Electronic
Research Archive

https://www.aimspress.com/journal/era

ERA, 34(6): 4037–4050.
DOI: 10.3934/era.2026181
Received: 26 February 2026
Revised: 14 April 2026
Accepted: 22 April 2026
Published: 14 May 2026

Research article

Generalized word metrics and a Mazur–Ulam-type theorem for gyrogroups

Teerapong Suksumran*

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

* **Correspondence:** Email: teerapong.suksumran@cmu.ac.th.

Abstract: We investigate the metric geometry of gyrogroups, a class of group-like structures whose binary operation is generally nonassociative. In particular, we extend the notion of the word metric from finitely generated groups to gyrogroups. This extension enables any gyrogroup to be viewed as a metric space, providing a suitable framework for proving a Mazur–Ulam-type theorem and for analyzing its algebraic and combinatorial structure via the associated right Cayley graph in a manner analogous to the classical setting of groups.

Keywords: word metric; generating set; gyrogroup; Cayley graph; isometry
Mathematics Subject Classification: Primary 05C25; Secondary 20N05, 51F99

---

1. Introduction

The word metric defined on a finitely generated group plays a fundamental role in connecting algebraic and geometric viewpoints through its realization in Cayley (di)graph representation [1, 2]. Given a finite generating set S of a group Γ, the word metric measures the minimal number of generators in S needed to express each element in Γ, which is translated into a geometric distance within the Cayley graph $\mathrm{Cay}(\Gamma, S)$. This metric endows the group with a natural metric space structure, allowing the study of geometric properties such as the growth rate, isoperimetric inequalities, and quasi-isometries of finitely generated groups. Consequently, the word metric provides the foundation of geometric group theory, enabling deep insights into how algebraic properties of groups are reflected in the large-scale geometry of their Cayley graphs [3, 4].

Gyrogroups, whose binary operations are in general nonassociative, provide a suitable generalization of groups that capture nonassociative structures arising in the study of Einstein's velocity addition. Subsequently, gyrogroups offer an algebraic framework for studying analytic hyperbolic geometry via the concept of gyroassociativity, which is a generalized form of associativity governed by gyrations [5–8]. The presence of gyrogroup structures allows algebraic methods to be extended beyond associative settings. For instance, in [9], the authors successfully extended the notion of a word metric to the setting of strongly generated gyrogroups and then generalized the Švarc–Milnor lemma (also known as the fundamental lemma of geometric group theory) for gyrogroups. In the same spirit, this work represents a further step toward extending classical results from group theory to the framework of gyrogroups by appropriately addressing the issue of nonassociativity.

2. Preliminaries

The basic theory of gyrogroups can be found in, for instance [8, 10]. Suppose that G is a nonempty set equipped with a binary operation $\oplus$ defined on G. A map $f : G \to G$ is called an *automorphism* of $(G, \oplus)$ if it is a bijection such that $f(a \oplus b) = f(a) \oplus f(b)$ for all $a, b \in G$. The pair $(G, \oplus)$ is called a *gyrogroup* if the following conditions hold: there is an element e, called an *identity*, such that $e \oplus a = a$ for all $a \in G$; for any element $a \in G$, there is an element $\ominus a$ in G, called an *inverse* of a, such that $\ominus a \oplus a = e$; for each pair of a and b, there is an automorphism $\mathrm{gyr}\,[a, b]$ of $(G, \oplus)$, called a *gyroautomorphism*, such that $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b](c)$ for all $c \in G$; and for all $a, b \in G$, $\mathrm{gyr}\,[a \oplus b, b] = \mathrm{gyr}\,[a, b]$. Let G be a gyrogroup. In the case when $\mathrm{gyr}\,[a, b]$ becomes the identity automorphism of G for all $a, b \in G$, $(G, \oplus)$ forms a group, and we say that G is *degenerate*. The *coaddition* of G, denoted by $\boxplus$, is defined by $a \boxplus b = a \oplus \mathrm{gyr}\,[a, \ominus b](b)$ for all $a, b \in G$. Also, define $a \boxminus b = a \boxplus (\ominus b)$. We say that a subset H of G is a *subgyrogroup* of G if H forms a gyrogroup under the operation inherited from G. If H is a subgyrogroup of G, then the *index* of H in G is defined as the number of distinct left cosets $a \oplus H$ in G, where $a \oplus H = \{a \oplus h \colon h \in H\}$. A subgyrogroup H of G is called an *L-subgyrogroup* if $\mathrm{gyr}\,[a, h](H) = H$ for all $a \in G, h \in H$. For each $g \in G$, the map L_g is defined as $L_g(x) = g \oplus x$ for all $x \in G$, called the *left gyrotranslation* by g.

Let G be a gyrogroup. A subset A of G is called a *generating set* for G if the smallest subgyrogroup of G containing A is equal to G, that is, if

$$G = \bigcap \{H : A \subseteq H \text{ and } H \text{ is a subgyrogroup of } G\}.$$

Suppose that $a_i \in G$ for all $i \in \mathbb{N}$. As in [11], we inductively define the following notation: $S_1^{\ell}(a_1) = a_1$, $S_1^{r}(a_1) = a_1$,

$$\begin{aligned} S_n^{\ell}(a_1, a_2, \ldots, a_n) &= a_1 \oplus S_{n-1}^{\ell}(a_2, a_3, \ldots, a_n), \\ S_n^{r}(a_1, a_2, \ldots, a_n) &= S_{n-1}^{r}(a_1, a_2, \ldots, a_{n-1}) \oplus a_n. \end{aligned}$$

Then, S_n^{ℓ} and S_n^{r} are maps from G^n to G for all $n \in \mathbb{N}$. For each nonempty subset A of G, define $\ominus A = \{\ominus a \colon a \in A\}$. Also, define $\ominus \emptyset = \emptyset$. For convenience, define ${\oplus}A = A \cup \ominus A$, called the *symmetric hull* of A. A subset A of G is called a *left generating set* for G if for each $g \in G$, there are $a_1, a_2, \ldots, a_n$ in ${\oplus}A$ such that $g = S_n^{\ell}(a_1, a_2, \ldots, a_n)$ [11]. A right generating set can be defined in a similar fashion. A subset A of G is *gyro-invariant* if $\mathrm{gyr}\,[a, b](A) = A$ for all $a, b \in G$.

