-2}u = f(u),$$

then the function $c \mapsto E_c$ is strictly decreasing in a right neighborhood of c if $\lambda > 0$, and the function $c \mapsto E_c$ is strictly increasing in a left neighborhood of c if $\lambda < 0$.

Proof. For any $t > 0$ and $s \in \mathbb{R}$, we set $u_{t,s} := s * (tu) = e^{Ns/p}tu(e^s x) \in S_{ct}$ when $u \in S_c$, and it is clear that $u_{t,s} \to u$ in E as $(t, s) \to (1, 0)$. Define

$$\begin{aligned} a(t,s) &= I(u_{t,s}) = I(s * tu) \\ &= \frac{1}{p}t^p e^{ps}\int_{\mathbb{R}^N}|\nabla u|^p dx + \frac{1}{p}t^p\int_{\mathbb{R}^N}V\left(\frac{x}{e^s}\right)|u|^p dx - e^{-Ns}\int_{\mathbb{R}^N}F\left(te^{\frac{Ns}{p}}u(x)\right)dx. \end{aligned}$$

We compute that

$$\begin{aligned}\frac{\partial}{\partial t}a(t,s) &= t^{p-1}e^{ps}\int_{\mathbb{R}^N}|\nabla u|^p dx + t^{p-1}\int_{\mathbb{R}^N}V\left(\frac{x}{e^s}\right)|u|^p dx\\ &\quad - e^{-Ns}\int_{\mathbb{R}^N}f\left(te^{\frac{Ns}{p}}u(x)\right)\cdot e^{\frac{Ns}{p}}u(x)dx\\ &= \frac{1}{t}I'(u_{t,s})u_{t,s}\end{aligned}$$

and

$$I'(u)u = \|\nabla u\|_p^p + \int_{\mathbb{R}^N}V(x)|u|^p dx - \int_{\mathbb{R}^N}f(u)udx = -\lambda\|u\|_p^p = -\lambda c.$$

If $\lambda > 0$, then there exists a constant $\bar{\delta} > 0$ small enough such that

$$\frac{\partial}{\partial t}a(t,s) < 0 \quad \text{for any } (t,s) \in \left(1, 1+\bar{\delta}\right] \times \left[-\bar{\delta}, \bar{\delta}\right].$$

It follows from the mean value theorem that

$$a(t,s) = a(1,s) + (t-1)\frac{\partial}{\partial t}a(\xi,\eta) < a(1,s), \tag{3.8}$$

where $(\xi,\eta) \in \left(1, 1+\bar{\delta}\right] \times \left[-\bar{\delta}, \bar{\delta}\right]$. Because the mapping $u \mapsto s_u$ is continuous, one has $s_{tu} \to s_u = 0$ as $t \to 1$. In particular, for any c' in a right neighborhood of c, we set

$$t := \frac{c'}{c} \in \left(1, 1+\bar{\delta}\right] \quad \text{and} \quad s := s_{tu} \in \left[-\bar{\delta}, \bar{\delta}\right].$$

Therefore, we conclude that

$$E_{c'} \le a(t, s_{tu}) < a(1, s_{tu}) = I(s_{tu} * u) \le I(u) = E_c.$$

In the case of $\lambda < 0$, we can similarly obtain the desired result. □

In the next, in order to establish the existence of a normalized ground state of (1.1); inspired by [31], we introduce the homotopy stable family to construct a Palais–Smale sequence for the constrained functional $I|_{S_c}$ at the level E_c. To begin, we introduce the homotopy stable family introduced in [[31], Section 5].

Definition 3.9. *[[31], Definition 5.1] Let B be a closed subset of X. We shall say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with extended boundary B if for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times B)$, we have that $\eta(\{1\} \times A) \in \mathcal{F}$.*

We remark that because $B = \varnothing$ is admissible, it is sufficient to adopt the usual convention $\sup(\varnothing) = -\infty$.

Inspired by [30], we define the functional $\Psi : E \setminus \{0\} \to \mathbb{R}$ by

$$\begin{aligned}\Psi(u) &= I(s_u * u)\\ &= \frac{1}{p}e^{ps_u}\int_{\mathbb{R}^N}|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N}V\left(\frac{x}{e^{s_u}}\right)|u|^p dx - e^{-Ns_u}\int_{\mathbb{R}^N}F\left(e^{\frac{Ns_u}{p}}u(x)\right)dx,\end{aligned}$$

where $s_u \in \mathbb{R}$ is a unique number mentioned by Lemma 3.3. Moreover, it is clear that $\Psi(u)$ is of class C^1, and