Let G be a gyrogroup, and let S be a subset of G. As in Definition 2.8 of [12], the *right Cayley digraph* of G with respect to S is defined as the digraph with vertex set G and arc set given by $\{(g, g \oplus s) \colon s \in S\}$, denoted by $\overrightarrow{\text{R-Cay}}\,(G, S)$. The *right Cayley graph* of G with respect to S is defined as the underlying graph of $\overrightarrow{\text{R-Cay}}\,(G, S)$, denoted by $\text{R-Cay}\,(G, S)$. The basic theory of graphs and digraphs can be found in, for instance, [13]. We close this section with an important identity in a gyrogroup that corresponds to the familiar group identity $(gh)^{-1}(gk) = h^{-1}k$. This identity proves useful in Section 3. We also mention a few results on gyrogroups for reference.

Proposition 2.1. *Let G be a gyrogroup. Then,*

$$\ominus(g \oplus y) \oplus (g \oplus x) = \operatorname{gyr}[g, y](\ominus y \oplus x) \tag{2.1}$$

for all $g, x, y \in G$.

Proof. Using Theorems 2.32, 2.34, and 2.15 in [8], we obtain that

$$\begin{aligned}\ominus(g \oplus y) \oplus (g \oplus x) &= \operatorname{gyr}[g, y](\ominus y \ominus g) \oplus (g \oplus x)\\ &= \operatorname{gyr}[g, y]((\ominus y \ominus g) \oplus \operatorname{gyr}[y, g](g \oplus x))\\ &= \operatorname{gyr}[g, y]((\ominus y \ominus g) \oplus \operatorname{gyr}[\ominus y, \ominus g](g \oplus x))\\ &= \operatorname{gyr}[g, y](\ominus y \oplus x),\end{aligned}$$

as required.

Proposition 2.2 (Proposition 1, Section 3 of [11]). *If* φ *is a homomorphism from a gyrogroup G to a gyrogroup H, then*

$$\varphi(S^{\ell}_n(a_1, a_2, \dots, a_n)) = S^{\ell}_n(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n))$$

and

$$\varphi(S^{r}_n(a_1, a_2, \dots, a_n)) = S^{r}_n(\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n))$$

for all $n \in \mathbb{N}$ *and for all* $a_1, a_2, \dots, a_n \in G$.

Proposition 2.3 (Proposition 2, Section 3 of [11]). *Let G be a gyrogroup with a nonempty subset A. Suppose that* $\operatorname{gyr}[a, b](\oplus A) \subseteq \oplus A$ *for all* $a, b \in G$.

(1) *For all* $m, n \in \mathbb{N}$, *if* $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$ *are in* $\oplus A$, *then there are* $k_1, k_2, \dots, k_{m+n}$ *in* $\oplus A$ *such that*

$$S^{\ell}_m(a_1, a_2, \dots, a_m) \oplus S^{\ell}_n(b_1, b_2, \dots, b_n) = S^{\ell}_{m+n}(k_1, k_2, \dots, k_{m+n}).$$

(2) *For each* $n \in \mathbb{N}$, *if* $a_1, a_2, \dots, a_n$ *are in* $\oplus A$, *then there are* $k_1, k_2, \dots, k_n$ *in* $\oplus A$ *such that* $\ominus S^{\ell}_n(a_1, a_2, \dots, a_n) = S^{\ell}_n(k_1, k_2, \dots, k_n)$.

Proposition 2.4 (Proposition 3, Section 3 of [11]). *Let G be a gyrogroup with* $g \in G$, *and suppose that* $\emptyset \neq A \subseteq G$. *If* $\operatorname{gyr}[a, b](\oplus A) \subseteq \oplus A$ *for all* $a, b \in G$, *then for all* $a_1, a_2, \dots, a_n \in \oplus A$, *there are* $b_1, b_2, \dots, b_n$ *in* $\oplus A$ *such that* $S^{\ell}_{n+1}(g, a_1, a_2, \dots, a_n) = S^{r}_{n+1}(g, b_1, b_2, \dots, b_n)$.

3. Main results

In Section 3.2 of [9], the authors successfully extended the notion of a word metric, originally defined for finitely generated groups, to the setting of strongly generated gyrogroups. In this section, we build upon and refine that result. Specifically, we show that the existence of a left generating set whose symmetric hull is gyro-invariant allows for the definition of a metric analogous to the word metric on finitely generated groups. This naturally motivates the study of right Cayley graphs as a means of visualizing the structure of gyrogroups. We refer the reader to [12] for previous work on Cayley (di)graphs of gyrogroups. Suppose that S is a left generating set for a gyrogroup G such that $\oplus S$ is gyro-invariant. For all $x, y \in G$, define

$$d_S^w(x, y) = \begin{cases} 0 & \text{if } x = y; \\ \min\{n \in \mathbb{N} \colon \ominus y \oplus x = S_n^\ell(a_1, a_2, \ldots, a_n), a_i \in \oplus S\} & \text{if } x \neq y. \end{cases} \tag{3.1}$$

Note that if $x, y \in G$ and $x \neq y$, then $\ominus y \oplus x = S_n^\ell(a_1, a_2, \ldots, a_n)$ for some $a_1, a_2, \ldots, a_n$ in $\oplus S$, as S is a left generating set for G. Hence, $d_S^w(x, y)$ is meaningful by the well-ordering principle. We remark that d_S^w depends on the set S. Furthermore, we obtain the following theorem, whose proof strongly relies on Propositions 2.2 and 2.3.

Theorem 3.1. *Suppose that S is a left generating set for a gyrogroup G such that $\oplus S$ is gyro-invariant. Then, d_S^w is a metric on G so that (G, d_S^w) forms a metric space.*

Proof. Clearly, $d_S^w(x, y) \geq 0$ for all $x, y \in G$, and $d_S^w(x, x) = 0$ for all $x \in G$. Now, suppose that $d_S^w(x, y) = 0$. If $x \neq y$, then $d_S^w(x, y)$ would be a positive integer, a contradiction. Hence, $x = y$.

Let $x, y \in G$. If $x = y$, then $d_S^w(x, y) = 0 = d_S^w(y, x)$. Therefore, assume that $x \neq y$. There are $a_1, a_2, \ldots, a_n$ in $\oplus S$ for which $\ominus y \oplus x = S_n^\ell(a_1, a_2, \ldots, a_n)$ and $n = d_S^w(x, y)$. By part 2 of Proposition 2.3, there are $k_1, k_2, \ldots, k_n$ in $\oplus S$ such that $\ominus(\ominus y \oplus x) = \ominus S_n^\ell(a_1, a_2, \ldots, a_n) = S_n^\ell(k_1, k_2, \ldots, k_n)$. Hence,

$$\operatorname{gyr}[\ominus y, x](\ominus x \oplus y) = S_n^\ell(k_1, k_2, \ldots, k_n).$$

It follows from Proposition 2.2 that

$$\begin{aligned} \ominus x \oplus y &= \operatorname{gyr}[x, \ominus y](S_n^\ell(k_1, k_2, \ldots, k_n)) \\ &= S_n^\ell(\operatorname{gyr}[x, \ominus y](k_1), \operatorname{gyr}[x, \ominus y](k_2), \ldots, \operatorname{gyr}[x, \ominus y](k_n)). \end{aligned}$$

Because $\oplus S$ is gyro-invariant, $\operatorname{gyr}[x, \ominus y](k_1)$, $\operatorname{gyr}[x, \ominus y](k_2), \ldots, \operatorname{gyr}[x, \ominus y](k_n)$ belong to $\oplus S$. Thus, $d_S^w(y, x) \leq n = d_S^w(x, y)$ by minimality of $d_S^w(y, x)$. This, in fact, proves that $d_S^w(u, v) \leq d_S^w(v, u)$ for all $u, v \in G$. Hence, $d_S^w(x, y) \leq d_S^w(y, x)$, and so equality holds.

Assume that $x \neq y$ and $y \neq z$. Then, there are $a_1, a_2, \ldots, a_m$ in $\oplus S$ for which $\ominus y \oplus x = S_m^\ell(a_1, a_2, \ldots, a_m)$ and $m = d_S^w(x, y)$. In a similar fashion, one finds $b_1, b_2, \ldots, b_n$ in $\oplus S$ such that $\ominus z \oplus y = S_n^\ell(b_1, b_2, \ldots, b_n)$ and $n = d_S^w(y, z)$. We obtain that

$$\begin{aligned} \ominus z \oplus x &= (\ominus z \oplus y) \oplus \operatorname{gyr}[\ominus z, y](\ominus y \oplus x) \\ &= S_n^\ell(b_1, b_2, \ldots, b_n) \oplus \operatorname{gyr}[\ominus z, y](S_m^\ell(a_1, a_2, \ldots, a_m)) \\ &= S_n^\ell(b_1, b_2, \ldots, b_n) \oplus S_m^\ell(\operatorname{gyr}[\ominus z, y](a_1), \operatorname{gyr}[\ominus z, y](a_2), \ldots, \operatorname{gyr}[\ominus z, y](a_m)), \end{aligned}$$

noting that the first equality follows from Theorem 2.15 of [8], and the third equality follows from Proposition 2.2. By part 2 of Proposition 2.3, there are $k_1, k_2, \ldots, k_{m+n}$ in $\oplus S$ such that

$$\begin{aligned} &S_n^\ell(b_1, b_2, \ldots, b_n) \oplus S_m^\ell(\operatorname{gyr}[\ominus z, y](a_1), \operatorname{gyr}[\ominus z, y](a_2), \ldots, \operatorname{gyr}[\ominus z, y](a_m)) \\ &= S_{m+n}^\ell(k_1, k_2, \ldots, k_{m+n}). \end{aligned}$$

It follows that $d_S^w(x, z) \leq m + n = d_S^w(x, y) + d_S^w(y, z)$ by minimality of $d_S^w(x, z)$.

3.1. An analog of the Mazur–Ulam theorem for gyrogroups

The Mazur–Ulam theorem states that every surjective distance-preserving mapping between real normed vector spaces is an affine transformation, and in particular any distance-preserving mapping that sends the origin to the origin is linear. A gyrogroup version of the Mazur–Ulam theorem and related work can be found in, for instance, [14–16]. One of the main goals of this subsection is to prove an analog of the Mazur–Ulam theorem for gyrogroups endowed with their generalized word metric. We begin with proving fundamental properties of the metric mentioned in Theorem 3.1. To verify that d^w_S can be induced by a gyronorm (see Definition 2 in [17]) in the standard way, we first prove the following theorem, which is important in its own right.

Theorem 3.2. *Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant.*

(1) *Then, L_g is an isometry of (G, d^w_S) for all $g \in G$.*

(2) *If ϕ is an automorphism of G such that $\phi(\oplus S) = \oplus S$, then ϕ is an isometry of (G, d^w_S).*

Proof. Let $g \in G$. By Theorem 10 of [18], L_g is a bijection from G to G. Let $x, y \in G$. Assume that $x \neq y$. This implies that $L_g(x) \neq L_g(y)$. Then, there are $k_1, k_2, \ldots, k_n$ in $\oplus S$ such that $\ominus y \oplus x = S^{\ell}_n(k_1, k_2, \ldots, k_n)$. By Propositions 2.1 and 2.2,

$$\begin{aligned}\ominus L_g(y) \oplus L_g(x) &= \ominus(g \oplus y) \oplus (g \oplus x)\\ &= \operatorname{gyr}[g, y](\ominus y \oplus x)\\ &= \operatorname{gyr}[g, y](S^{\ell}_n(k_1, k_2, \ldots, k_n))\\ &= S^{\ell}_n(\operatorname{gyr}[g, y](k_1), \operatorname{gyr}[g, y](k_2), \ldots, \operatorname{gyr}[g, y](k_n)).\end{aligned}$$

Because $\operatorname{gyr}[g, y](k_1), \operatorname{gyr}[g, y](k_2), \ldots, \operatorname{gyr}[g, y](k_n)$ belong to $\oplus S$, it follows that

$$d^w_S(L_g(x), L_g(y)) \leq n = d^w_S(x, y)$$

by minimality of $d^w_S(L_g(x), L_g(y))$. This, in fact, proves that $d^w_S(L_h(u), L_h(v)) \leq d^w_S(u, v)$ for all $h, u, v \in G$. Hence, $d^w_S(x, y) = d^w_S(L_{\ominus g}(L_g(x)), L_{\ominus g}(L_g(y))) \leq d^w_S(L_g(x), L_g(y))$, and so equality holds. This proves part 3.2.

Let $x, y \in G$. Assume that $x \neq y$. Then, there are $k_1, k_2, \ldots, k_n$ in $\oplus S$ such that $\ominus y \oplus x = S^{\ell}_n(k_1, k_2, \ldots, k_n)$. Because ϕ preserves the gyrogroup operation,

$$\ominus\phi(y) \oplus \phi(x) = \phi(\ominus y \oplus x) = \phi(S^{\ell}_n(k_1, k_2, \ldots, k_n)) = S^{\ell}_n(\phi(k_1), \phi(k_2), \ldots, \phi(k_n)),$$

and so $d^w_S(\phi(x), \phi(y)) \leq n = d^w_S(x, y)$. Because ϕ^{-1} is also an automorphism of G such that $\phi^{-1}(\oplus S) = \oplus S$, it follows that $d^w_S(x, y) = d^w_S(\phi^{-1}(\phi(x)), \phi^{-1}(\phi(y))) \leq d^w_S(\phi(x), \phi(y))$, and so equality holds. This proves part 3.2.