$$\begin{aligned} d\Psi(u)[v] &= e^{ps_u}\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla v dx + \int_{\mathbb{R}^N} V\left(\frac{x}{e^{s_u}}\right)|u|^{p-2}uvdx \\ &- e^{-Ns_u}\int_{\mathbb{R}^N} f\left(e^{\frac{Ns_u}{p}}u\right)e^{\frac{Ns_u}{p}}vdx \\ &= \int_{\mathbb{R}^N} |\nabla(s_u * u)|^{p-2}\nabla(s_u * u)\nabla(s_u * v)dx \\ &+ \int_{\mathbb{R}^N} V(x)|s_u * u|^{p-2}(s_u * u)(s_u * v)dx - \int_{\mathbb{R}^N} f(s_u * u)(s_u * v)dx \\ &= dI(s_u * u)[s_u * v] \end{aligned}$$

for all $s_u * v \in T_{s_u * u}S_c$. In addition, for any $c > 0$, we can define the constrained functional

$$J := \Psi|_{S_c} : S_c \to \mathbb{R}.$$

It is clear that the constrained functional is class of C^1 and

$$dJ(u)[v] = d\Psi(u)[v] = dI(s_u * u)[s_u * v]$$

for all $u \in S_c$ and $v \in T_uS_c$. The following result show the existence of a Palais–Smale sequence at the level $E_{c,\mathcal{F}}$.

Lemma 3.10. *Assume that $\mathcal{F}$ is a homotopy stable family of compact subsets of S_c with $B = \emptyset$, and let*

$$E_{c,\mathcal{F}} = \inf_{A\in\mathcal{F}}\max_{u\in A} J(u).$$

If $E_{c,\mathcal{F}} > 0$, then there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_c$ for the constrained functional $I|_{S_c}$ at the level $E_{c,\mathcal{F}}$.

Proof. Let $\{A_n\}$ be an arbitrary minimizing sequence at the level $E_{c,\mathcal{F}}$. We define a continuous mapping

$$\eta : (0,1) \times S_c \to S_c,$$

$$(t,u) \mapsto (ts_u) * u.$$

We observe that $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times S_c)$. It follows from Definition 3.9 that

$$M_n = \eta(1, A_n) = \{s_u * u | u \in A_n\} \in \mathcal{F}.$$

Based on Lemma 3.3(i), it is obvious that $\{M_n\} \subset \mathcal{P}_c$. In addition,

$$\max_{M_n} J(u) = \max_{A_n} J(u) \to E_{c,\mathcal{F}},$$

so $\{M_n\}$ is another minimizing sequence of $E_{c,\mathcal{F}}$. By Theorem 3.4 in [31], there exists a sequence $\{v_n\} \subset S_c$ such that

(i) $\lim_{n\to\infty} J(v_n) = E_{c,\mathcal{F}}$.

(ii) $\lim\limits_{n\to\infty} \|dJ(v_n)\|_{E^*} = 0.$

(iii) $\lim\limits_{n\to\infty} \mathrm{dist}_E(v_n, M_n) = 0.$

Set $u_n = s_{v_n} * v_n$, and then, $\{u_n\} \in \mathcal{P}_c$, and $\|\nabla u_n\|_p^p = e^{ps_{v_n}} \|\nabla v_n\|_P^p$. Due to Lemma 3.4(iii), $e^{-ps_{v_n}} = \|\nabla v_n\|_p^p \,/\, \|\nabla u_n\|_p^p$ is well-defined, and $\|\nabla u\|_p^p$ is bounded from below by a positive constant. Furthermore, $\{M_n\} \subset \mathcal{P}_c \subset S_c$ for any n, and

$$\inf \max_{M_n} J = \inf \max_{M_n} I \to E_{c,\mathcal{F}}.$$

By virtue of Lemma 3.4(ii), we know that $\{M_n\}$ is bounded in E for every $n \in N$. On the other hand, it follows from $\lim\limits_{n\to\infty} \mathrm{dist}_E(v_n, M_n) = 0$ that $\|\nabla v_n\|_p$ is bounded. Combining with the above argument, we know that $e^{-ps_{v_n}}$ is bounded, that is, s_{v_n} is bounded from below. Therefore, for any $\varphi \in E$, there exists a constant C such that

$$\left\|(-s_{v_n} * \varphi)\right\|_p \le C \|\varphi\|_p \,, \left\|\nabla(-s_{v_n} * \varphi)\right\|_p \le C \|\nabla\varphi\|_p \,.$$

Based on the initial analysis, we obtain

$$\lim_{n\to+\infty} I(u_n) = \lim_{n\to+\infty} J(u_n) = \lim_{n\to+\infty} J(v_n) = E_{c,\mathcal{F}}.$$