In light of part 3.2 of Theorem 3.2, the metric d^w_S is left invariant under the gyrogroup operation, but it need not be right invariant. Moreover, as suggested by Theorem 11 of [17], neither gyroaddition nor negation is necessarily continuous with respect to the topology induced by d^w_S.

Theorem 3.3. *Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant. Then, the function $\|\cdot\|_S^w$ defined by the formula*

$$\|x\|_S^w = d_S^w(e, x) \tag{3.2}$$

for all $x \in G$ is a gyronorm on G that induces the same metric d_S^w.

Proof. The theorem follows directly from Theorems 3.1 and 3.2, together with Theorem 9 in [17].

Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant. In light of Theorem 3.3, we obtain that

(1) $\|x\|_S^w \geq 0$ for all $x \in G$, and $\|x\|_S^w = 0$ if and only if $x = e$;

(2) $\|\ominus x\|_S^w = \|x\|_S^w$ for all $x \in G$;

(3) $\|x \oplus y\|_S^w \leq \|x\|_S^w + \|y\|_S^w$ for all $x, y \in G$; and

(4) $\|\mathrm{gyr}\,[a, b](x)\|_S^w = \|x\|_S^w$ for all $a, b, x \in G$.

Furthermore, in light of Theorem 3.3, every finite gyrogroup can be viewed as a normed gyrogroup in a nontrivial way. As we will see shortly, it is reasonable to refer to the metric d_S^w as a word metric. Let $n \in \mathbb{N}$. The element $S_n^\ell(h_1, h_2, \ldots, h_n)$ with $h_1, h_2, \ldots, h_n$ in $\oplus S$ is called a *left-minimal word* of length n with respect to S if $S_n^\ell(h_1, h_2, \ldots, h_n) \neq S_m^\ell(k_1, k_2, \ldots, k_m)$ for all $k_1, k_2, \ldots, k_m \in \oplus S$ and for all $m < n$. The identity of G is defined to be a left-minimal word of length zero with respect to S. Conversely, any left-minimal word of length zero is the identity. By the well-ordering principle, every element of G can be expressed as a left-minimal word with respect to S. The following lemma justifies the use of the term “left-minimal”.

Lemma 3.4. *Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant. Let $h_1, h_2, \ldots, h_n \in \oplus S$. Then, $S_n^\ell(h_1, h_2, \ldots, h_n)$ is a left-minimal word of length n with respect to S if and only if $\|S_n^\ell(h_1, h_2, \ldots, h_n)\|_S^w = n$.*

Proof. The case when $n = 0$ is clear. Therefore, assume that $v = S_n^\ell(h_1, h_2, \ldots, h_n)$ is a left-minimal word of length $n > 0$ with respect to S. Note that $v \neq e$. Because $\ominus e \oplus v = S_n^\ell(h_1, h_2, \ldots, h_n)$, it follows that $\|v\|_S^w = d_S^w(e, v) = d_S^w(v, e) \leq n$. Assume to the contrary that $\|v\|_S^w = m < n$. Thus, $v = S_m^\ell(k_1, k_2, \ldots, k_m)$ for some $k_1, k_2, \ldots, k_m \in \oplus S$, which is a contradiction because v is a left-minimal word of length n with respect to S. Hence, $\|v\|_S^w = n$ as claimed.

Next, suppose that $\|S_n^\ell(h_1, h_2, \ldots, h_n)\|_S^w = n > 0$. Set $v = S_n^\ell(h_1, h_2, \ldots, h_n)$. Let $m < n$. Assume to the contrary that there are $k_1, k_2, \ldots, k_m \in \oplus S$ such that $v = S_m^\ell(k_1, k_2, \ldots, k_m)$. From the minimality of $d_S^w(v, e)$, we obtain that $d_S^w(v, e) \leq m < n$, a contradiction. Therefore, v is a left-minimal word of length n with respect to S.

Using Lemma 3.4, one can prove the following proposition, which gives a characterization of a permutation of G preserving the gyronorm $\|\cdot\|_S^w$.

Proposition 3.5. *Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant. Suppose that ρ is a permutation of G. Then, the following statements are equivalent:*

(I) $\|\rho(x)\|_S^w = \|x\|_S^w$ *for all $x \in G$;*

(II) *for all integers $n \geq 0$, ρ sends a left-minimal word of length n with respect to S to a left-minimal word of length n with respect to S.*

It turns out that any isometry of (G, d_S^w), which preserves the gyrogroup identity, also preserves the corresponding gyronorm, as shown in the following proposition. In contrast, any nonidentity left gyrotranslation is an isometry of (G, d_S^w) that does not preserve the corresponding gyronorm because if $g \neq e$, then $\|e\|_S^w = 0$, whereas $\|L_g(e)\|_S^w = \|g\|_S^w > 0$.

Proposition 3.6. *Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant. If ρ is an isometry of (G, d_S^w) such that $\rho(e) = e$, then $\|\rho(x)\|_S^w = \|x\|_S^w$ for all $x \in G$.*

Proof. Let $x \in G$. Then, $\|\rho(x)\|_S^w = d_S^w(e, \rho(x)) = d_S^w(\rho(e), \rho(x)) = d_S^w(e, x) = \|x\|_S^w$, which completes the proof.

Example 3.7. In Example 8 of [10], a finite gyrogroup of order 15, denoted by G_{15}, is presented, with its gyroaddition and gyration tables given explicitly in Tables 6 and 7 of [10]. In this example, we point out that the converse of Proposition 3.6 is not, in general, true. Consider the gyrogroup G_{15}, and let $T = \{1, 4\}$. Then, T is a left generating set for G_{15}. Furthermore, $S = \{1, 4, 5, 6, 7, 10\}$ is a gyro-invariant left generating set for G_{15}. As $\ominus S = \{2, 3, 8, 9, 11, 14\}$, it follows that

$$
\begin{aligned}
\{g \in G_{15} \colon \|g\|_S^w = 0\} &= \{0\},\\
\{g \in G_{15} \colon \|g\|_S^w = 1\} &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14\},\\
\{g \in G_{15} \colon \|g\|_S^w = 2\} &= \{12, 13\},
\end{aligned}
$$

because $12 = 1 \oplus 11$ and $13 = 1 \oplus 8$. Define a map ρ by

$$
\rho(i) = \begin{cases} i & \text{if } i \in G_{15} \setminus \{2, 5\};\\ 2 & \text{if } i = 5;\\ 5 & \text{if } i = 2. \end{cases}
$$

Because $\|2\|_S^w = \|5\|_S^w$, it follows that $\|\rho(i)\|_S^w = \|i\|_S^w$ for all $i \in G_{15}$. Hence, ρ is a gyronorm-preserving bijection from G_{15} to G_{15}. However, ρ is not an isometry of (G_{15}, d_S^w) because $d_S^w(1, 2) = 1$, whereas $d_S^w(\rho(1), \rho(2)) = d_S^w(1, 5) = 2$.

In view of Proposition 3.6, we obtain the following theorem that describes factorization of an isometry of (G, d_S^w), analogous to the well-known Mazur–Ulam theorem for normed linear spaces.

Theorem 3.8. *Let G be a gyrogroup with a left generating set S such that $\oplus S$ is gyro-invariant. Then, every isometry σ of (G, d_S^w) can be factored as $L_a \circ \rho$, where $a \in G$, and ρ is a gyronorm-preserving isometry.*

Proof. By Theorem 11 of [18], $\sigma = L_a \circ \rho$, where $\rho \in \mathrm{Sym}(G)$ with $\rho(e) = e$. By part 3.2 of Theorem 3.2, L_a is an isometry of (G, d_S^w). Hence, $\rho = L_a^{-1} \circ \sigma$ is an isometry of (G, d_S^w). By Proposition 3.6, $\|\rho(x)\|_S^w = \|x\|_S^w$ for all $x \in G$. Thus, ρ is a gyronorm-preserving isometry.

3.2. *Cayley graphs of gyrogroups and their generalized word metrics*

In this section, we show that the right Cayley graph R-Cay $(G, {\oplus}S)$ is a combinatorial object that naturally represents the space (G, d_S^w), where G is a finite gyrogroup, and S is a left generating set for G whose symmetric hull is gyro-invariant. We begin by proving the following lemma, which provides a characterization of an edge in a right Cayley graph of a gyrogroup.

Lemma 3.9. *Let G be a gyrogroup with $\emptyset \neq S \subseteq G$. Then, $\{x, y\}$ is an edge in the right Cayley graph* R-Cay (G, S) *if and only if there is $s \in S$ for which $y = x \oplus s$ or $y = x \boxminus s$.*

Proof. Suppose that $\{x, y\}$ is an edge in R-Cay (G, S). In the case when (x, y) is an arc in R-$\overrightarrow{\text{Cay}}\,(G, S)$, we obtain that $y = x \oplus s$ for some $s \in S$. In the case when (y, x) is an arc in R-$\overrightarrow{\text{Cay}}\,(G, S)$, we obtain that $x = y \oplus s$ for some $s \in S$. Applying a right cancellation law in a gyrogroup yields $y = x \oplus \text{gyr}\,[x, s](\ominus s) = x \boxminus s$ as required.

To prove the converse, let $x, y \in G$. Suppose that $y = x \oplus s$ or $y = x \oplus \text{gyr}\,[x, s](\ominus s)$ for some $s \in S$. In the former case, we obtain that (x, y) is an arc in R-$\overrightarrow{\text{Cay}}\,(G, S)$. In the latter case, we obtain that $y = x \oplus \text{gyr}\,[x, s](\ominus s) = x \boxplus (\ominus s)$, and so $x = y \oplus s$ by a right cancellation law. Thus, (y, x) is an arc in R-$\overrightarrow{\text{Cay}}\,(G, S)$.

We are now in a position to characterize a left generating set via a right Cayley graph, under the condition of gyro-invariance, as follows. This result refines Theorem 4.8 in [19].

Theorem 3.10. *Let G be a gyrogroup with $S \subseteq G$. If ${\oplus}S$ is gyro-invariant, then the following statements are equivalent:*

(1) *The right Cayley graph* R-Cay $(G, {\oplus}S)$ *is connected;*

(2) *S is a right generating set for G;*

(3) *S is a left generating set for G.*

Proof. The theorem holds trivially when $G = \{e\}$ or $S = \emptyset$. Therefore, assume that $G \neq \{e\}$ and $S \neq \emptyset$.

To prove the implication (3.10) ⇒ (3.10), suppose that $g \in G$. In the case when $g = e$, we write $g = \ominus s \oplus s = S_2^r(\ominus s, s)$ for some $s \in S$. Assume that $g \neq e$. By assumption, there is a path from e to g, say, $e = g_0, g_1, g_2, \dots, g_{n-1}, g_n = g$. By Lemma 3.9, $g_i = g_{i-1} \oplus k_i$ with k_i in ${\oplus}S$ for $i = 1, 2, \dots, n$. By substitution, $g = g_n = S_n^r(k_1, k_2, \dots, k_n)$. This shows that S is a right generating set for G.

The implication (3.10) ⇒ (3.10) follows directly from Corollary 1 in Section 3 of [11].

To prove the implication (3.10) ⇒ (3.10), let g and h be distinct elements of G. By assumption $\ominus g \oplus h = S_n^\ell(k_1, k_2, \dots, k_n)$ for some $k_1, k_2, \dots, k_n \in {\oplus}S$. By a left cancellation law in a gyrogroup, $h = g \oplus S_n^\ell(k_1, k_2, \dots, k_n) = S_{n+1}^\ell(g, k_1, k_2, \dots, k_n)$. By Proposition 2.4, $S_{n+1}^\ell(g, k_1, k_2, \dots, k_n) = S_{n+1}^r(g, k_1', k_2', \dots, k_n')$ for some $k_1', k_2', \dots, k_n' \in {\oplus}S$. Define $g_0 = g$, and for $i = 1, 2, \dots, n$, define $g_i = g_{i-1} \oplus k_i'$. By Lemma 3.9, $\{g_{i-1}, g_i\}$ is an edge in R-Cay $(G, {\oplus}S)$ for $i = 1, 2, \dots, n$. Because $g_n = h$, it follows that $g = g_0, g_1, g_2, \dots, g_{n-1}, g_n = h$ contains a path from g to h.

It is well-known in geometric group theory that if d_W is the word metric defined on a finitely generated group (Γ, S), then $d_W(g, h)$ is equal to the length of the shortest path from g to h in Cay (Γ, S) and is equal to the minimum number of generators from S needed to express $g^{-1}h$. We generalize this result to the setting of gyrogroups as follows.

Theorem 3.11. *Suppose that S is a left generating set for a nontrivial gyrogroup G such that $\oplus S$ is gyro-invariant. Then, the distance between two arbitrary points g and h in G measured by the metric d^w_S coincides with the shortest length of a path joining vertices g and h in* R-Cay $(G, \oplus S)$.