Below, we prove that $\{u_n\}$ is a Palais–Smale sequence for I at the level $E_{c,\mathcal{F}}$. For any $\psi \in T_{u_n}S_c$, it is easy to check that $(-s_{v_n}) * \psi \in T_{v_n}S_c$, and $\left\|(-s_{v_n} * \psi)\right\| \le C\|\psi\|$. We take $\|\cdot\|_{u,E^*}$ to denote the norm of $(T_uS_c)^*$; then,

$$\begin{aligned}
\|dI(u_n)\|_{u_n,E^*} &= \sup_{\psi\in T_{u_n S_c}, \|\psi\|\le 1} |dI(u_n)[\psi]| \\
&= \sup_{\psi\in T_{u_n S_c}, \|\psi\|\le 1} \left|dI(s_{v_n} * v_n)[s_{v_n} * ((-s_{v_n}) * \psi)]\right| \\
&= \sup_{\psi\in T_{u_n S_c}, \|\psi\|\le 1} \left|dJ(v_n)[(-s_{v_n}) * \psi]\right| \\
&\le \|dJ(v_n)\|_{v_n,E^*} \cdot \sup_{\psi\in T_{u_n S_c}, \|\psi\|\le 1} \left\|(-s_{v_n}) * \psi\right\| \\
&\le C \|dJ(v_n)\|_{v_n,E^*} \,.
\end{aligned}$$

Because $\{v_n\} \subset S_c$ is a Palais–Smale sequence of J at the level $E_{c,\mathcal{F}}$, we yield that $\|dI(u_n)\|_{u_n,E^*} \to 0$. Therefore, we have proved that $\{u_n\}$ is a Palais-Smale sequence for I at the level $E_{c,\mathcal{F}}$. □

Lemma 3.11. *There exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_c$ for $I|_{S_c}$ at the level $E_c = \inf\limits_{u\in\mathcal{P}_c} I(u)$.*

Proof. In virtue of Lemma 3.4(iv), we know $E_c > 0$. By Lemma 3.10, we only need to prove $E_c = E_{c,\mathcal{F}}$. First, we note that

$$E_{c,\mathcal{F}} = \inf_{A\in\mathcal{F}} \max_{u\in A} J(u) = \inf_{u\in\mathcal{P}_c} I(s_u * u).$$

On the one hand, for any $u \in S_c$, $s_u * u \in \mathcal{P}_c$ and $I(s_u * u \ge E_c)$, $E_{c,\mathcal{F}} \ge E_c$ is implied. On the other hand, for any $u \in \mathcal{P}_c$ taking $s_u = 0$, we have $I(u) = I(0 * u) \ge E_{c,\mathcal{F}}$, and thus, $E_c \ge E_{c,\mathcal{F}}$. Combining the above two aspects, we obtain $E_c = E_{c,\mathcal{F}}$. □

Lemma 3.12. *Assume that $\{u_n\} \in \mathcal{P}_c$ is any bounded Palais–Smale sequence for $I|_{S_c}$ at the level E_c. Then, there exists $(u, \lambda) \in (S_c, \mathbb{R}^+)$ such that up to a subsequence, still denote by $\{u_n\}$, $u_n \to u \in E$ and*

$$-\Delta_p u + V(x)|u|^{p-2}u + \lambda|u|^{p-2}u = f(u).$$

Proof. Because $\{u_n\} \subset S_c$ is bounded in E, we obtain the existence of $\lim\limits_{n\to\infty} \|\nabla u_n\|_p^p$, $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^p dx$, and $\lim\limits_{n\to\infty} \|u_n\|_p^p$. Moreover, $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx$ and $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n dx$ exist according to (F_2), (3.1), and the Gagliardo–Nirenberg inequality. By applying Lemma 3 from [35] and $\|dI(u_n)\|_{u_n,E^*} \to 0$, we have

$$-\Delta_p u_n + V(x)|u_n|^{p-2}u_n + \lambda_n |u_n|^{p-2}u_n - f(u_n) \to 0 \quad \text{on} \quad E^* \quad \text{as} \quad n \to \infty,$$

where

$$\lambda_n = -\frac{\langle I'(u_n), u_n\rangle}{c} = \frac{1}{c}\left(\int_{\mathbb{R}^N} f(u_n)u_n dx - \|\nabla u_n\|_p^p - \int_{\mathbb{R}^N} V(x)|u_n|^p dx\right)$$

is bounded. Thus there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$.

Next, we claim that $\{u_n\}$ is nonvanishing. By contradiction that $\{u_n\}$ is vanishing, by Lemma I.1 in [36], we get $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for α between p and $Np/(N-p)$. Afterward in virtue of Lemma 2.3 in [11], we derive that $F(u_n) \to 0$ in $L^1(\mathbb{R}^N)$, and $f(u_n)u_n \to 0$ in $L^1(\mathbb{R}^N)$, and therefore, $\int_{\mathbb{R}^N} \bar{F}(u_n)dx \to 0$. Combining with $P(u_n) \to 0$ and (V_2), we get

$$(1-\sigma_2)\, \|\nabla u_n\|_p^p \le \|\nabla u_n\|_p^p - \int_{\mathbb{R}^N} W(x)|u_n|^p dx \to \frac{N}{p}\int_{\mathbb{R}^N} \bar{F}(u_n)dx \to 0.$$

Hence,

$$I(u_n) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u_n|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p dx - \int_{\mathbb{R}^N} F(u_n)dx \to 0,$$

which contradicts with $E_c > 0$. Therefore, $\{u_n\}$ is nonvanishing. Then, there exist $\{y_n^1\} \in \mathbb{R}^N$ and $u^1 \in E \setminus \{0\}$ such that the translated sequence

$$v_n(x) = u_n(x + y_n)$$

satisfies (up to a subsequence)

$$\begin{cases} v_n(x) \rightharpoonup u^1 & \text{in} \quad E; \\ v_n(x) \to u^1 & \text{in} \quad L^r_{loc}(\mathbb{R}^N) \quad \text{for all} \quad 1 \le r < p^*; \\ v_n(x) \to u^1 & a.e. \quad \text{in} \quad \mathbb{R}^N. \end{cases} \tag{3.9}$$

By translation, we have

$$\|\nabla v_n\|_p = \|\nabla u_n\|_p,\ \|v_n\|_p = \|u_n\|_p,\ \int_{\mathbb{R}^N} F(v_n)dx = \int_{\mathbb{R}^N} F(u_n)dx$$

and

$$\int_{\mathbb{R}^N} V(x)|v_n|^p dx = \int_{\mathbb{R}^N} V(x-y_n)|u_n|^p dx,\ \int_{\mathbb{R}^N} W(x)|v_n|^p dx = \int_{\mathbb{R}^N} W(x-y_n)|u_n|^p dx.$$

From $P(u_n) = 0$, we obtain

$$P(v_n) = \int_{\mathbb{R}^N} [W(x) - W(x-y_n)]|u_n|^p dx. \tag{3.10}$$

Subsequently, we employ the Lions concentration–compactness principle to prove the boundedness of $\{y_n\}$. Assume by contradiction that $|y_n| \to \infty$. Because $v_n \rightharpoonup u^1$, there exist $R > 0$ and $\delta > 0$ such that

$$\int_{B_R(0)} |v_n|^p dx \geq \delta > 0, \int_{B_R(y_n)} |u_n|^p dx \geq \delta > 0 \text{ for all sufficiently large } n. \tag{3.11}$$

Now, apply the Lions concentration–compactness principle [36] toLemma I.1 to $\{u_n\}$. Because $\{u_n\}$ is bounded in E, there exists a subsequence such that one of the following three cases occurs:

(i) Compactness: there exists $z_n \in \mathbb{R}^N$ such that $u_n(x + z_n)$ converges strongly in $L^p(\mathbb{R}^N)$;
(ii) Vanishing: $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p dx = 0$ for all $R > 0$;
(iii) Dichotomy: there exist $\alpha \in (0, 1)$ and $R_n \to \infty$ such that

$$\left|\int_{B_{R_n}(0)} |u_n|^p dx - \alpha c\right| \to 0, \int_{B_{2R_n} \backslash B_{R_n}(0)} |u_n|^p dx \to 0,$$

$$\int_{\mathbb{R}^N \backslash B_{2R_n}(0)} |u_n|^p dx \to (1 - \alpha)c.$$

(3.11) excludes the vanishing case.

Assume dichotomy occurs. By the Lions concentration-compactness principle [36] Theorem II.1, for the energy functional I at level c, we claim that the strict subadditivity inequality

$$E_c < E_{\alpha c} + E_{(1-\alpha)c} \tag{3.12}$$

holds for all $\alpha \in (0, 1)$. For any $\varepsilon > 0$, choose $u_1 \in \mathcal{P}_{\alpha c}$, $u_2 \in \mathcal{P}_{(1-\alpha)c}$ such that $I(u_1) < E_{\alpha c} + \varepsilon/2$, $I(u_2) < E_{(1-\alpha)c} + \varepsilon/2$. By translation, we may assume $\text{supp}(u_1) \cap \text{supp}(u_2) = \emptyset$. Set $u = u_1 + u_2$, and then $\|u\|_p^p = \alpha c + (1 - \alpha)c = c$ and $I(u) = I(u_1) + I(u_2) < E_{\alpha c} + E_{(1-\alpha)c} + \varepsilon$. By Lemma 3.3, there exists a unique $s_u \in \mathbb{R}$ such that $s_u * u \in \mathcal{P}_c$ and $\Psi_u(s_u) \geq \Psi_u(0) = I(u)$. Hence,

$$E_c \leq I(s_u * u) = \Psi_u(s_u) \leq I(u) < E_{\alpha c} + E_{(1-\alpha)c} + \varepsilon.$$

Letting $\varepsilon \to 0^+$ gives $E_c \leq E_{\alpha c} + E_{(1-\alpha)c}$.