Proof. Let $g, h \in G$ with $g \neq h$. By Theorem 3.10, R-Cay $(G, \oplus S)$ is connected. Hence, the shortest length of a path joining vertices g and h in R-Cay $(G, \oplus S)$ exists, say, m. Suppose that $d^w_S(h, g) = n$. Then, $\ominus g \oplus h = S^\ell_n(k_1, k_2, \dots, k_n)$ with k_i in $\oplus S$, and $\ominus g \oplus h \neq S^\ell_t(a_1, a_2, \dots, a_t)$ for all $t < n$ and for all $a_i \in \oplus S$. As in the proof of Theorem 3.10,

$$g = g_0, g_1, g_2, \dots, g_{n-1}, g_n = h$$

contains a path from g to h. Hence, $m \leq n$ by minimality of m. Suppose by contradiction that $m < n$. Let $g = h_1, h_2, \dots, h_m = h$ be a path from g to h. Note that $m \geq 2$ because $g \neq h$. Furthermore, $h_i = h_{i-1} \oplus k'_i$ with k'_i in $\oplus S$ for $i = 2, 3, \dots, m$. By substitution, $h = h_m = S^r_m(g, k'_2, k'_3, \dots, k'_m)$. By Proposition 2.4, $S^r_m(g, k'_2, k'_3, \dots, k'_m) = S^\ell_m(g, k''_2, k''_3, \dots, k''_m)$ for some $k''_2, k''_3, \dots, k''_m \in \oplus S$. Hence, $h = g \oplus S^\ell_{m-1}(k''_2, k''_3, \dots, k''_m)$, which implies that $\ominus g \oplus h = S^\ell_{m-1}(k''_2, k''_3, \dots, k''_m)$, contrary to the minimality of n. This shows that $m = n$, and the theorem follows.

In light of Theorem 3.11, the family of isometries of the space (G, d^w_S) coincides with the family of graph automorphisms of R-Cay $(G, \oplus S)$, as shown below.

Theorem 3.12. *Suppose that S is a left generating set for a nontrivial gyrogroup G such that $\oplus S$ is gyro-invariant. Then, σ is an isometry of (G, d^w_S) if and only if σ is a graph automorphism of* R-Cay $(G, \oplus S)$.

Proof. The forward implication follows from Theorem 3.11 and the observation that an edge corresponds to a path of length one between two vertices. Suppose conversely that σ is an automorphism of R-Cay $(G, \oplus S)$. Let $x, y \in G$ with $x \neq y$. Assume that $d^w_S(x, y) = n$. Hence, $n > 0$, and there is a path $x = g_0, g_1, \dots, g_n = y$. Hence, $\{g_{i-1}, g_i\}$ is an edge in R-Cay $(G, \oplus S)$ for $i = 1, 2, \dots, n$. By assumption, $\{\sigma(g_{i-1}), \sigma(g_i)\}$ is an edge in R-Cay $(G, \oplus S)$ for $i = 1, 2, \dots, n$, and so $\sigma(x) = \sigma(g_0), \sigma(g_1), \dots, \sigma(g_n) = \sigma(y)$ contains a path from $\sigma(x)$ to $\sigma(y)$ of length at most n. This implies that $d^w_S(\sigma(x), \sigma(y)) \leq n = d^w_S(x, y)$. Because σ^{-1} is also an automorphism of R-Cay $(G, \oplus S)$, it follows that $d^w_S(x, y) = d^w_S(\sigma^{-1}(\sigma(x)), \sigma^{-1}(\sigma(y))) \leq d^w_S(\sigma(x), \sigma(y))$, and so equality holds.

It turns out that the right Cayley graph R-Cay $(G, \oplus S)$ is vertex-transitive whenever S is a left generating set for a gyrogroup G whose symmetric hull is gyro-invariant. We obtain this fact as a consequence of Theorems 3.2 and 3.12.

Theorem 3.13. *Suppose that S is a left generating set for a nontrivial gyrogroup G such that $\oplus S$ is gyro-invariant. Then,* R-Cay $(G, \oplus S)$ *is vertex-transitive.*

Proof. Let $u, v \in G$. By Lemma 2.19 of [8], there is $g \in G$ for which $g \oplus u = v$. In fact, $g = v \boxplus (\ominus u) = v \oplus \mathrm{gyr}\,[v, u](\ominus u)$. By part 3.2 of Theorem 3.2, L_g is an isometry of (G, d^w_S). Furthermore L_g sends u to v because $L_g(u) = g \oplus u = v$. By Theorem 3.12, L_g is an automorphism of R-Cay $(G, \oplus S)$, and the proof completes.

The following proposition presents the metric space analog of Theorem 3.13.

Proposition 3.14. *Suppose that S is a left generating set for a nontrivial gyrogroup G such that $\oplus S$ is gyro-invariant. Then, (G, d_S^w) is isometrically homogeneous.*

Proof. As in the proof of Theorem 3.13, for all points u, v in G, $L_{v \oplus \mathrm{gyr}\,[v,u](\ominus u)}$ is an isometry of (G, d_S^w) that sends u to v.

We now derive several consequences from the fact that R-Cay $(G, \oplus S)$ is vertex-transitive.

Proposition 3.15. *Suppose that S is a left generating set for a nontrivial finite gyrogroup G such that $\oplus S$ is gyro-invariant. Then,* R-Cay $(G, \oplus S)$ *is $|\oplus S|$-regular.*

Proof. By Theorem 3.13, R-Cay $(G, \oplus S)$ is vertex-transitive. Hence, by Lemma 1.3.1 of [13], every vertex of R-Cay $(G, \oplus S)$ has the same degree. Hence, R-Cay $(G, \oplus S)$ is regular. Moreover, by a left cancellation law in a gyrogroup, $g \oplus k = g \oplus k'$ if and only if $k = k'$ for all $g \in G, k \in \oplus S$. Hence, any vertex in R-Cay $(G, \oplus S)$ is of degree $|\oplus S|$.

According to Proposition 3.15, all unit spheres in (G, d_S^w) are equinumerous, as shown in the following corollary. In fact, this corollary is the metric space analog of Proposition 3.15.

Corollary 3.16. *Suppose that S is a left generating set for a nontrivial finite gyrogroup G such that $\oplus S$ is gyro-invariant. For all $x_0 \in G$, $|\{x \in G \colon d_S^w(x_0, x) = 1\}| = |\oplus S|$.*

Proof. By Proposition 3.15, each vertex x_0 in R-Cay $(G, \oplus S)$ has $|\oplus S|$ neighbors, and these neighbors have a distance of one from x_0.

Corollary 3.17. *Suppose that S is a left generating set for a nontrivial finite gyrogroup G such that $\oplus S$ is gyro-invariant. Then, the edge-connectivity of* R-Cay $(G, \oplus S)$ *equals $|\oplus S|$.*

Proof. The corollary follows from Proposition 3.15 and Lemma 3.3.3 of [13].

Corollary 3.18. *Suppose that S is a left generating set for a nontrivial finite gyrogroup G such that $\oplus S$ is gyro-invariant. Then, the vertex-connectivity of* R-Cay $(G, \oplus S)$ *is greater than or equal to* $2/3(|\oplus S| + 1)$.