If equality holds, then the above argument forces $s_u = 0$ and $u \in \mathcal{P}_c$ with $I(u) = E_c$. But then, $\Psi_u''(0) < 0$ by Lemma 3.2, contradicting the fact that $s_u = 0$ is a global maximum of Ψ_u (a maximum would require $\Psi_u''(0) \leq 0$). Thus, the inequality is strict. This completes the proof of (3.12).

Next, we construct splittting sequences in the dichotomy case. Take a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfying $0 \leq \varphi \leq 1, \varphi(x) = 1$ for $|x| \leq 1, \varphi(x) = 0$ for $|x| \geq 2$ and $|\nabla\varphi| \leq 2$. Define

$$\varphi_n^1(x) = \varphi\left(\frac{x}{R_n}\right), \quad \varphi_n^2(x) = 1 - \varphi\left(\frac{x}{R_n}\right)$$

and set

$$u_n^1 = \varphi_n^1 u_n, \quad u_n^2 = \varphi_n^2 u_n.$$

Then, we have

- $u_n = u_n^1 + u_n^2$;

- $\operatorname{supp}(u_n^1) \subset B_{2R_n}(0), \operatorname{supp}(u_n^2) \subset \mathbb{R}^N \setminus B_{R_n}(0)$;
- $\|u_n^1\|_p^p \to \alpha c, \quad \|u_n^2\|_p^p \to (1-\alpha)c, \quad \int_{B_{2R_n} \setminus B_{R_n}} |u_n|^p \, dx \to 0.$

From $\nabla u_n^i = \varphi_n^i \nabla u_n + u_n \nabla \varphi_n^i$, we get

$$\int_{\mathbb{R}^N} |u_n \nabla \varphi_n^i|^p dx \le \frac{C}{R_n^p} \int_{R_n < |x| < 2R_n} |u_n|^p dx \to 0.$$

Hence,

$$\|\nabla u_n\|_p^p = \|\nabla u_n^1\|_p^p + \|\nabla u_n^2\|_p^p + o(1). \tag{3.13}$$

By the disjointness of supports,

$$\int_{\mathbb{R}^N} V(x)|u_n|^p dx = \int V|u_n^1|^p dx + \int V|u_n^2|^p dx + o(1), \tag{3.14}$$

$$\int_{\mathbb{R}^N} W(x)|u_n|^p dx = \int W|u_n^1|^p dx + \int W|u_n^2|^p dx + o(1). \tag{3.15}$$

On the other hand, by the Brezis–Lieb lemma and the growth condition in (F_2),

$$\int_{\mathbb{R}^N} F(u_n)\, dx = \int F(u_n^1)\, dx + \int F(u_n^2)\, dx + o(1), \tag{3.16}$$

$$\int_{\mathbb{R}^N} \bar{F}(u_n)\, dx = \int \bar{F}(u_n^1)\, dx + \int \bar{F}(u_n^2)\, dx + o(1). \tag{3.17}$$

Based on $P(u_n) = 0$ and the above discussions, we obtain

$$P(u_n^1) + P(u_n^2) = o(1), \;\; I(u_n) = I(u_n^1) + I(u_n^2) + o(1). \tag{3.18}$$

By Lemma 3.3, for each u_n^i, there exists a unique $s_n^i \in \mathbb{R}$ such that

$$s_n^i * u_n^i \in \mathcal{P}_{\|u_n^i\|_p^p} \text{ and } I(s_n^i * u_n^i) \ge E_{\|u_n^i\|_p^p}.$$

By the mean value theorem,

$$I(u_n^i) - I(s_n^i * u_n^i) = \int_{s_n^i}^0 P(t * u_n^i)\, dt.$$

The boundedness of the sequence $\{u_n^i\}$ implies that there exists a constant $C_1 > 0$ such that

$$|P(t * u_n^i)| \le C_1$$

for all n and all $|t| \le |s_n^i|$. Hence,

$$|I(u_n^i) - I(s_n^i * u_n^i)| \le C_1 |s_n^i|. \tag{3.19}$$

Because $P(t * u_n^i)$ is strictly decreasing in t (as $\Psi_u''(t) < 0$), there exists a constant $\kappa > 0$ such that