Proof. The corollary follows from Proposition 3.15 and Theorem 3.4.2 of [13].

It is known in algebraic graph theory that a Cayley graph Cay (Γ, S), where Γ is a finite group and S is a symmetric generating set for Γ not containing the identity of Γ, is bipartite if and only if Γ has a subgroup of index 2 that is disjoint from S. See, for instance, Lemma 4 of [20]. We extend this result to the setting of gyrogroups in the following theorem. In particular, this theorem relates a graph-theoretic property of gyrogroups to an algebraic property of gyrogroups.

Theorem 3.19. *Suppose that S is a left generating set for a nontrivial finite gyrogroup G not containing e such that $\oplus S$ is gyro-invariant. Then,* R-Cay $(G, \oplus S)$ *is bipartite if and only if G has an L-subgyrogroup of index* 2 *that is disjoint from $\oplus S$.*

Proof. Suppose that R-Cay $(G, \oplus S)$ is bipartite. Hence, there are disjoint subsets A and B of G such that $G = A \cup B$, and every edge connects a vertex in A to a vertex in B. Without loss of generality, assume that $e \in A$. This implies that $\oplus S \subseteq B$ because $\{e, k\}$ is an edge for all $k \in \oplus S$. Set $g = S_n^\ell(k_1, k_2, \ldots, k_n)$ with k_i in $\oplus S$. We claim that if n is even, then $g \in A$. By Proposition 2.4, $g = S_n^r(k_1', k_2', \ldots, k_n')$

with k'_i in $\oplus S$. Note that $k'_1 \in B$ implies $k'_1 \oplus k'_2 \in A$ because $\{k'_1, k'_1 \oplus k'_2\}$ is an edge. This in turn implies that $(k'_1 \oplus k'_2) \oplus k'_3 \in B$. Hence, $((k'_1 \oplus k'_2) \oplus k'_3) \oplus k'_4 \in A$. Continuing this process shows that $S^r_n(k'_1, k'_2, \ldots, k'_n) \in A$. This also implies that if n is odd, then g lies in B. As a consequence of this fact, if $S^\ell_n(k_1, k_2, \ldots, k_n) = S^\ell_m(k'_1, k'_2, \ldots, k'_m)$ with k_i, k'_i in $\oplus S$, then n and m have the same parity. Hence, A consists precisely of elements of the form $S^\ell_n(k_1, k_2, \ldots, k_n)$, where n is even and $k_i \in \oplus S$; and B consists precisely of elements of the form $S^\ell_n(k_1, k_2, \ldots, k_n)$, where n is odd and $k_i \in \oplus S$. By Proposition 2.3, A forms a subgyrogroup of G. Furthermore, $B = g \oplus A$ for all $g \in G \setminus A$. Hence, A partitions G into disjoint left cosets A and $g \oplus A$ for some $g \in B$. It follows that A forms an L-subgyrogroup of G of index 2. Because $\oplus S \subseteq B$, we obtain that $A \cap \oplus S = \emptyset$.

($\Leftarrow$) Suppose that H is an L-subgyrogroup of G of index 2 that is disjoint from $\oplus S$. Hence, there is $g \in G$ for which $(g \oplus H) \cap H = \emptyset$ and $G = H \cup (g \oplus H)$. Note that $g \notin H$. Let $u_1, u_2 \in H$ with $u_1 \neq u_2$, and let $v_1, v_2 \in g \oplus H$ with $v_1 \neq v_2$. Assume to the contrary that $\{u_1, u_2\}$ is an edge in R-Cay$\,(G, \oplus S)$. Hence, there is $k \in \oplus S$ for which $u_2 = u_1 \oplus k$ or $u_2 = u_1 \oplus \mathrm{gyr}\,[u_1, k](\ominus k)$. In the former case, $k = \ominus u_1 \oplus u_2 \in H$, a contradiction. In the latter case, $k = \ominus \mathrm{gyr}\,[k, u_1](\ominus u_1 \oplus u_2) \in H$ (because H is an L-subgyrogroup), a contradiction. Next, assume to the contrary that $\{v_1, v_2\}$ is an edge in R-Cay$\,(G, \oplus S)$. Note that $v_1 = g \oplus u'_1$ and $v_2 = g \oplus u'_2$ for some $u'_1, u'_2 \in H$. By part 3.2 of Theorem 3.2 and by Theorem 3.12, L_g is an automorphism of R-Cay$\,(G, \oplus S)$ so that $\{u'_1, u'_2\}$ is an edge, a contradiction. Now, assume that $\{u, v\}$ is an edge. Then, as shown above, u and v cannot both lie in H, nor can they both lie in $g \oplus H$. This shows that H and $g \oplus H$ are partite sets for G.

In Example 1 of [10], a finite gyrogroup of order 8, denoted by G_8, is presented, with its gyroaddition and gyration tables given explicitly in Tables 3 and 4 of [10]. For example, the right Cayley graph of G_8 with respect to $\{1, 2, 4, 6\}$ (that is, $S = \{1, 4, 6\}$) is depicted in Figure 1, which is the complete bipartite graph $K_{4,4}$. Hence, G_8 has an L-subgyrogroup of index 2, which is $\{0, 3, 5, 7\}$.

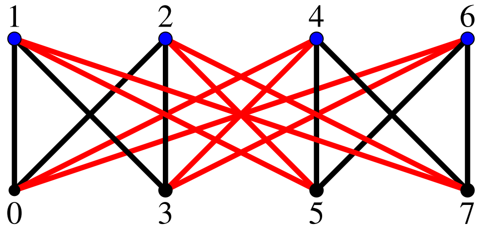

Figure 1. The right Cayley graph of G_8 with respect to $\{1, 2, 4, 6\}$.

It can be proved that an L-subgyrogroup of a gyrogroup G of index 2 must be normal in G. Moreover, every normal subgyrogroup is an L-subgyrogroup. Hence, Theorem 3.19 implies that if S is a left generating set for a nontrivial finite gyrogroup G not containing e such that $\oplus S$ is gyro-invariant, then R-Cay$\,(G, \oplus S)$ is bipartite if and only if G has a normal subgyrogroup of index 2 that is disjoint from $\oplus S$. We demonstrate the usefulness of Theorem 3.19 in the following example.