$$|P(u_n^i)| = |P(0 * u_n^i) - P(s_n^i * u_n^i)| \ge \kappa |s_n^i|.$$

Thus,

$$|s_n^i| \le \kappa^{-1} |P(u_n^i)|. \tag{3.20}$$

Suppose $\limsup |P(u_n^1)| = \delta > 0$. Then, there exists a subsequence such that $|P(u_n^1)| \geq \delta/2$. By 3.18, it follows that $|P(u_n^2)| \geq \delta/4$ and $P(u_n^2) = -P(u_n^1) + o(1)$. Using (3.18)–(3.20) and $I(s_n^i * u_n^i) \geq E_{\|u_n^i\|_p^p}$, we obtain

$$I(u_n) \geq E_{\|u_n^1\|_p^p} + E_{\|u_n^2\|_p^p} - C_1\kappa^{-1}(|P(u_n^1)| + |P(u_n^2)|) + o(1).$$

Taking the limit and using the continuity of E_c (Lemma 3.7), we get

$$E_c \geq E_{\alpha c} + E_{(1-\alpha)c} - C_1\kappa^{-1}\delta. \tag{3.21}$$

By the strict subadditivity (3.12) from the Lions principle, $E_c < E_{\alpha c} + E_{(1-\alpha)c}$. Hence, we must have $\delta = 0$, that is, $P(u_n^1) \to 0$. Similarly, $P(u_n^2) \to 0$.

From $P(u_n^i) \to 0$ and (3.20), we have $s_n^i \to 0$. Substituting into (3.19) yields

$$I(u_n^i) = I(s_n^i * u_n^i) + o(1) \geq E_{\|u_n^i\|_p^p} + o(1). \tag{3.22}$$

Inserting (3.22) into (3.18) gives

$$I(u_n) \geq E_{\|u_n^1\|_p^p} + E_{\|u_n^2\|_p^p} + o(1).$$

Taking the limit and using the continuity of E_c, we obtain

$$E_c \geq E_{\alpha c} + E_{(1-\alpha)c},$$

which contradicts the strict subadditivity (3.12). Therefore dichotomy cannot occur.

Because vanishing and dichotomy are excluded, the only possibility is compactness, so there exists $z_n \in \mathbb{R}^N$ such that $u_n(x + z_n)$ converges strongly in L^p. By (3.11), the balls $B_R(y_n)$ and $B_R(z_n)$ must intersect (otherwise, the mass would disperse to infinity, contradicting strong convergence); hence, $|y_n - z_n| \leq 2R$. Strong convergence implies that $\{z_n\}$ is bounded (otherwise the translated sequence would vanish), so $\{y_n\}$ is bounded. In addition, there exist $y_0, z_0 \in \mathbb{R}^N$ such that $y_n \to y_0$, $z_n \to z_0$, and $u_n(x + z_n) \to u^1(x + z_0 - y_0)$ in $L^p(\mathbb{R}^N)$. Setting $u(x) = u^1(x - y_0)$, we obtain $\|u_n - u\|_p \to 0$ and $u \in S_c$.

From the convergence established in front, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, using the strong convergence $u_n \to u$ in $L^p(\mathbb{R}^N)$, the continuity of $V(x)$, and the dominated convergence theorem controlled by (F_2), we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi \, dx + \lambda \int_{\mathbb{R}^N} |u|^{p-2}u\varphi \, dx = \int_{\mathbb{R}^N} f(u)\varphi \, dx.$$

By density, u satisfies

$$-\Delta_p u + V(x)|u|^{p-2}u + \lambda|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N \tag{3.23}$$

and $P(u) = 0$.

Applying the Brezis–Lieb lemma to the functional I, we obtain

$$E_c = \lim_{n\to\infty} I(u_n) = I(u) + \lim_{n\to\infty} I_0(u_n - u),$$

where $I_0(u) = 1/p\|\nabla u\|_p^p - \int F(u)\, dx$.

Due to $I(u_n) \to E_c$, it follows that

$$E_c = I(u) + \lim_{n\to\infty} I_0(u_n - u). \tag{3.24}$$

On the one hand, because $u \in \mathcal{P}_c$, by Lemma 3.7, we have $I(u) \geq E_c$. On the other hand, from $\|u_n - u\|_p \to 0$ and (F_2), we obtain that $\liminf_{n\to\infty} I_0(u_n - u) \geq 0$. Combining with (3.24), we finally obtain $I(u) = E_c$ and $\|\nabla u_n\|_p \to \|\nabla u\|_p$.