Example 3.20. In Example 18 of [18], a finite gyrogroup of order 16, denoted by K_{16}, is presented, with its gyroaddition and gyration tables given explicitly in Tables 1 and 2 of [18]. In the gyrogroup K_{16}, $S = \{4, 8, 9\}$ is a left generating set for K_{16}. Moreover, $\oplus S = \{4, 6, 8, 9\}$ is gyro-invariant. The right Cayley graph of K_{16} with respect to $\{4, 6, 8, 9\}$ is depicted in Figure 2. Note that R-Cay$\,(K_{16}, \oplus S)$ is bipartite, as it is 2-colorable. By Theorem 3.19, K_{16} has an L-subgyrogroup of index 2 not containing $\{4, 6, 8, 9\}$, which is $\{0, 1, 2, 3, 12, 13, 14, 15\}$.

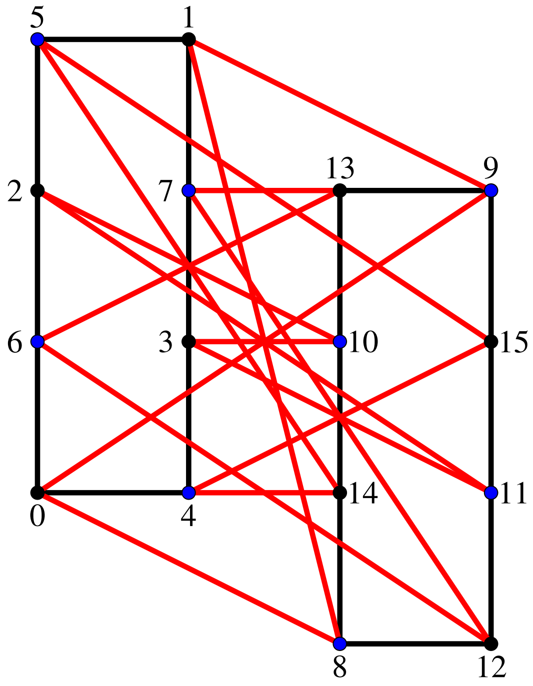

Figure 2. The right Cayley graph of K_{16} with respect to $\{4, 6, 8, 9\}$.

4. Conclusions

In this work, we developed a generalized word metric, arising from left generating sets with gyro-invariant symmetric hulls, on gyrogroups, thereby extending a fundamental geometric concept from group theory to the nonassociative setting of gyrogroups. This framework enabled us to establish an analog of the Mazur–Ulam theorem, showing that every isometry of a gyrogroup equipped with the generalized word metric factors into a left gyrotranslation and a gyronorm-preserving isometry. Furthermore, we demonstrated that the metric geometry of such gyrogroups is reflected in their right Cayley graphs, whose structural properties, vertex-transitivity and bipartiteness, correspond directly to algebraic characteristics of the underlying gyrogroups. These results provide new tools for studying the interplay between algebraic and geometric features of gyrogroups.

Use of AI tools declaration

The author declares that Artificial Intelligence (AI) tools were used in the preparation of this manuscript only for language editing and grammar correction. The author takes full responsibility for the content, interpretation, and conclusions of this work.

Acknowledgments

The author sincerely thanks the reviewers for their careful reading of the manuscript and for their insightful comments and suggestions, which have significantly improved the clarity and overall quality of the article. The author would like to thank Watchareepan Atiponrat and Jaturon Wattanapan for their collaboration and their fruitful discussions. This research project was supported by Fundamental Fund 2026 (No. 214486), Chiang Mai University and also Thailand Science Research and Innovation (TSRI) (FRB690042/0162).

Conflict of interest

The author declares no conflict of interest.

References

1. C. Löh, *Geometric Group Theory: An Introduction*, Springer, Cham, 2017.
2. P. Patel, On the residual finiteness growths of particular hyperbolic manifold groups, *Geom. Dedicata*, **185** (2016), 87–103. https://doi.org/10.1007/s10711-016-0169-x
3. M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer, Berlin, 1999.
4. W. Lück, Survey on geometric group theory, *Münster J. Math.*, **1** (2008), 73–108, 2008.
5. S. Kim, J. Lawson, Unit balls, Lorentz boosts, and hyperbolic geometry, *Results Math.*, **63** (2013), 1225–1242. https://doi.org/10.1007/s00025-012-0265-7
6. A. Ungar, *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*, Springer, 2001. https://doi.org/10.1007/0-306-47134-5_11
7. A. Ungar, Einstein's velocity addition law and its hyperbolic geometry, *Comput. Math. Appl.*, **53** (2007), 1228–1250. https://doi.org/10.1016/j.camwa.2006.05.028
8. A. Ungar, *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific, Hackensack, NJ, 2008. https://doi.org/10.1142/12478
9. J. Wattanapan, W. Atiponrat, T. Suksumran, Extension of the Švarc–Milnor lemma to gyrogroups, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, **115** (2021), 122. https://doi.org/10.1007/s13398-021-01062-y
10. T. Suksumran, *Essays in Mathematics and its Applications: In Honor of Vladimir Arnold*, Springer, Switzerland, 2016. https://doi.org/10.1007/978-3-319-31338-2
11. T. Suksumran, A simple bound on the ℓ-rank of finite gyrogroups, *Eur. J. Pure Appl. Math.*, **19** (2026), 7399.
12. L. Bussaban, A. Kawekhao, S. Suantai, Cayley graphs of gyrogroups, *Quasigroups Relat. Syst.*, **27** (2019), 25–32.
13. C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer, New York, 2001.
14. T. Abe, K. Watanabe, Finitely generated gyrovector subspaces and orthogonal gyrodecomposition in the Möbius gyrovector space, *J. Math. Anal. Appl.*, **449** (2017), 77–90. https://doi.org/10.1016/j.jmaa.2016.11.039

15. T. Abe, O. Hatori, Generalized gyrovector spaces and a Mazur–Ulam theorem, *Publ. Math. Debrecen*, **87** (2015), 393–413.
16. T. Abe, Gyrometric preserving maps on Einstein gyrogroups, Möbius gyrogroups and proper velocity gyrogroups, *Nonlinear Funct. Anal. Appl.*, **19** (2014), 1–17.
17. T. Suksumran, *Mathematical Analysis and Applications*, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-31339-5
18. T. Suksumran, K. Wiboonton, Isomorphism theorems for gyrogroups and L-subgyrogroups, *J. Geom. Symmetry Phys.*, **37** (2015), 67–83. https://doi.org/10.7546/jgsp-37-2015-67-83
19. R. Maungchang, P. Khachorncharoenkul, K. Prathom, T. Suksumran, On transitivity and connectedness of Cayley graphs of gyrogroups, *Heliyon*, **7** (2021), e07049. https://doi.org/10.1016/j.heliyon.2021.e07049
20. R. van Dam, M. Jazaeri, On bipartite distance-regular Cayley graphs with small diameter, *Electron. J. Combin.*, **29** (2022). https://doi.org/10.37236/10757