Finally, for $u \in \mathcal{P}_c$, it follows from (V_1) and (V_2), (2.2), and (2.3) that

$$\begin{aligned}
\lambda \|u\|_p^p &= \int_{\mathbb{R}^N} f(u)udx - \|\nabla u\|_p^p - \int_{\mathbb{R}^N} V(x)|u|^p dx \\
&\geq \int_{\mathbb{R}^N} f(u)udx - \|\nabla u\|_p^p \geq \frac{\beta}{\beta - p}\int_{\mathbb{R}^N} \bar{F}(u)dx - \|\nabla u\|_p^p \\
&= \frac{\beta}{\beta - p}\frac{p}{N}\left(\|\nabla u\|_p^p - \int_{\mathbb{R}^N} W(x)|u|^p dx\right) - \|\nabla u\|_p^p \\
&\geq \left(\frac{\beta p}{(\beta - p)N}(1 - \sigma_2) - 1\right)\|\nabla u\|_p^p > 0.
\end{aligned}$$

Therefore, we deduce that $\lambda > 0$ (because of Lemma 3.4(iii)). □

Proof of Theorem 1.2. In virtue of Lemma 3.4(ii), Lemmas 3.10 and 3.11, we obtain a bounded Palais–Smale sequence $\{u_n\} \subset \mathcal{P}_c$ for the constrained functional $I|_{S_c}$ at the level $E_c > 0$. This together with Lemma 3.12 gives the existence of a ground state $\bar{u} \in S_c$ at the level E_c.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by Natural Science Foundation of Heilongjiang Province (Mathematical Modeling and Spatiotemporal Dynamics of Oncolytic Virus-Based Combination Therapy for Cancer), No. PL2025A018.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. Y. M. Chen, L. Stacey, R. Murali, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.*, **66** (2006), 1383–1406. https://doi.org/10.1137/050624522 .
2. J. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Elliptic Equations, Research Notes in Mathematics, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1985.

3. N. Mastorakis, H. Fathabadi, On the solution of p-laplacian for non-newtonian fluid flow, *WSEAS Trans. Math.*, **8** (2009), 238–245.
4. D. Motreanu, M. Tanaka, Sign-changing and constant-sign solutions for p- laplacian problems with jumping nonlinearities, *J. Differ. Equations*, **249** (2010), 3352–3376. https://doi.org/10.1016/j.jde.2010.08.017
5. W. Bao, Y. Cai, Mathematical theory and numerical methods for bose- einstein condensation, *Kinet. Relat. Models.*, **6** (2013),1135. https://doi.org/10.3934/krm.2013.6.1
6. L. Jeanjean, S. Lu, A mass supercritical problem revisited, *Calc. Var. Part. Differ. Equations*, **59** (2020), 1–43. https://doi.org/10.1007/s00526-020-01828-z
7. F. Mehats, C. Sparber, Dimension reduction for rotating bose-einstein condensates with anisotropic confinement, *Discrete Contin. Dyn. Syst*, **36** (2016), 5097–5118. https://doi.org/10.3934/dcds.2016021
8. L. Jeanjean, T. Luo, Z. Wang, Multiple normalized solutions for quasi-linear schrödinger equation, *J. Differ. Equations*, **259** (2015), 3894–3928. https://doi.org/10.1016/j.jde.2015.05.008
9. N. Soave, Normalized ground states for the nls equation with combined nonlinearities, *J. Differ. Equations*, **269** (2020), 6941–6987. https://doi.org/10.1016/j.jde.2020.05.016
10. N. Soave, Normalized ground states for the nls equation with combined nonlinearities: The sobolev critical case, *J. Funct. Anal.*, **279** (2020), 108610. https://doi.org/10.1016/j.jfa.2020.108610
11. Z. Zhang, Z. Zhang, Normalized solutions to p-Laplacian equations with combined nonlinearities, *Nonlinearity*, **35** (2022), 5621–5663. https://doi.org/10.1088/1361-6544/ac902c
12. W. Wang, Q. Li, J. Zhou, Y. Li, Normalized solutions for p-laplacian equations with a L^2-supercritical growth, *Ann. Funct. Anal.*, **12** (2021), 9. https://doi.org/10.1007/s43034-020-00101-w
13. Q. Lou, Z. Zhang, Multiplicity and concentration of normalized solutions to p-laplacian equations, *Z. Angew. Math. Phys.*, **75** (2024), 81. https://doi.org/10.1007/s00033-024-02219-6
14. C. Wang, J. Sun, Normalized solutions for the p-laplacian equation with a trapping potential, *Adv. Nonlinear Anal.*, **12** (2023), 20220291. https://doi.org/10.1515/anona-2022-0291
15. J. Zhang, C. Lei, J. Lei, The existence and nonexistence of normalized solutions for a p-laplacian equation, *Appl. Math. Lett.*, **148** (2024), 108890. https://doi.org/10.1016/j.aml.2023.108890
16. L. Gu, X. Zeng, H. Zhou, Eigenvalue problems for p-laplacian equation with trapping potentials, *Nonlinear Anal.*, **148** (2017), 212–227. https://doi.org/10.1016/j.na.2016.10.002
17. N. Ikoma, Y. Miyamoto, Stable standing waves of nonlinear schrödinger equations with potentials and general nonlinearities, *Calc. Var. Part. Differ. Equations*, **59** (2020), 48. https://doi.org/10.1007/s00526-020-1703-0
18. T. Bartsch, R. Molle, M. Rizzi, G. Verzini, Normalized solutions of mass supercritical schrödinger equations with potential, *Commun. Partial Differ. Equations*, **46** (2021), 1729–1756. https://doi.org/10.1080/03605302.2021.1893747
19. Y. Ding, X. Zhong, Normalized solution to the schrödinger equation with potential and general nonlinear term: Mass super-critical case, *J. Differ. Equations*, **334** (2022), 194–215. https://doi.org/10.1016/j.jde.2022.06.013

20. Q. He, Z. Lv, Z. Tang, The existence of normalized solutions to the kirchhoff equation with potential and sobolev critical nonlinearities, *J. Geom. Anal.*, **33** (2023), 236. https://doi.org/10.1007/s12220-023-01298-7
21. L. Chergui, T. Gou, H. Hajaiej, Existence and dynamics of normalized solutions to nonlinear schrödinger equations with mixed fractional laplacians, *Calc. Var. Part. Differ. Equations*, **62** (2023), 208. https://doi.org/10.1007/s00526-023-02548-w
22. M. Xiang, Y. Ma, Existence and stability of normalized solutions for nonlocal double phase problems, *J. Geom. Anal.*, **34** (2024), 1–29. https://doi.org/10.1007/s12220-023-01497-2
23. S. Chen, X. Tang, A comprehensive review on the existence of normalized solutions for four classes of nonlinear elliptic equations, *Opuscula Math.*, **45** (2025), 739–763. https://doi.org/10.7494/OpMath.2025.45.6.739
24. M. Shu, L. Wen, H. Yang, Normalized solutions for planar schrödinger- poisson equation with mixed nonlinearities, *Bull. Math. Sci.*, **15** (2025), 2550008. https://doi.org/10.1142/S1664360725500080
25. L. Wei, Y. Song, Normalized solutions for critical schrödinger equations involving (2,q)-laplacian, *Opuscula Math.*, **45** (2025), 685–716. https://doi.org/10.7494/OpMath.2025.45.5.685
26. A. Abouelregal, M. Marin, C. Andreas, The influence of a non-local Moore–Gibson–Thompson heat transfer model on an underlying thermoelastic material under the model of memory-dependent derivatives, *Contin. Mech. Thermodyn.*, **35** (2023), 545–562. https://doi.org/10.1007/s00161-023-01195-y
27. A. Abouelregal, M. Marin, A. Öchsner, A modified spa-tiotemporal nonlocal thermoelasticity theory with higher-order phase delays for a viscoelastic micropolar medium exposed to short-pulse laser excitation, *Contin. Mech. Thermodyn.*, **37** (2024). https://doi.org/10.1007/s00161-024-01342-z
28. S. Vlase, I. Negrean, M. Marin, M. Scutaru, Energy of accelerations used to obtain the motion equations of a three-dimensional finite element, *Symmetry*, **12** (2020), 321. https://doi.org/10.3390/sym12020321
29. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.*, **28** (1997), 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1
30. T. Bartsch, N. Soave, Multiple normalized solutions for a competing system of schrödinger equations, *Calc. Var. Part. Differ. Equations*, **58** (2019), 22. https://doi.org/10.1007/s00526-018-1476-x
31. N. Ghoussoub, *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press, Cambridge, 1993. https://doi.org/10.1017/CBO9780511551703
32. M. Agueh, Sharp gagliardo-nirenberg inequalities via p-laplacian type equations, *Nonlinear Differ. Equations Appl.*, **15** (2008), 457–472. https://doi.org/10.1007/s00030-008-7021-4
33. S. Pohozaev, Eigenfunctions of the equation $\delta u + \lambda f(u) = 0$, *Sov. Math. Dokl.*, **6** (1965), 1408–1411.
34. P. Pucci, J. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007. https://doi.org/10.1007/978-0-8176-4733-9_11
35. H. Berestycki, P. Lions, Nonlinear scalar field equations ii: existence of infinitely many solutions, *Arch. Rational Mech. Anal.*, **82** (1983), 347–375. https://doi.org/10.1007/BF00250556

36. P. Lions, The concentration-compactness principle in the calculus of variations. the locally compact case, part 1, *Ann. Inst. H. Poincaré Anal. non linéaire*, **1** (1984), 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0